Large N limits of the tensor HCIZ integral

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Joint work with Benoit Collins and Razvan Gurau

1 – Motivations

2 – First asymptotics and graphs

3 – Large N limits

4 – Application to the detection of entanglement

1 – Motivations

The Harish-Chandra–Itzykson–Zuber (HCIZ) integral

$$I_N(A,B;t) = \int_{U(N)} dU\ e^{t\mathrm{Tr}[AUBU^\dagger]}$$

- *U* : *N* x *N* unitary matrix
- *dU* : Haar measure
- *A*, *B* : *N* x *N* matrices (self-adjoint)

- Study random matrix models whose measure is invariant upon conjugation by unitary matrices [=unitarily invariant], apart from a term of the form $\exp(A^p B^q)$

→ 2-matrix models, models with external source, Kontsevitch-like models, etc…

- Properties of unitarily invariant matrix-valued functions through Fourier transform…

- also quantum physics, optimization/sampling, data analysis, and so on…see arxiv…

Through graph expansion:

- Generating function for monotone Hurwitz numbers (enumerative geometry)…

- A, B generic of rank N + large N limit: identify the class of planar graphs…

- A of small rank, B generic of rank N + large N limit: derive some of the key results of free-probability, which describes the distribution of random matrices in the limit of infinite size…

Generalization: the tensor HCIZ integral

With Benoit Collins and Razvan Gurau in: 2010.13661 & 2201.12778.

Generalization adapted to ''colored'' random tensor models, whose distribution is invariant upon conjugation by tensor products of unitary matrices [LU-invariant].

$$I_{D,N}(A,B;t) = \int_{U(N)^D} [dU] \, e^{t \mathrm{Tr}[AUBU^\dagger]}$$

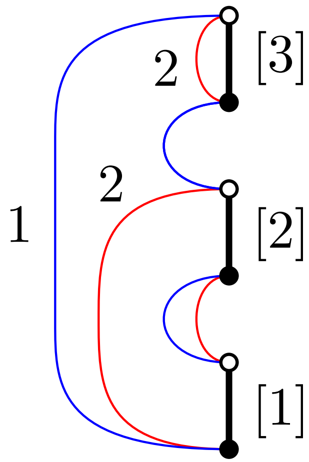
- $U = U_1 \otimes \cdots \otimes U_D$
- U_c : *N* x *N* unitary matrix
- $[dU]$: product of *D* Haar measures
- *A*, *B* : tensors with *D* inputs and *D* outputs, each of size *N*

Generalization of the points of the previous slide for colored tensor models…?

→ Today: large *N* limit and graph expansion (the last 2 points), and maybe some quantum physics.

2 – First asymptotics and graphs

Labeled (D+1)-colored graphs ("bubbles"…)

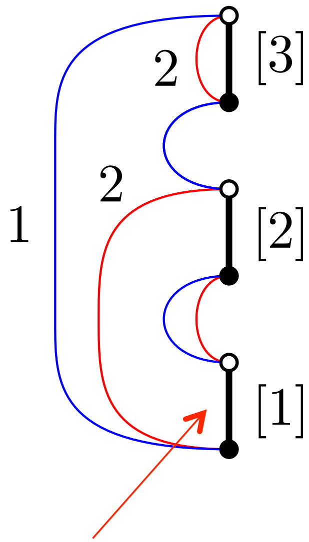


D=2

Labeled (D+1)-colored graph

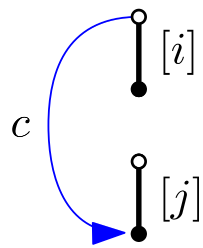
- Black and white vertices
- Edges colored from 1 to D+1, only link black and white vertices
- Edges of color D+1 are labeled from 1 to n (→ thick edges)

Labeled (D+1)-colored graphs ("bubbles"…)



Thick edge has color D+1

Labeled (D+1)-colored graph ⟷ *D* permutations:
$\vec{\sigma} = (\sigma_1, \ldots, \sigma_D) \in S_n^D$

