Non-oriented maps and BKP hierarchy

Applications to the $O(N)$-BGW model and recurrence formulas

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Or. . .

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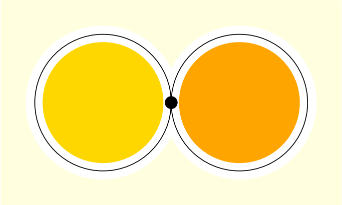
- How many matrix integrals can one cram into a 1 hour presentation?
- A lot
- Too many for your sake
- Rewrite you well-known matrix integrals using algebraic combinatorics

Orientable maps

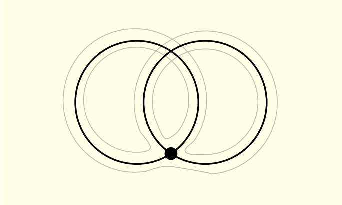
- Graphs "properly" embedded in surfaces
- No crossings
- Graph complement is disjoint union of disks, i.e. faces
- Euler's relation relates topology to combinatorics

$$2 - 2g = F - E + V$$

- Plane projection (hence crossings)
- Find faces by following edges and corners at vertices



$g = 0$

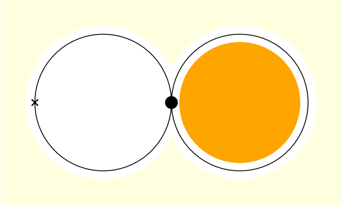


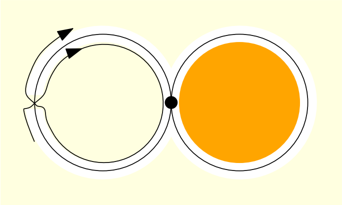
$g = 1$

The two non contractible cycles of torus

- Encoded by permutations $\to$ rep. theory of $\mathfrak{S}_n$

- Maps on non-necessarily orientable surfaces
- Half-integer genus counting handles and cross-caps
- Drawing with **twisted** edges





One face less than sphere, one more than torus
$g = 1/2$ Projective plane

- Matchings instead of permutations $\rightarrow$ rep. theory of $H_n \diagdown \mathfrak{S}_{2n} \diagup H_n$

Recurrences for 3 families of orientable maps

- $t_g^n = |\{\#$ triangulations of genus g with n triangles$\}|$
  [Kazakov-Kostov-Nekrasov 99 in appendix, Goulden-Jackson 08]

$$(n+1)t_g^n = 4n(3n-2)(3n-4)t_{g-1}^{n-1} + 4 \sum_{\substack{i+j=n-2 \\ h+k=g}} (3i+2)(3j+2)t_h^i t_k^j$$

- $m_g^n = |\{\#$ maps of genus g with n edges & weight u per vertex$\}|$
  [Carrell-Chapuy 15, Kazarian-Zograf 15]

$$(n+1)m_g^n = 2(1+u)(2n-1)m_g^{n-1} + \frac{1}{2}(2n-3)(2n-2)(2n-1)m_{g-1}^{n-2} + 3 \sum_{\substack{i+j=n-2 \\ h+k=g}} (2i+1)(2j+1)m_h^i m_k^j$$

- $b_g^n = \{\#$ bip. maps, weight u per white vertex & v per black vertex$\}|$
  [Kazarian-Zograf 15]

$$(n+1)b_g^n = \alpha^n(u,v)b_{g-1}^n + \beta^n(u,v)b_{g-2}^n + \gamma^n b_{g-2}^{n-1} + \sum_{\substack{i+j=n-2 \\ h+k=g}} \mu_i \mu_j b_h^i b_k^j$$

What's interesting?

More

- Corollaries: maps with given number of faces and vertices, bijections [Carrell-Chapuy 15]
- 1-face maps: linear eqs (reproduce Harer-Zagier & Adrianov)
- Application: Local limits of maps of high genus [Budzinski-Louf 19-20]
- Constellations [Louf 19]
- Robustness? e.g. quadrangulations? Not known...
- Same 3 families but non-oriented [VB-Chapuy-Dołęga 21]

What's remarkable

- Extremely simple
- Mysterious origin
  - Tutte equations? Schaeffer-like bijections?
  - No: **integrable hierarchy**
  - Bijective explanation for 1-face maps [Chapuy-Féray-Fusy C-decorated trees]
  - Bijection for planar maps [Louf 18]
- Computational efficiency

Plan

Orientable maps and constellations

- KP hierarchy
- What is it? How to derive it?
  - From matrix models for maps
  - From Virasoro constraints
- Monotone Hurwitz numbers and the Brézin-Gross-Witten (BGW) and IZ integrals
- Enumerative applications

b-conjecture and β-ensemble

Non-oriented maps and constellations

- BKP hierarchy for non-oriented maps, bip maps and monotone Hurwitz
- Applications to $O(N)$-BGW model
- Enumerative applications

