The F-theorem in the melonic limit

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Joint work with D. Benedetti, R.Gurau and D. Lettera - arXiv:2111.11792

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c-, a- and F-theorems

- Under the RG flow between fixed points: always decrease
- Count degrees of freedom
- $d = 2$: c-theorem $\to$ central charge [Zamolodchikov '86]
- $d = 4$: a-theorem $\to$ Weyl anomaly coefficient a [Cardy '88; Komargodski, Schwimmer '11]
- $d = 3$: No anomaly ! Is there a quantity decreasing along the RG flow ?

F-theorem

- Free energy on the sphere
- Sphere: regulates IR divergences
- UV divergences: F is the finite part of the free energy
- Proof using relation between free energy and entanglement entropy [Casini, Huerta]
- Role of unitarity ?

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- Role of unitarity ?

Long-range $O(N)^3$ model $\Rightarrow$ Non-trivial example

Outline

Going to the sphere

- Conformally flat metric

$$g_{\mu\nu}(x) = \Omega(x)^2 \delta_{\mu\nu} \ , \quad \Omega(x) = \frac{2a}{(1+x^2)}$$

- Transformation of primary fields

$$\mathcal{O}(x) \to \Omega(x)^{-\Delta_{\mathcal{O}}} \mathcal{O}(x)$$

- In practice: flat distance $\to$ chordal distance

$$s(x,y) = 2a \frac{|x-y|}{(1+x^2)^{1/2}(1+y^2)^{1/2}} = |x-y| \Omega(x)^{1/2} \Omega(y)^{1/2}$$

Long-range models on the sphere

Scalar Laplacian on the sphere

- Eigenmodes: spherical harmonics
- Eigenvalues

$$\omega_n = \frac{n(n+d-1)}{a^2}\ , D_n = \frac{(n+d-2)!(2n+d-1)}{n!(d-1)!}$$

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Long-range Laplacian on the sphere $\left(-\partial^2\right)^\zeta$

- Naively: fractional Laplacian with exponent $0 < \zeta < 1$
- Careful analysis:

$$\omega_n^{(\zeta)} = a^{-2\zeta} \frac{\Gamma(n + \frac{d}{2} + \zeta)}{\Gamma(n + \frac{d}{2} - \zeta)}$$

Flow between two Gaussian CFTs

$$S_{\text{Gauss}}[\phi] = \underbrace{\frac{1}{2}\int \mathrm{d}^d x\, \phi(x)(-\partial^2)^{\zeta}\phi(x)}_{\text{Generalized free field theory}} + \underbrace{\frac{\lambda}{2}\int \mathrm{d}^d x\, \phi(x)(-\partial^2)\phi(x)}_{\text{Short-range free action}}$$

- $0 < \zeta < 1$
- Two-point function $(p^{2\zeta} + \lambda p^2)^{-1}$
  $\rightarrow\ p^{-2\zeta}$ when $p \rightarrow 0$
  $\rightarrow\ p^{-2}$ when $p \rightarrow \infty$
- RG flow between long-range free action in the IR and short-range in the UV

Free energy

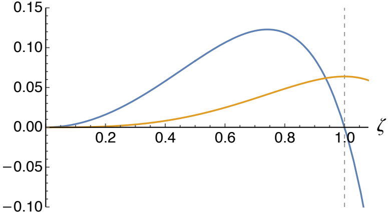
$$F = \frac{1}{2}\,\mathrm{Tr}[\ln C^{-1}] = \frac{1}{2}\sum_{n\geq 0} D_n \,\ln\left(\omega_n^{(\zeta)}\right)$$

- Compare free-energy at the fixed points
- GFFT with different values of ζ
- Study the variations of F with respect to ζ

$$\frac{dF}{d\zeta} = -\zeta\frac{\sin(\pi\zeta)}{\sin(\pi d/2)}\frac{\Gamma(d/2-\zeta)\Gamma(d/2+\zeta)}{\Gamma(1+d)}$$

Variation of the free energy for $d = 3$

$$\frac{dF}{d\zeta} = \frac{1}{24}\pi\zeta\left(1 - 4\zeta^2\right)\tan(\pi\zeta)$$



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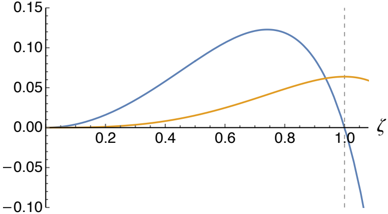
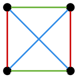


- Free energy decreases along the RG flow: respects the F-theorem
- $\zeta > 1$: UV and IR exchanged
- Trivial counter-example for **non-unitary** theories

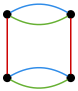
The long-range $O(N)^3$ model

$$S[\varphi] = \frac{1}{2}\int d^d x \, \varphi_{abc}(x)(-\Delta)^{\zeta}\varphi_{abc}(x) + S^{\text{int}}[\varphi]$$

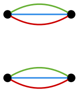
- $O(N)^3$ tensor model with quartic interactions [Carrozza, Tanasa, ...]