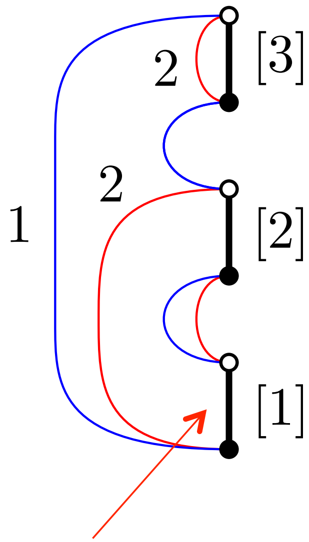


⟷ $\sigma_c(i) = j$

Here: $\sigma_1 = (123) \quad \& \quad \sigma_2 = (12)(3)$

→ Also use the notation $\vec{\sigma}$ to denote the labeled graph

Labeled (D+1)-colored graphs ("bubbles"…)



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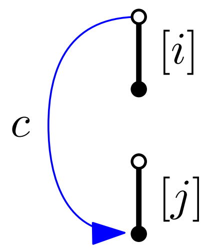

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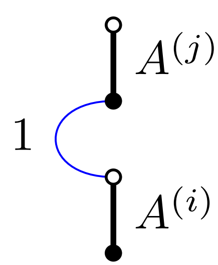
<u>Trace-invariants</u>

$\mathrm{Tr}_{\vec{\sigma}}(A)$ polynomial obtained by taking *n* labeled copies of the tensor *A* and summing the indices according to the permutations $\sigma_1, \dots, \sigma_D$



⟷

$$\sum_{p=1}^{N} A^{(i)}_{k_1,\dots,k_D\ ;\ p,l_2,\dots,l_D} A^{(j)}_{p,k'_2,\dots k'_D\ ;\ l'_1,\dots,l'_D}$$

First asymptotics

$$U = U_1 \otimes \cdots \otimes U_D$$

- **Moments (easy).**

$$\int [dU] (\mathrm{Tr}(AUBU^\dagger))^n = N^{-2nD} \sum_{\vec{\sigma},\vec{\tau}} \mathrm{Tr}_{\vec{\sigma}}(A)\, \mathrm{Tr}_{\vec{\tau}^{-1}}(B)\, N^{\sum_{c=1}^{D} \#(\sigma_c \tau_c^{-1})} \prod_{c=1}^{D} \mathsf{M}[\sigma_c \tau_c^{-1}](1 + O(1/N^2))$$

$\#(\nu)$: number of cycles of the permutation

$\mathsf{M}[\nu]$: a signed product of Catalan numbers

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$\#(\nu)$: number of cycles of the permutation

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- **Cumulants (hard).** Uses Weingarten calculus and a good dose of combinatorics.

With the notation:

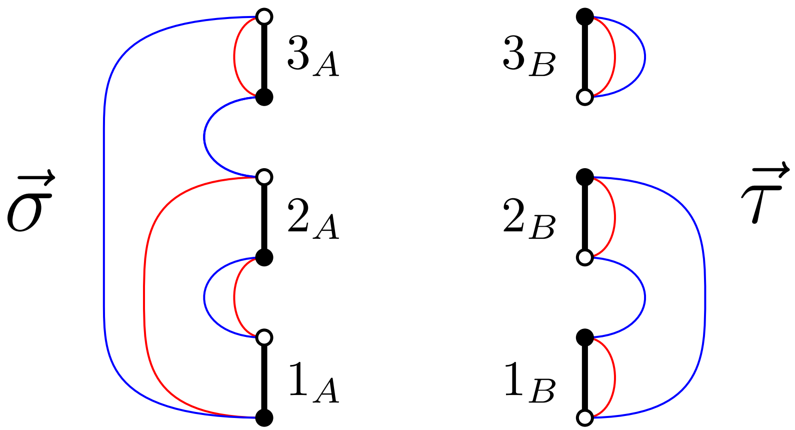
$$\log \int [dU]\; e^{t \mathrm{Tr}\left(AUBU^\dagger\right)} = \sum_{n \geq 1} \frac{t^n}{n!} C_n \left(\mathrm{Tr}[AUBU^\dagger]\right)$$

$$C_n \left(\mathrm{Tr}(AUBU^\dagger)\right) = N^{-2nD} \sum_{\vec{\sigma},\vec{\tau}} \mathrm{Tr}_{\vec{\sigma}}(A)\, \mathrm{Tr}_{\vec{\tau}^{-1}}(B)\, N^{\sum_{c=1}^D \#(\sigma_c \tau_c^{-1}) - 2(\mathsf{cc}(\vec{\sigma},\vec{\tau}) - 1)} f[\vec{\sigma},\vec{\tau}] (1 + O(N^{-2}))$$