Symmetric functions

- Indeterminates $x_1, x_2, x_3, \ldots$
- **Power-sum** functions

$$p_k(x_1, x_2, \ldots) = \sum_{i \geq 1} x_i^k$$

- If $M = \mathrm{diag}(x_1, x_2, \ldots)$ then $p_k(M) = \mathrm{tr}\, M^k$
- Let $\lambda \vdash n$ a partition: $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ with $\sum_{i \geq 1} \lambda_i = n$
- Power-sum basis of symmetric functions

$$p_\lambda(x_1, x_2, \ldots) = p_{\lambda_1}(x_1, x_2, \ldots) p_{\lambda_2}(x_1, x_2, \ldots) \cdots$$

- Basis of homogeneous polynomials of degree n

Examples

- Elementary symmetric function $e_2 = \sum_{i<j} x_i x_j$
- Degree 2: partitions (2) and $(1,1)$

$$\sum_{i<j} x_i x_j = \frac{1}{2}(p_1^2 - p_2)$$

- Complete homogeneous symmetric function $h_3 = \sum_{i\leq j\leq k} x_i x_j x_k$
- Degree 3: partitions $(3), (2,1)$ and $(1,1,1)$

$$\sum_{i\leq j\leq k} x_i x_j x_k = \frac{1}{3} p_3 + \frac{1}{2} p_2 p_1 + \frac{1}{6} p_1^3$$

- Not writing x-dependance bc indpdt variables
- Beware! If finite number of x_i, say x_1, x_2, then

$$\frac{1}{6} p_1^3 - \frac{1}{2} p_2 p_1 + \frac{1}{3} p_3 = 0$$

Schur functions

- Power-sums are orthogonal for Hall scalar product on symmetric functions

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda$$

with $z_\lambda = \prod_i n_i! i^{n_i}$ automorphism factor

- In power-sum basis

$$\langle f(p_1, p_2, \dots), g(p_1, p_2, \dots) \rangle = f\left(\frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, \dots\right) g(p_1, p_2, \dots)|_{p_1=p_2=\dots=0}$$

- Unique orthonormal $\mathbb{Z}$-basis: **Schur** functions

$$s_\lambda = \sum_\mu \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu$$

with $\chi_\lambda(\mu)$ character of symm. group in irrep λ evaluated in conjugacy class μ

- Generating series

$$e^{V(x)} = e^{\sum_{k\geq 1} \frac{p_k}{k} x^k} = \sum_k s_{(k)}(p_1, p_2, \dots) x^k$$

and **Jacobi-Trudi** formula

$$s_\lambda = \det(s_{(\lambda_i - i + j)})_{1\leq i,j\leq n}$$

The truth about KP

Integrable hierarchies as Schur exp. with special coefficients

- Let $A \in GL(\infty)$, e.g. bi-infinite matrix s.t. $A - I$ has finite number of non-zero entries
- Determinants are well-defined
- Solutions of **KP** hierarchy

$$\tau(p_1, p_2, \dots) = \sum_{\lambda} \det(A^{\lambda_1 - 1, \lambda_2 - 2, \dots}_{-1, -2, \dots}) \; s_\lambda$$

- Solutions of **BKP** hierarchy [Kac-van de Leur, Carrell]

$$\tau_n(p_1, p_2, \dots) = \sum_{\lambda} \mathrm{Pf}(A^{\lambda_1 + n - 1, \lambda_2 + n - 2, \dots}_{\lambda_1 + n - 1, \lambda_2 + n - 2, \dots}) \; s_\lambda$$

n is the charge (matrix size in matrix models)

KP equation

- Minor determinants satisfy special eqs, **Plücker** relations
- Say $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$

$$A_{12}A_{34} + A_{14}A_{23} - A_{13}A_{24} = 0$$

- Same in $GL(\infty)$

$$\underbrace{A^{-1,-2,-3\dots}_{-1,-2,-3\dots}}_{\lambda=\emptyset}\underbrace{A^{1,0,-3\dots}_{-1,-2,-3\dots}}_{\lambda=(2,2)} + A^{0,-1,-3\dots}_{-1,-2,-3\dots}A^{1,-2,-3\dots}_{-1,-2,-3\dots} - A^{0,-2,-3\dots}_{-1,-2,-3\dots}A^{1,-1,-3\dots}_{-1,-2,-3\dots} = 0$$

- Minor dets as scalar products

$$\langle s_\emptyset, \tau\rangle\langle s_{(2,2)}, \tau\rangle + \langle s_{(2)}, \tau\rangle\langle s_{(1,1)}, \tau\rangle - \langle s_{(2,1)}, \tau\rangle\langle s_{(1)}, \tau\rangle = 0$$