Tetrahedron



Pillow



Double-trace

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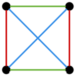

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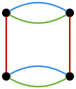

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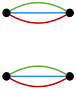


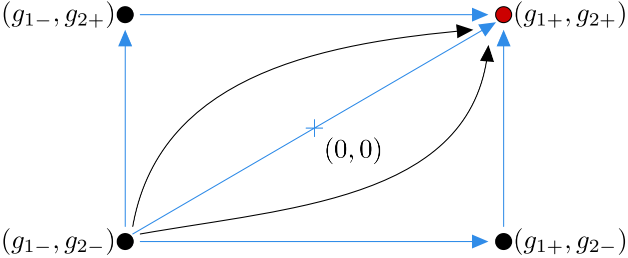
Double-trace

- $0 < \zeta < 1$: long-range model with $d < 4$ fixed
- Canonical dimension of the field: $\Delta = \frac{d-2\zeta}{2}$
- Marginal case: $\zeta = \frac{d}{4}$

RG trajectories

At large N:

- Tetrahedral coupling g does not flow
- Four lines of fixed point paramatrized by g
- One infrared attractive fixed point, stable and strongly interacting
- Explicit renormalization group trajectory from UV to IR fixed point



Further properties

- Exact computation of the two-point function in the large-N limit
- Four-point function: geometric series in a Bethe-Salpeter kernel
- No local stress-energy tensor but **conformally invariant** fixed points
- Strong indications of **unitarity** at the large-N fixed points
- Breaking of unitarity at NLO

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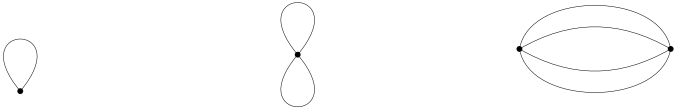
- What happens when putting this model on the sphere ?
- Does it respect the F-theorem ?

Schwinger-Dyson equation

Three types of vacuum 2PI-diagrams occurring at large N



$$G^{-1}(x,y) = C^{-1}(x,y) + \left(m^{d/2} + \lambda_2 G(x,x)\right) \frac{\delta(x-y)}{\sqrt{g(x)}} + \lambda^2 G(x,y)^3$$

- Metric $\sqrt{g(x)} = \left(\frac{2a}{1+x^2}\right)^d$
- Covariance on the sphere:

$$C(x,y) = \frac{c(\Delta)}{s(x,y)^{2\Delta}}\,, \qquad c(\Delta) = \frac{\Gamma(\Delta)}{2^{d-2\Delta}\pi^{d/2}\Gamma(\frac{d}{2}-\Delta)}$$

Solution of the Schwinger-Dyson equation

- Tune bare mass to cancel tadpole and divergent part of melonic integral
- SDE solved by:

$$G_\star(x,y) = \mathcal{Z}\, C(x,y) = \mathcal{Z}\,\frac{c(\Delta)}{s(x,y)^{2\Delta}}$$

$$\mathcal{Z} = 1 + \lambda^2 \mathcal{Z}^4 \frac{4\Gamma(1-d/4)}{d(4\pi)^d \Gamma(3d/4)}$$

$\rightarrow$ Same equation as in flat space

- Square root singularity at λ_c: model defined for $g < g_c \equiv \lambda_c\, \mathcal{Z}(\lambda_c)^2$

Leading order

$$F_{LO} = N^3 \left(\frac{1}{2} \mathcal{Z} \, \mathrm{Tr}[C^{-1} C] + \frac{1}{2} \mathrm{Tr}[\ln(\mathcal{Z}^{-1} C^{-1})] + \frac{m^{2\zeta}}{2} \mathcal{Z} \int_x C(x,x) + \frac{\lambda_2 \mathcal{Z}^2}{4} \int_x C(x,x)^2 + \frac{\lambda^2 \mathcal{Z}^4}{8} \int_{x,y} C(x,y)^4 \right)$$