Number of transitivity classes of the group generated by all permutations

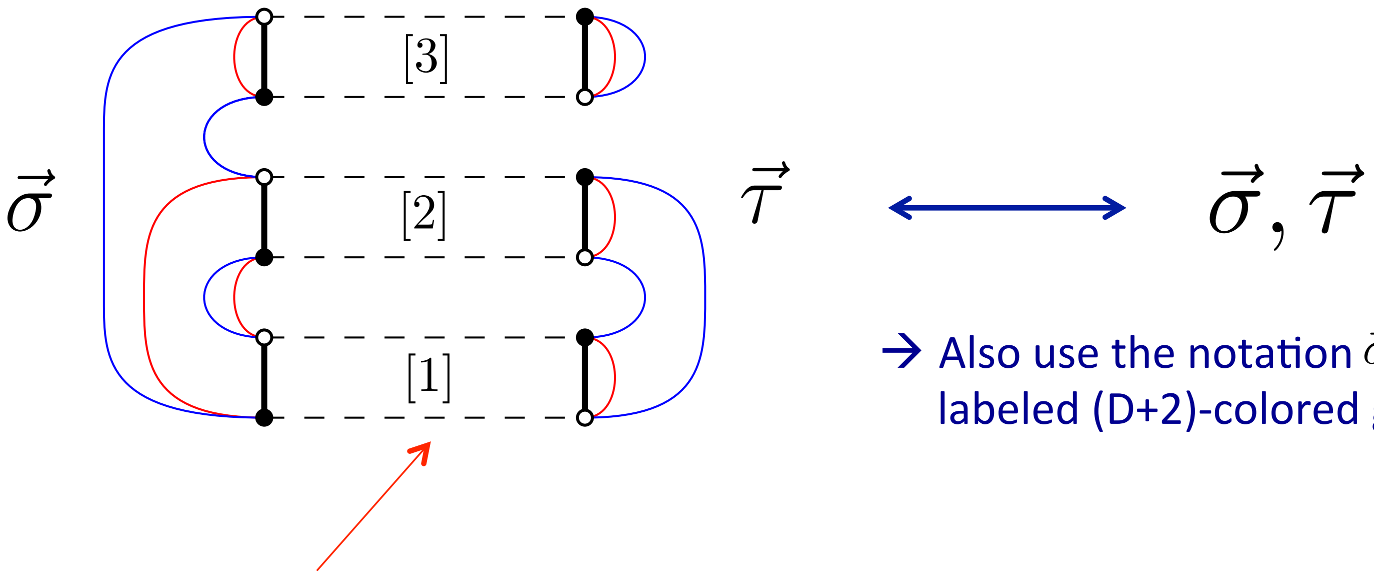
A generalization of connected genus 0 monotone double Hurwitz numbers

Labeled (D+2)-colored graphs (Feynman graphs)



$\longleftrightarrow \quad \vec{\sigma}, \vec{\tau}$

Labeled (D+2)-colored graphs (Feynman graphs)



→ Also use the notation $\vec{\sigma}, \vec{\tau}$ to denote the labeled (D+2)-colored graph