- Evaluate scalar products using $\langle s_\lambda, \tau\rangle = s_\lambda(\frac{\partial}{\partial p_1}, \frac{2\partial}{\partial p_2}, \dots)\tau|_{p_1=p_2=\dots=0}$
- KP equation (with $F = \ln\tau$)

$$-F_{3,1} + F_{2,2} + \frac{1}{2}F_{1,1}^2 + \frac{1}{12}F_{1,1,1,1} = 0 \qquad \left(F_i = \frac{\partial F}{\partial p_i}\right)$$

How to use the KP equation

- Generating series $F(t, p_1, p_2, \dots)$ of maps with weight p_k on faces of degree k
- Use KP equation for F as a black box $-F_{3,1} + F_{2,2} + \frac{1}{2}F_{1,1}^2 + \frac{1}{12}F_{1,1,1,1} = 0$
- Two problems:
  1. Infinite set of solutions... How to pick the correct one?
  2. Eliminate partial derivatives
- Tutte equations a.k.a. Virasoro constraints
- For maps (use Schwinger-Dyson on Hermitian 1-matrix model)

$$\sum_{k=0}^{n} \langle \mathrm{tr}\, M^k \,\mathrm{tr}\, M^{n-k} \rangle - \frac{1}{2t^2}\langle \mathrm{tr}\, M^{n+2} \rangle + \sum_k p_k \langle \mathrm{tr}\, M^{n+k} \rangle = 0$$

- As differential eqs $\langle \mathrm{tr}\, M^n \rangle = \frac{k\partial}{\partial p_k} e^F \Rightarrow L_n e^F = 0$

$$L_n = \frac{-1}{t^2}\frac{(n+2)\partial}{\partial p_{n+2}} + \sum_{k+l=n} \frac{kl\partial^2}{\partial p_k \partial p_l} + \sum_{k\geq 1} p_k \frac{(k+n)\partial}{\partial p_{k+n}} + 2u\frac{n\partial}{\partial p_n} + up_1\delta_{n,-1} + u^2\delta_{n,0}$$

- Specialization to eliminate the p_k
  - Maps & bip maps: $p_i \to z$ for all i
  - Triangulations: $p_i \to z\delta_{i,3}$ on maps

A calculation

- Specialization operator: $\overline{A}(t,z) = A(t, \vec{p} = (z, z, \dots))$
- Let us determine $\overline{F_1}$ as a function of t
- Consider the constraint L_{-1} and specialize it

$$L_{-1} = -\frac{\partial}{\partial p_1} + t^2 \sum_{n\geq 2}(n-1)p_n \frac{\partial}{\partial p_{n-1}} + t^2 u p_1$$

$$\underset{\text{specialization}}{\Rightarrow} \quad \overline{F_1} = t^2 z \sum_{n\geq 2}(n-1)\overline{F_{n-1}} + t^2 uz$$

- Homogeneity equation

$$\frac{t\partial}{\partial t}F = \sum_{n\geq 1} p_n \frac{n\partial}{\partial p_n} F \quad \Rightarrow \quad t\frac{d\overline{F}}{dt} = z\sum_{n\geq 1} n\overline{F_n}$$

- It comes

$$\overline{F_1} = t^3 \frac{d\overline{F}}{dt} + t^2 uz$$

Theorem

- KP eq + Finite set of Virasoro constraints + Homogeneity
- Specialization

$$\Rightarrow \qquad \text{Polynomial}\left(z, t, F, \frac{d}{dt}\right) = 0$$

- Observed in appendix by [Kazakov-Kostov-Nekrasov 99]

Non-oriented maps and BKP equation

- RHS of KP equation becomes non-zero

$$-F_{3,1}(u) + F_{2^2}(u) + \frac{1}{2}F_{1^2}(u)^2 + \frac{1}{12}F_{1^4}(u) = S(u)e^{F(u+2)+F(u-2)-2F(u)}$$

- Shifts on u (matrix size in matrix models)

Recurrence relations for non-oriented maps

Theorem [VB-Chapuy-Dołęga 21]

For non-oriented maps, bip. maps and triangulations

1. Recurrence with non-polyn. coefficients, coming from ODEs with **shifts**

$$\frac{d}{dt}\big(F(t,z,u+2)+F(t,z,u-2)-2F(t,z,u)\big)\ P\left(\frac{d}{dt},F\right)=Q\left(\frac{d}{dt},F\right)$$

   Non-polynomial coeff. because extract coeff. from shifted series $[u^r](u+2)^k$

2. Recurrence from **ODEs** (no shifts), e.g. for maps

$$P\left(\frac{dF}{dt},\dots,\frac{d^6F}{dt^6}\right)=0$$

   with explicit P of degree 5

- Large expression (still reasonably bounded number of pages), but everything is fast to evaluate!
- Maple worksheet and html version on G. Chapuy's page