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- $C(x,x) = 0$ by analytic continuation
- Melon integral: set $\Delta = \frac{d-\epsilon}{4}$

$$M_\epsilon = \frac{a^{2\epsilon} \, \Gamma(\frac{d+\epsilon}{4})^4 \, \Gamma(-\frac{d}{2} + \epsilon)}{2^{3d-1} \, \pi^{d-1/2} \, \Gamma(\frac{d-\epsilon}{4})^4 \, \Gamma(\frac{d+1}{2}) \, \Gamma(\epsilon)} \xrightarrow[\epsilon \to 0]{} 0$$

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$\Rightarrow$ Reduces to N^3 times the free energy of the GFFT

Next-to-next-to-leading order

- NLO: only contribution is a *figure eight* diagram $\rightarrow$ vanishes
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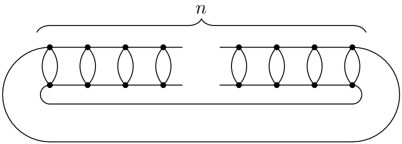

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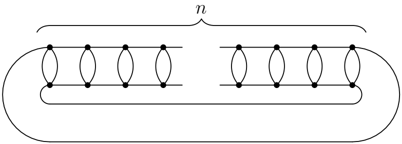
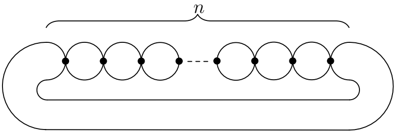


- Chains of bubbles with pillow vertices



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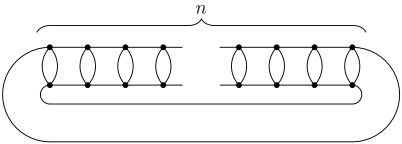

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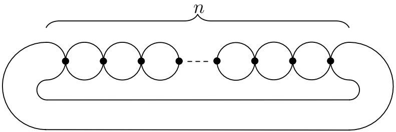
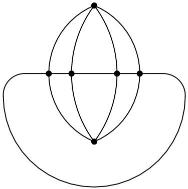


- Mixing of chains and ladders

Special diagram

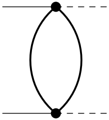
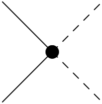


- Gives **finite** contribution to the free energy
- Only depends on the tetrahedral coupling: same value at all the fixed points
- **No role** in checking the F-theorem

Resumming the ladders

$$F_{\mathrm{NNLO}} = \frac{N^2}{2}\left(\mathrm{Tr}[\ln(\mathbb{I} - K_1)] + \mathrm{Tr}[K_1]\right)$$

with K_1 the four-point kernel:

$K_1 = -\lambda^2$  $-\ \lambda_1$ 

Non-trivial resummation $\rightarrow$ use **conformal partial wave expansion**

Conformal partial wave expansion

$$\Psi_{h,J}^{\Delta,\Delta,\widetilde{\Delta},\widetilde{\Delta}} \sim |\psi_n >< \psi_n|$$

- $|\psi_n >$: basis for bilocal functions (three-point functions)
- CPW: basis for conformal four-point functions
- Labeled by the scaling dimension h in the principal series

$$\mathcal{P}_+ = \left\{ h | h = \frac{d}{2} + ir,\; r \in \mathbb{R}_+ \right\}$$

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- Standard CFT technique for four-point functions
- First application to the free energy
- In practice: insert resolution of the identity

CPW expansion of the free energy

$$F_{\mathrm{NNLO}} = \frac{N^2}{2} \sum_{J \in \mathbb{N}_0} \int_{\frac{d}{2}}^{\frac{d}{2}+\mathrm{i}\,\infty} \frac{\mathrm{d}h}{2\pi\,\mathrm{i}} \rho(h,J) \left(\ln(1-k(h,J)) + k(h,J) \right)$$
$$\times \mathcal{N}^{\Delta}_{h,J} \mathcal{N}^{\widetilde{\Delta}}_{\tilde{h},J} \, \mathrm{Tr}[\Psi^{\Delta,\Delta,\widetilde{\Delta},\widetilde{\Delta}}_{h,J}]$$

- ρ, $\mathcal{N}$ known conformal quantities
- $k(h,J)$ kernel eigenvalues

$$k(h,J) = -\frac{g^2}{(4\pi)^d} \frac{\Gamma(-\frac{d}{4} + \frac{h+J}{2})\Gamma(\frac{d}{4} - \frac{h-J}{2})}{\Gamma(\frac{3d}{4} - \frac{h-J}{2})\Gamma(\frac{d}{4} + \frac{h+J}{2})},$$