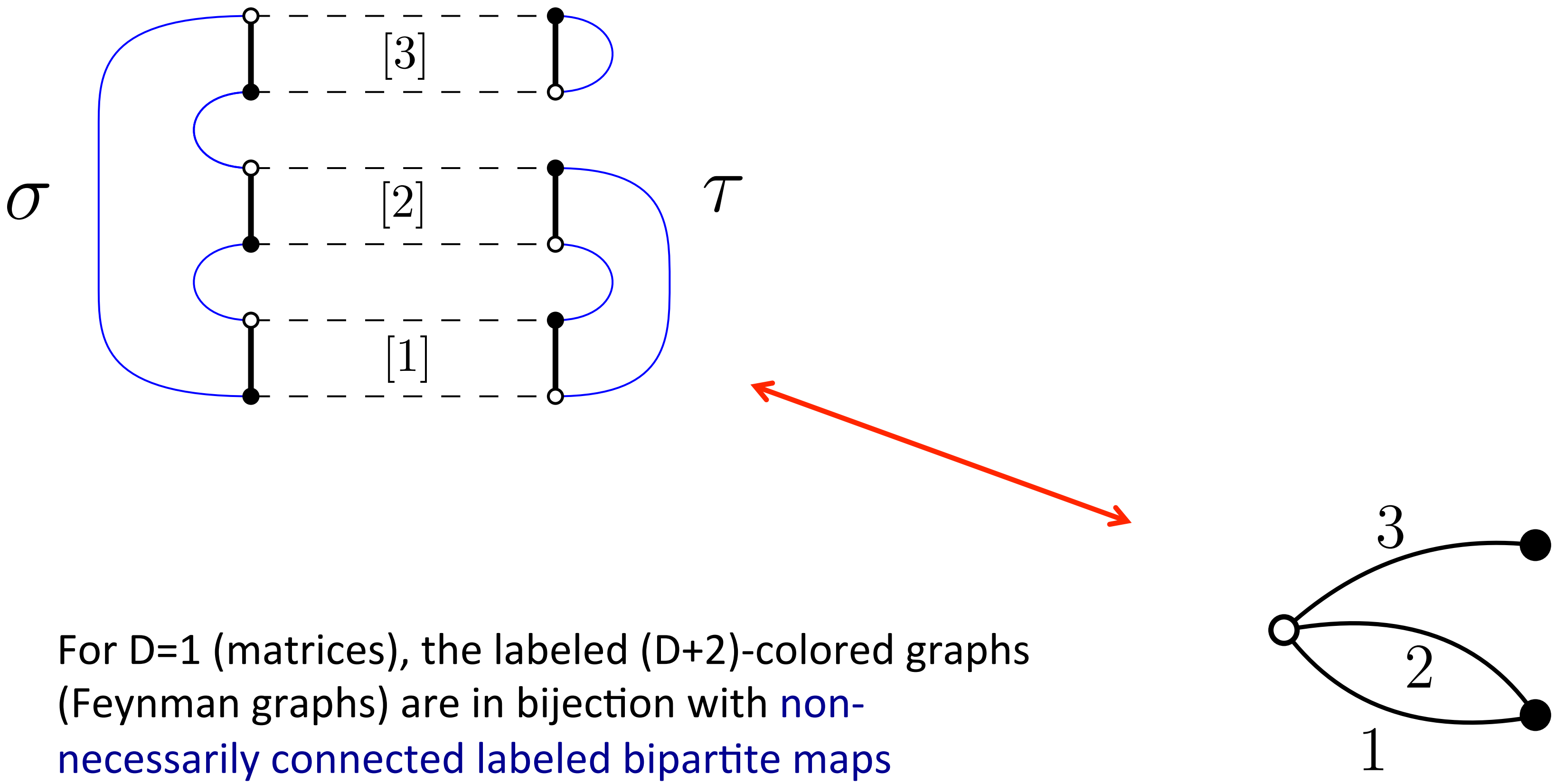
Then in the formula:

$$C_n\left(\mathrm{Tr}(AUBU^\dagger)\right) = N^{-2nD} \sum_{\vec{\sigma},\vec{\tau}} \mathrm{Tr}_{\vec{\sigma}}(A)\,\mathrm{Tr}_{\vec{\tau}^{-1}}(B)\, N^{\sum_{c=1}^{D} \#(\sigma_c\tau_c^{-1}) - 2(\mathrm{cc}(\vec{\sigma},\vec{\tau})-1)}\, f[\vec{\sigma},\vec{\tau}](1+O(N^{-2}))$$

→ $\mathrm{cc}(\vec{\sigma},\vec{\tau})$: number of connected components of the graph $\vec{\sigma},\vec{\tau}$

→ $\sum_{c=1}^{D} \#(\sigma_c\tau_c^{-1}) = F_0(\vec{\sigma},\vec{\tau})$: total number of bicolored cycles of color 0c (some of the faces)

Feynman graphs in D=1



For D=1 (matrices), the labeled (D+2)-colored graphs (Feynman graphs) are in bijection with non-necessarily connected labeled bipartite maps