A table

Rooted maps of genus g with n edges, orientable or not

$$h_n^g = \frac{2}{(n+1)(n-2)} \Bigg(n(2n-1)(2h_{n-1}^g + h_{n-1}^{g-1/2}) + \frac{(2n-3)(2n-2)(2n-1)(2n)}{2} h_{n-2}^{g-1}$$

$$+12 \sum_{\substack{g_1=0..g \\ g_1+g_2=g}} \sum_{\substack{n_1=0..n \\ n_1+n_2=n}} \frac{(2n_2-1)(2n_1-1)n_1}{2} h_{n_2-1}^{g_2} h_{n_1-1}^{g_1} - \sum_{\substack{g_1=0..g \\ g_1+g_2=g}} \sum_{\substack{n_1=0..n-1 \\ n_1+n_2=n}} \sum_{\substack{g_0=0..g_1 \\ g_1-g_0\in\mathbb{N}}} \binom{n_1+2-2g_0}{n_1-2g_1} 2^{2(1+g_1-g_0)} h_{n_1}^{g_0}$$

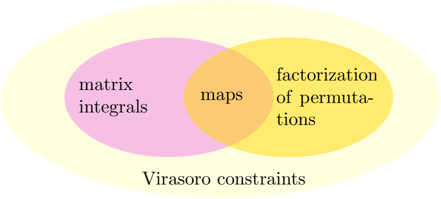
$$\Bigg(\frac{(2n_2-1)(2n_2-2)(2n_2-3)}{2} h_{n_2-2}^{g_2-1} - \delta_{(n_2,g_2)\neq(n,g)} \frac{n_2+1}{4} h_{n_2}^{g_2} + \frac{2n_2-1}{2} (2h_{n_2-1}^{g_2} + h_{n_2-1}^{g_2-1/2})$$

$$+ 6 \sum_{\substack{g_3=0..g_2 \\ g_3+g_4=g_2}} \sum_{\substack{n_3=0..n_2 \\ n_3+n_4=n_2}} \frac{(2n_3-1)(2n_4-1)}{4} h_{n_3-1}^{g_3} h_{n_4-1}^{g_4} \Bigg) \Bigg)$$

$n\backslash g$	5/2	3	7/2	4
5	8229	0	0	0
6	516958	166377	0	0
7	19381145	13093972	4016613	0
8	562395292	595145086	382630152	113044185
9	13929564070	20431929240	20549348578	12704958810
10	309411522140	587509756150	818177659640	790343495467
11	6344707786945	14923379377192	26881028060634	35918779737610
12	122357481545872	345651571125768	770725841809552	1330964564940140
13	2247532739398856	7452363840633244	19946409152977346	42611002435124552
14	39681114425793904	151717486205709730	476412224477845444	1220973091185233106
15	677939355268197412	2946794762696249280	10665684328125155376	32054128913697072040
16	11265765391845733784	55029552840385680100	226357454725004343024	783804517126931727890

Some solutions of KP hierarchy

It is enough to prove expansion on Schur with minor coefficients



(More cyclic types available with factorizations)

- Direct calculation for matrix integrals and factorizations of permutations
- Virasoro constraints technique well-known
- First application to prove BKP in [VB-Chapuy-Dołęga]

Matrix integral

- Integral for maps

$$\tau = \int_{\mathcal{H}_N} dM \; e^{-\frac{1}{2t^2}\,\mathrm{tr}\, M^2 + \mathrm{tr}\, V(M)} = \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} dx_i e^{-\frac{x_i^2}{2t^2} + V(x_i)} \right) |\det(x_j^{i-1})|^2$$

- Use Andreev's formula

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} dx_i \rho(x_i) \right) \det f_i(x_j) \det g_i(x_j) = \det \left(\int_{\mathbb{R}} dx \rho(x) f_i(x) g_j(x) \right)_{1 \le i,j \le N}$$

with $f_i(x) = g_i(x) = x^{i-1}$ and $\rho(x) = e^{-\frac{x^2}{2t^2}}$

- Use $e^{V(x)} = \sum_k h_k x^k$, then Cauchy-Binet formula

$$\det AB = \sum_I \det A_I \det B_I$$

and Jacobi-Trudi

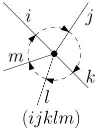
$$\tau = \sum_\lambda \det \left(\int_{\mathbb{R}} dx e^{-\frac{x^2}{2t^2}} x^{\lambda_i + N - i + j - 1} \right)_{1 \le i,j \le N} s_\lambda$$