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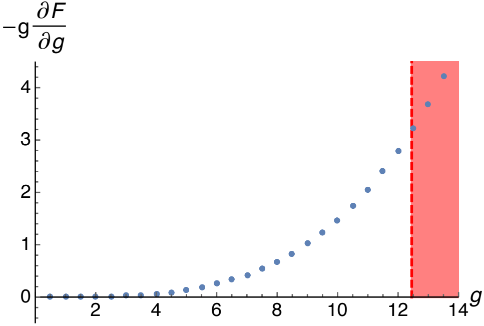
- Consider $-g\frac{\partial F_{\mathrm{NNLO}}}{\partial g}$
- Shift dimension of shadow operators to regulate divergences
- One finite sum remaining: can be computed numerically

Renormalized sphere free energy

For $d = 3$, $g = 1$ and $a = 1$:

$$-g \frac{\partial}{\partial g} F_{\mathrm{NNLO}} = 7.57 \times 10^{-4} \, N^2$$

For different values of g up to g_c:



The non-normalizable contribution

- IR: CPW expansion restricted to the principal series
- UV: One primary operator has dimension below $d/2 \to$ Add a non-normalizable state

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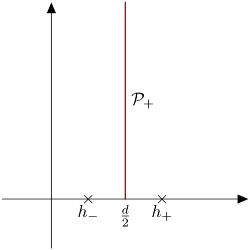
- Solutions of $k(h, 0) = 1$



- Physical dimension on the left of the contour
- UV: add minus the residue at $h = h_-$
- More intuitive from a perturbative point of view

The non-normalizable contribution: perturbative check

$$g_{1\pm} = \pm\sqrt{g^2}\big(1 + \mathcal{O}(g^2)\big) + g^2\big(\psi(1) + \psi(d/2) - 2\psi(d/4) + \mathcal{O}(g^2)\big)$$

- From the UV to the IR, goes from $g_{1-} \simeq -\sqrt{g^2}$ to $g_{1+} \simeq \sqrt{g^2}$

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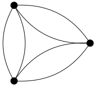
- One contribution at order $|g|^3$:



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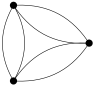
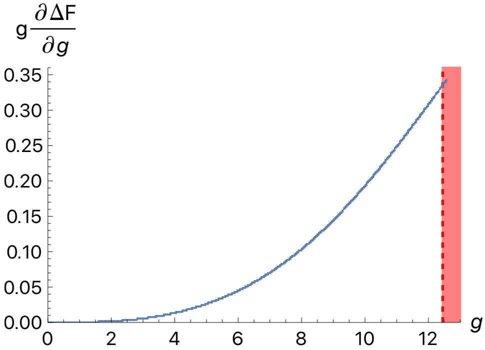


- Variation of free energy:

$$g\frac{\partial}{\partial g}\left(F^{UV}_{\mathrm{NNLO}} - F^{IR}_{\mathrm{NNLO}}\right) = 16\frac{\Gamma(-d/2)|g|^3}{2^{3d}\pi^{3d/2}\Gamma(d)}N^2 + \mathcal{O}(|g|^5)$$

$\Rightarrow$ Positive for $2 < d < 4$

Numerical evaluation at finite g



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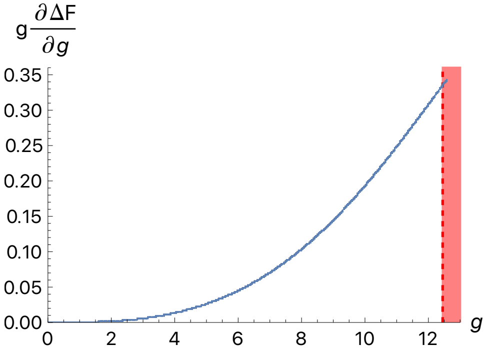

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$\Rightarrow$ The long-range $O(N)^3$ model satisfies the F-theorem

Conclusion

- F-theorem: for CFTs in $d = 3$, the sphere free energy decreases along the RG flow
- Proof using relation with entanglement entropy
- Relies heavily on unitarity
- Long-range $O(N)^3$ model satisfies the F-theorem
- Use of conformal partial wave expansion to resum ladders
- Further hint of the unitarity of the model at large N ?
- Lower orders in $1/N$?