F_0 : number of faces

3 – Large *N* limits

Scaling ansatz for the trace-invariants

$$C_n\left(\mathrm{Tr}(AUBU^\dagger)\right) = N^{-2nD} \sum_{\vec{\sigma},\vec{\tau}} \mathrm{Tr}_{\vec{\sigma}}(A)\,\mathrm{Tr}_{\vec{\tau}^{-1}}(B)\, N^{F_0(\vec{\sigma},\vec{\tau})-2(\mathrm{cc}(\vec{\sigma},\vec{\tau})-1)}\, f[\vec{\sigma},\vec{\tau}](1+o(1))$$

$$U = U_1 \otimes \cdots \otimes U_D$$

Depends on size *N* of the tensors

We have to make an ansatz regarding the asymptotic scaling in *N* of the trace-invariant:

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D=1

$$\mathrm{Tr}_\sigma(A) = \prod_{\gamma \text{ cycle of } \sigma} \mathrm{Tr}(A^{\mathrm{length}(\gamma)}) \sim_{N\to\infty} N^{\beta\#(\sigma)+\delta n}\, \mathrm{tr}_\sigma(a)$$

...for a matrix of rank N^β, whose non-vanishing eigenvalues have the same order in *N*

Rescaled trace-invariant

e.g.:

- Normalized identity: $\mathrm{Tr}_\sigma(\mathrm{id}/N) = N^{\#(\sigma)-n}$
- Wishart random matrix: $\langle \mathrm{Tr}_\sigma(W/N^2)\rangle_W \sim_{N\to\infty} N^{\#(\sigma)-n} \sum_{\pi\in\mathrm{NC}(n)} c^{\#(\pi)}$

The linear part in *n* does not play any role in the asymptotics and we leave it aside

Scaling ansatz for the trace-invariants

$$C_n\left(\mathrm{Tr}(AUBU^\dagger)\right) = N^{-2nD}\sum_{\vec\sigma,\vec\tau} \mathrm{Tr}_{\vec\sigma}(A)\,\mathrm{Tr}_{\vec\tau^{-1}}(B)\, N^{F_0(\vec\sigma,\vec\tau)-2(\mathrm{cc}(\vec\sigma,\vec\tau)-1)}\, f[\vec\sigma,\vec\tau](1+o(1))$$

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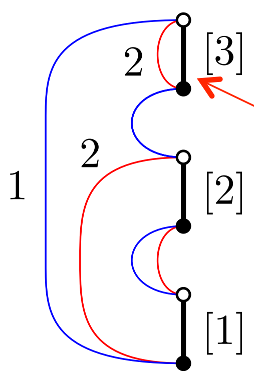
Rescaled trace-invariant

The scaling $s_A(\vec\sigma)$ a priori depends on the properties of the graph $\vec\sigma$...

We choose: $s_A(\vec\sigma) = \beta_A \sum_{c=1}^{D} \#(\sigma_c) + \epsilon_A \sum_{c_1<c_2} \#(\sigma_{c_1}\sigma_{c_2}^{-1}) = \beta_A\, F_{D+1}(\vec\sigma) + \epsilon_A\, F(\vec\sigma^{\widehat{D+1}})$



Thick edges have color *D+1*

Scaling ansatz for the trace-invariants

$$C_n\left(\mathrm{Tr}(AUBU^\dagger)\right) = N^{-2nD} \sum_{\vec\sigma,\vec\tau} \mathrm{Tr}_{\vec\sigma}(A)\,\mathrm{Tr}_{\vec\tau^{-1}}(B)\, N^{F_0(\vec\sigma,\vec\tau)-2(\mathrm{cc}(\vec\sigma,\vec\tau)-1)}\, f[\vec\sigma,\vec\tau](1+o(1))$$

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N.B. Other well-motivated choice: $s_A(\vec\sigma) = \beta_A\, F_{D+1}(\vec\sigma) + \epsilon_A\, \mathrm{cc}(\widehat{\vec\sigma^{D+1}})$

N.N.B. These assumptions are also motivated by applications to quantum information, see the last part.