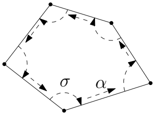
Permutation encoding of maps

Maps

- Label half-edges
- Permutation α maps half-edges from same edge
- σ for vertices and ϕ for faces

$$\sigma\alpha = \phi$$





Bipartite maps and constellations

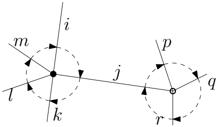
- Label edges
- $\sigma_\bullet$ around black vertices
- $\sigma_\circ$ around white vertices

$$\sigma_\bullet\sigma_\circ = \phi$$

- Constellations aka Hurwitz numbers

$$\sigma_1 \cdots \sigma_d = \phi$$

aka factorizations of permutations



Enumeration

- Infinite set of indeterminates $\vec{p} = (p_1, p_2, \dots)$
- Counting w.r.t. cyclic type λ of a permutation

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}$$

Example

- Counting permutations of given cycle type

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\lambda \vdash n} |C_\lambda| p_\lambda$$

- Orbit-stabilizer: $|C_\lambda| = n!/z_\lambda$

$$\sum_\lambda \frac{t^{|\lambda|}}{z_\lambda} p_\lambda = e^{\sum_{n\geq 1} \frac{p_n}{n} t^n} = e^{V(t)} = \sum_{n\geq 0} t^n s_{(n)}$$

Enumeration for constellations

via Frobenius formula

- For $d+1$ cyclic types

$$\tau(\vec{p}^{(1)},\dots,\vec{p}^{(d)},\vec{q}) = \sum_n \frac{t^n}{n!} \sum_{\lambda^{(1)},\dots,\lambda^{(d)},\mu\vdash n} C(\lambda^{(1)},\dots,\lambda^{(d)},\mu) p^{(1)}_{\lambda^{(1)}} \cdots p^{(d)}_{\lambda^{(d)}} q_\mu$$

- Cyclic types are invariant under conjugation, so is $\sigma_1 \cdots \sigma_d \phi^{-1} = 1$
- **Central** problem, solved with characters $\chi_\lambda(\mu)$ of symmetric group
- **Frobenius formula** counts number of factorisations of type $\lambda^{(1)},\dots,\lambda^{(d)},\mu$

$$C(\lambda^{(1)},\dots,\lambda^{(d)},\mu) = \sum_{\lambda\vdash n} \frac{n!}{\mathrm{hook}(\lambda)^{1-d}} \frac{\chi_\lambda(\lambda^{(1)})}{z_{\lambda^{(1)}}} \cdots \frac{\chi_\lambda(\lambda^{(d)})}{z_{\lambda^{(d)}}} \frac{\chi_\lambda(\mu)}{z_\mu}$$

- Schur magic: $s_\lambda(\vec{p}) = \sum_\mu \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu$

$$\tau(\vec{p}^{(1)},\dots,\vec{p}^{(d)},\vec{q}) = \sum_\lambda \frac{t^{|\lambda|}}{\mathrm{hook}(\lambda)^{1-d}} s_\lambda(\vec{p}^{(1)}) \cdots s_\lambda(\vec{p}^{(d)}) s_\lambda(\vec{q})$$

Quick summary

- I described two ways to prove KP for maps

Tension for non-oriented maps

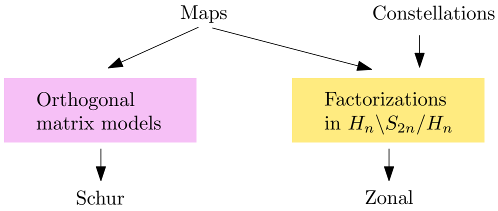
- Schur expansion to prove hierarchies
- Rep. theory of $H_n \backslash \mathfrak{S}_{2n} / H_n$ leads to expansion on **zonal** polynomials instead of Schur
  - For bipartite maps [Goulden-Jackson 96] and constellations [Chapuy-Dołęga 20, Ben Dali 21]

$$\tau(\vec{p}^{(1)}, \dots, \vec{p}^{(d)}, \vec{q}) = \sum_{\lambda} \frac{1}{\text{hook}(2\lambda)} Z_\lambda(\vec{p}^{(1)}) \cdots Z_\lambda(\vec{p}^{(d)}) Z_\lambda(\vec{q})$$

  - Proving zonal exp not too easy
  - No thms relating integrable hierarchies to zonal
- Do we have to relate zonal to Schur expansions **directly**?

Schur vs zonal

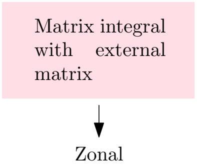
- Not a problem for maps bc matrix integrals expand directly on Schur!
- Proves BKP hierarchy for maps and bipartite maps [Adler-van Moerbeke, van de Leur 01]



- No BKP for non-oriented constellations?

Special case

- Matrix integrals with external matrix
- Includes $O(N)$-Itzykson-Zuber, $O(N)$-Brézin-Gross-Witten integrals



Orthogonal matrix integral

- Integral for non-oriented maps

$$\tau = \int_{\mathcal{S}_N} dM\ e^{-\frac{1}{2t^2}\operatorname{tr} M^2 + \operatorname{tr} V(M)} = \int_{\mathbb{R}^N} \left(\prod_{i=1}^N dx_i e^{-\frac{x_i^2}{2t^2} + V(x_i)} \right) |\operatorname{vdm}(x_i)|$$

- (van de Leur's?) trick $|\operatorname{vdm}(x_i)| = \operatorname{vdm}(x_i)\operatorname{Pf}(\operatorname{sgn}(x_i - x_j))$
- Use De Bruijn's formula (N even)

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^N dx_i \rho(x_i) \right) \det f_i(x_j) \operatorname{Pf}(h(x_i, x_j))$$

$$= \operatorname{Pf}\left(\int_{\mathbb{R}^2} dxdy\, \rho(x)\rho(y) f_i(x) h(x,y) f_j(y) \right)_{1 \le i,j \le N}$$

with $f_i(x) = x^{N-i}$ and $\rho(x) = e^{-\frac{x^2}{2t^2}}$ and $h(x,y) = \operatorname{sgn}(x,y)$

- Use $e^{V(x)} = \sum_k h_k x^k$, then Cauchy-Binet for Pfaffians and Jacobi-Trudi

$$\tau = \sum_\lambda \operatorname{Pf}\left(\int_{\mathbb{R}^2} dxdy e^{-\frac{x^2+y^2}{2t^2}} x^{\lambda_i + N - i} \operatorname{sgn}(x-y) y^{\lambda_j + N - j} \right)_{1 \le i,j \le N} s_\lambda$$

Interpolating orientable and non-oriented

- β-ensemble

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} dx_i e^{-\frac{x_i^2}{2t^2} + V(x_i)} \right) |\operatorname{vdm}(x_i)|^{\beta}$$

- Believed to not be integrable except at $\beta = 1, 2, 4$ for which genuine matrices exist
- Does it count things? β measuring non-orientability
- Natural expansions in which basis?

Jack symmetric functions: 1-parameter deformation $J_\lambda^{(\alpha)}$

- Orthogonal for a deformation of Hall scalar product

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda,\mu} z_\lambda \alpha^{\ell(\lambda)}$$

- $J_\lambda^{(1)} \sim s_\lambda$, related to rep. theory of $\mathfrak{S}_n$
- $J_\lambda^{(2)} = Z_\lambda$ zonal polynomials, related to rep. theory of $H_n \diagdown \mathfrak{S}_{2n} \diagup H_n$
- No representation theory for generic α

b-conjecture [Goulden-Jackson 96, Okounkov 97, Lacroix 09, Chapuy-Dołęga 20, Ben Dali 21-22]

- b-deformation of generating series: Schur $s_\lambda \to$ Jack $J^{(1+b)}$
  [Goulden-Jackson 96, Ben Dali 21]

$$\tau_b(\vec{p}^{(1)}, \dots, \vec{p}^{(d)}, \vec{q}) = \sum_\lambda \frac{J_\lambda^{(1+b)}(\vec{p}^{(1)}) \cdots J_\lambda^{(1+b)}(\vec{p}^{(d)}) J_\lambda^{(1+b)}(\vec{q})}{||J_\lambda^{(1+b)}||^2_{1+b}}$$

  - Orientable maps at $b = 0$
  - Non-oriented at $b = 1$
- Prove that it counts things
- Coefficients are polynomials in b with integer coefficients

$$\tau_b(\vec{p}^{(1)}, \dots, \vec{p}^{(d)}, \vec{q}) = \sum_n t^n \sum_{\lambda^{(1)}, \dots, \lambda^{(d)}, \mu \vdash n} \frac{c^\mu_{\lambda^{(1)}, \dots, \lambda^{(d)}}(b)}{(1+b)^{\ell(\mu)} z_\mu} p^{(1)}_{\lambda^{(1)}} \cdots p^{(d)}_{\lambda^{(d)}} q_\mu$$

- Difficult because no character theory for $b \neq 0, 1$
- Proof by [Chapuy-Dołęga 20] for $\vec{p}^{(2)} = (u_2, u_2, \dots), \dots, \vec{p}^{(d)} = (u_d, u_d, \dots)$

$$\sum_\lambda \left(\prod_{i=2}^{d} \prod_{\square \in \lambda} (u_i + (1+b)\text{-content}(\square)) \right) \frac{J_\lambda^{(1+b)}(\vec{p}^{(1)}) J_\lambda^{(1+b)}(\vec{q})}{||J_\lambda^{(1+b)}||^2_{1+b}}$$