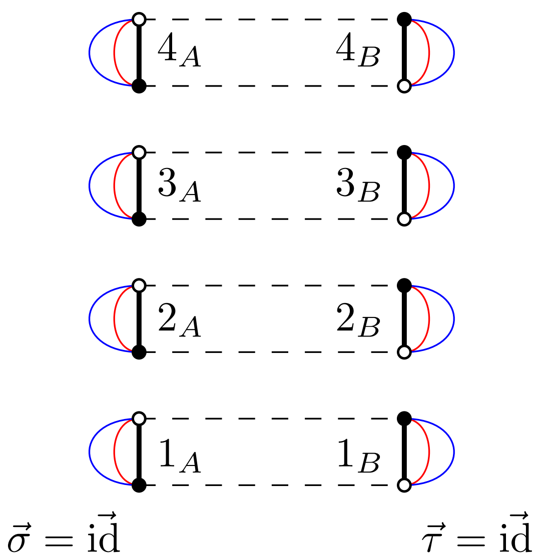
Moments

With these assumptions and for any $\epsilon_A, \epsilon_B, \beta_A, \beta_B \geq 0$:

$$\int [dU] \left(\mathrm{Tr}[AUBU^\dagger]\right)^n \sim_{N\to\infty} \left(\frac{\mathrm{Tr}(A)\mathrm{Tr}(B)}{N^D}\right)^n$$

$$U = U_1 \otimes \cdots \otimes U_D$$

Only dominant graph is the one with *n* connected components (all sigma*s* and tau*s* are the identity):



Large N limits of the cumulants - 1

$$\log \int [dU] \, \mathrm{e}^{t \mathrm{Tr}\left(AUBU^\dagger\right)} = \sum_{n \geq 1} \frac{t^n}{n!} C_n \left(\mathrm{Tr}[AUBU^\dagger]\right)$$

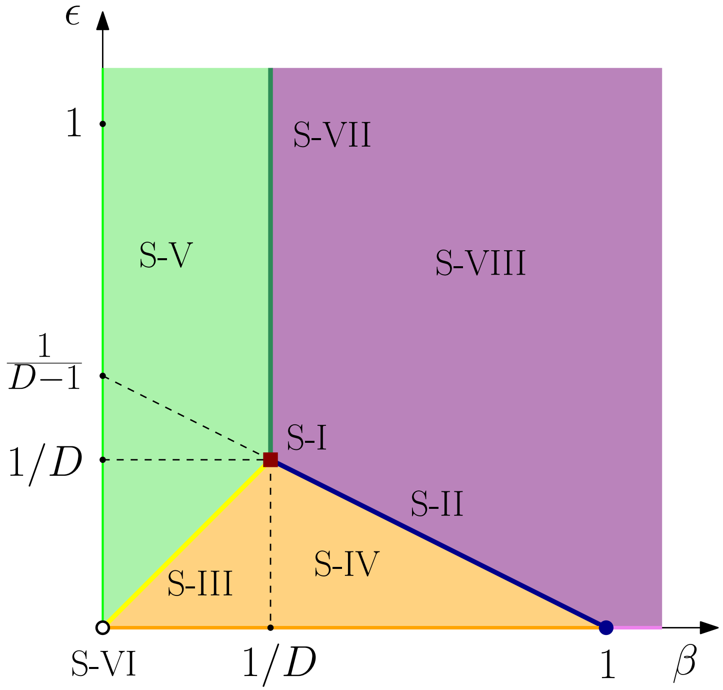
Large N limits of the cumulants - 1

2201.12778

Symmetric

$$\mathrm{Tr}_{\vec\sigma}(A) \sim N^{\beta F_{D+1}(\vec\sigma)+\epsilon F(\widehat{\vec\sigma^{D+1}})}\ \mathrm{tr}_{\vec\sigma}(a),$$

$$\mathrm{Tr}_{\vec\sigma}(B) \sim N^{\beta F_{D+1}(\vec\sigma)+\epsilon F(\widehat{\vec\sigma^{D+1}})}\ \mathrm{tr}_{\vec\sigma}(b),$$