Unitary integrals

Harish-Chandra–Itzykson–Zuber (HCIZ) integral

$$\tau_{HCIZ}(X,Y) = \int_{U(N)} dU\ e^{\mathrm{tr}(XUYU^\dagger)} = \int_{U(N)} dU\ e^{\mathrm{tr}(\mathrm{diag}(x_1,\ldots,x_N)U\,\mathrm{diag}(y_1,\ldots,y_N)U^\dagger)}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\le N}} \frac{s_\lambda(x_1,\ldots,x_N)s_\lambda(y_1,\ldots,y_N)}{\prod_{\square\in\lambda}(N+\mathrm{content}(\square))}$$

(Calculation can be continued using Cauchy-Binet)

Brézin-Gross-Witten (BGW) integral

$$\tau_{BGW}(X) = \int_{U(N)} dU\ e^{\mathrm{tr}(XU+X^\dagger U^\dagger)} = \sum_{\substack{\lambda \\ \ell(\lambda)\le N}} \frac{s_\lambda(XX^\dagger)}{\mathrm{hook}(\lambda)^2 s_\lambda(1^N)}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\le N}} \frac{s_\lambda(XX^\dagger)}{\mathrm{hook}(\lambda)\prod_{\square\in\lambda}(N+\mathrm{content}(\square))}$$

They satisfy KP as $N\to\infty$

HCIZ-BGW detailed calculation

- Traces are invariant under conjugation from $U(N)$
- Expand $e^{\mathrm{tr}(M)}$ on central functions, i.e. characters
- Then set $M = XUYU^\dagger$ and integrate over U using formulas for integrals of characters
- Characters of $GL(N)$ are Schur of their eigenvalues!
- Cauchy identity to expand on characters

$$e^{\mathrm{tr}\, M} = \sum_\lambda \frac{s_\lambda(M)}{\mathrm{hook}(\lambda)}$$

- Integral of character

$$\int_{U(N)} dU\ s_\lambda(XUYU^\dagger) = \frac{s_\lambda(X) s_\lambda(Y)}{s_\lambda(1^N)}$$

- For BGW use different integral

$$\int_{U(N)} dU\ s_\lambda(XU) s_\mu(X^\dagger U^\dagger) = \delta_{\lambda\mu} \frac{s_\lambda(XX^\dagger)}{s_\lambda(1^N)}$$

- Result of integral always in terms of Schur

Orthogonal integrals

What is $O(N)$-HCIZ?

- $O(N)$-Harish-Chandra is $X, Y \in \mathrm{Lie}(O(N))$: known from HC
- $O(N)$-Itzykson-Zuber is X, Y diagonal: not known! [Bergère-Eynard 08]

$O(N)$-BGW [Macdonald, Symmetric functions VII.3]

- Let $X = \mathrm{diag}(x_1, \dots, x_N)$

$$\tau_{BGW}^{O(N)}(X) = \int_{O(N)} dO\ e^{\mathrm{tr}(XO)} = \sum_\lambda \frac{1}{\mathrm{hook}(\lambda)} \int_{O(N)} dO\ s_\lambda(XO)$$

- Beware: Schur is not $O(N)$-character!

$$\int_{O(N)} dO\ s_\lambda(XO) = \frac{Z_{\lambda/2}(XX^\dagger)}{Z_{\lambda/2}(1^N)}$$

if $\ell(\lambda) \leq N$ and λ is even

- It comes

$$\tau_{BGW}^{O(N)}(X) = \sum_{\substack{\lambda \\ \ell(\lambda)\leq N}} \frac{Z_\lambda(XX^\dagger)}{\mathrm{hook}(2\lambda) Z_\lambda(1^N)}$$

BKP hierarchy

- Experimentally, 3 models satisfying BKP were found
- Non-oriented maps [van de Leur 01]
- Non-oriented bipartite maps [derivable from van de Leur 01]
- BGW-like function

$$\tau = \sum_{\lambda} \frac{Z_\lambda(2\vec{p})}{\mathrm{hook}(2\lambda) Z_\lambda(1^N)} \qquad \text{No bound and } N \text{ formal}$$

- van de Leur's trick not useful
- Transform zonal into Schur expansion!
- Nice expansions in both Schur and zonal?!
- Experimentally found

$$\tau = \sum_{\lambda} \frac{s_\lambda(\vec{p})}{\mathrm{hook}(\lambda)^2 o_\lambda(1^N)}$$

Thm [VB-Chapuy-Dołęga 21]

- Prove the equality

$$\sum_{\lambda} \frac{Z_\lambda(2\vec{p})}{\mathrm{hook}(2\lambda) Z_\lambda(1^N)} = \sum_{\lambda} \frac{s_\lambda(\vec{p})}{\mathrm{hook}(\lambda)^2 o_\lambda(1^N)}$$

- Prove Pfaffianity

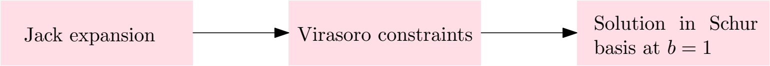
$$\frac{1}{\mathrm{hook}(\lambda)^2 o_\lambda(1^N)} = \prod_{k=1}^{n-1} (2k)!\,\mathrm{Pf}(a_{\lambda_i+n-i,\lambda_j+n-j})_{1\le i,j\le n}$$

where

$$a_{i,j} = \begin{cases} \frac{i-j}{4(i+j)\ \ i!^2 j!^2} & \text{for } \ i,j \ge 1, \\ \frac{1}{2i!^2} & \text{for } \ j \in \{-1,0\}, i > 0, \\ 1 & \text{for } \ j = -1, i = 0, \\ \frac{-1}{2j!^2} & \text{for } \ i \in \{-1,0\}, j > 0, \\ -1 & \text{for } \ i = -1, j = 0, \\ 0 & \text{for } \ i,j \in \{-1,0\}, i = j, \end{cases}$$

Proof

- No matrix model to use*



- Conclude with uniqueness
- *Virasoro constraints already appeared [Mironov-Morozov-Shakirov 11] as "pure gauge limit" of matrix model
- Here algebraic combinatorial proof
- For Pfaffianity, use Schur's Pfaffian formula

$$\prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{x_i + x_j} = \mathrm{Pf}\left(\frac{x_i - x_j}{x_i + x_j}\right)$$

- Virasoro constraints for generic b: proof of conjecture by [Féray 12]
- Relation between $Z_\lambda(1^N)$ and $o_\lambda(1^N)$? Proof of conjecture by [Oliveira-Novaes 20]

Pfaffian expression for $O(N)$-BGW

- Determinantal expression for $U(N)$-BGW
- If $XX^\dagger$ has eigenvalues $x_1, \dots, x_N$

$$\int_{U(N)} dU \; e^{\mathrm{tr}(XU + X^\dagger U^\dagger)} = \frac{\det\left(x_i^{\frac{N-j}{2}} I_{N-j}(2\sqrt{x_i})\right)}{\prod_{i<j}(x_i - x_j)}$$

- If XX^t has eigenvalues with multiplicity two $(x_1, x_1, x_2, x_2, \dots, x_n, x_n)$

$$\int_{O(2n)} dO \; e^{\mathrm{tr}(XO)} = \frac{\mathrm{Pf}\, M(x_i, x_j)}{\prod_{1 \leq i < j \leq n}(x_i - x_j)}$$

with

$$M(x,y) = \frac{1}{8}\int_0^1 \frac{dt'}{\sqrt{t'}}\Big(\sqrt{x} I_1(2\sqrt{t'x})\big(1 + I_0(2\sqrt{t'y})\big) - \sqrt{y} I_1(2\sqrt{t'y})\big(1 + I_0(2\sqrt{t'x})\big)\Big)$$

Conclusion

- New approach to prove hierarchies (Guess the Schur expansion + Prove the constraints are satisfied)
- Get ODEs by specializing (B)KP equations
- Still not a fully understood recipe
- Bijective explanation of KP?
- Still elusive $O(N)$-BGW and $O(N)$-IZ integral and β-deformations
- More non-oriented models satisfying BKP?

They exist! Not mythical creatures

$$-F_{3,1}(u)+F_{2^2}(u)+\tfrac{1}{2}F_{1^2}(u)^2+\tfrac{1}{12}F_{1^4}(u)=S(u)e^{F(u+2)+F(u-2)-2F(u)}$$

$$-2F_{4,1}(u)+2F_{3,2}(u)+2F_{2,1}(u)F_{1^2}(u)+\tfrac{1}{3}F_{2,1^3}(u)$$
$$=S(u)e^{F(u+2)+F(u-2)-2F(u)}(F_1(u+2)-F_1(u-2))$$

$$-6F_{5,1}(u)+4F_{4,2}(u)+2F_{3^2}(u)+4F_{3,1}(u)F_{1^2}(u)+\tfrac{2}{3}F_{3,1^3}(u)+4F_{2,1}(u)^2$$
$$+2F_{2^2}(u)F_{1^2}(u)+F_{2^2,1^2}(u)+\tfrac{1}{3}F_{1^2}(u)^3+\tfrac{1}{6}F_{1^4}(u)F_{1^2}(u)+\tfrac{1}{180}F_{1^6}(u)$$
$$=S(u)e^{F(u+2)+F(u-2)-2F(u)}\Big(F_{1^2}(u+2)+F_{1^2}(u-2)+2F_2(u+2)-2F_2(u-2)$$
$$+(F_1(u+2)-F_1(u-2))^2\Big)$$