Microscopic A

$$\mathrm{Tr}_{\vec\sigma}(A) \sim \mathrm{tr}_{\vec\sigma}(a),$$

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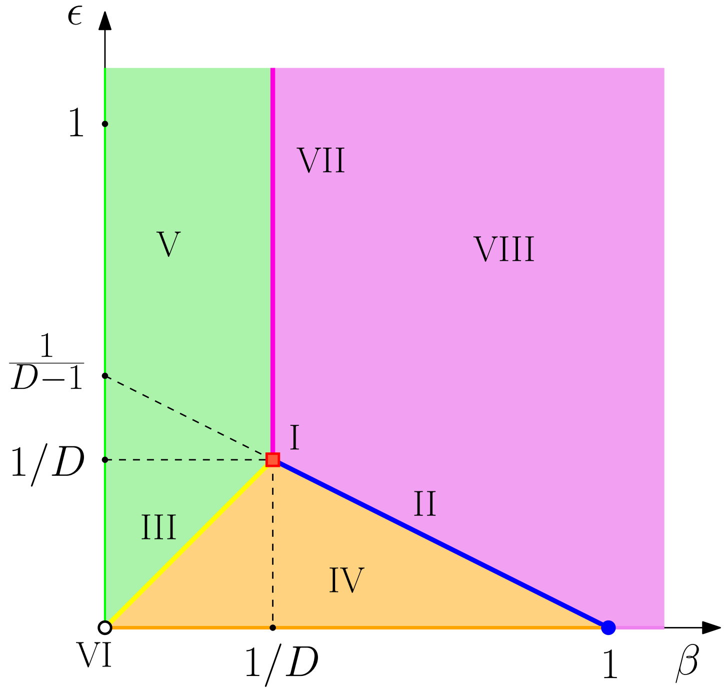
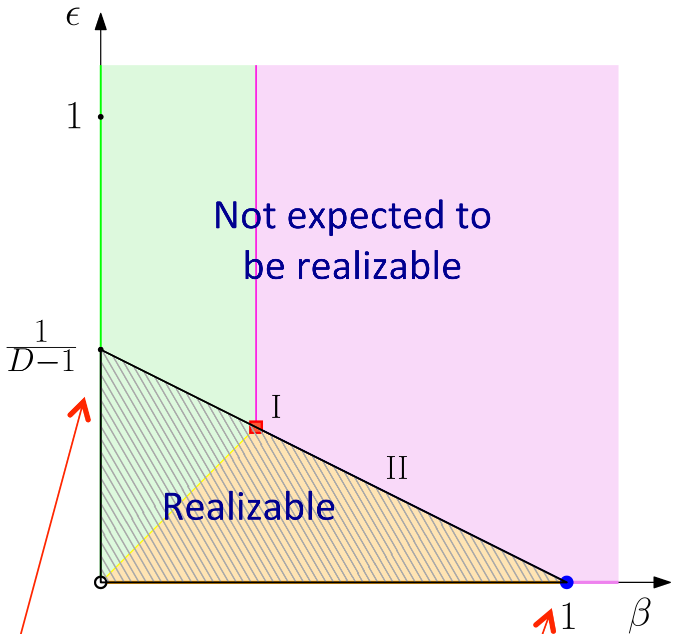


III, IV, V, VI: same dominant graphs for both

Large N limits of the cumulants - 2

For both scaling ansätze:



Should think that

$B = T \otimes \bar{T}$

is here, where *T* random tensor

Tensor products of generic rank N (random) matrices

$B = B_1 \otimes \ldots \otimes B_D$

Large N limits of the cumulants - 2

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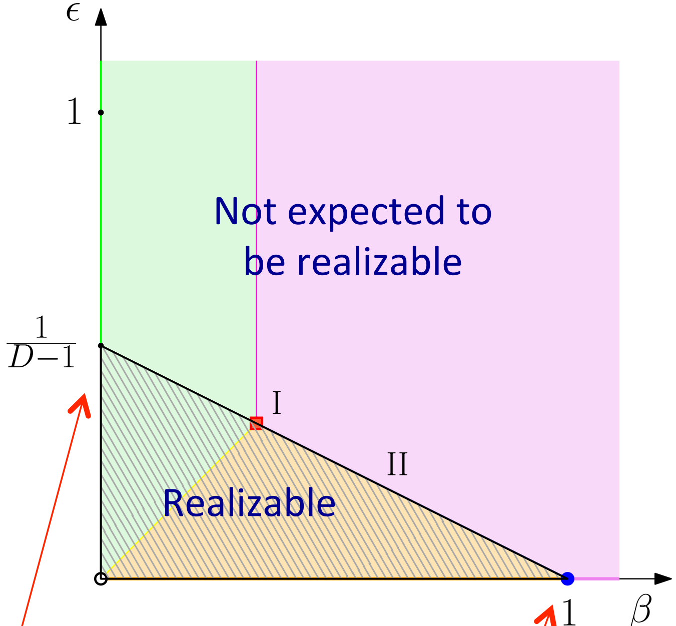

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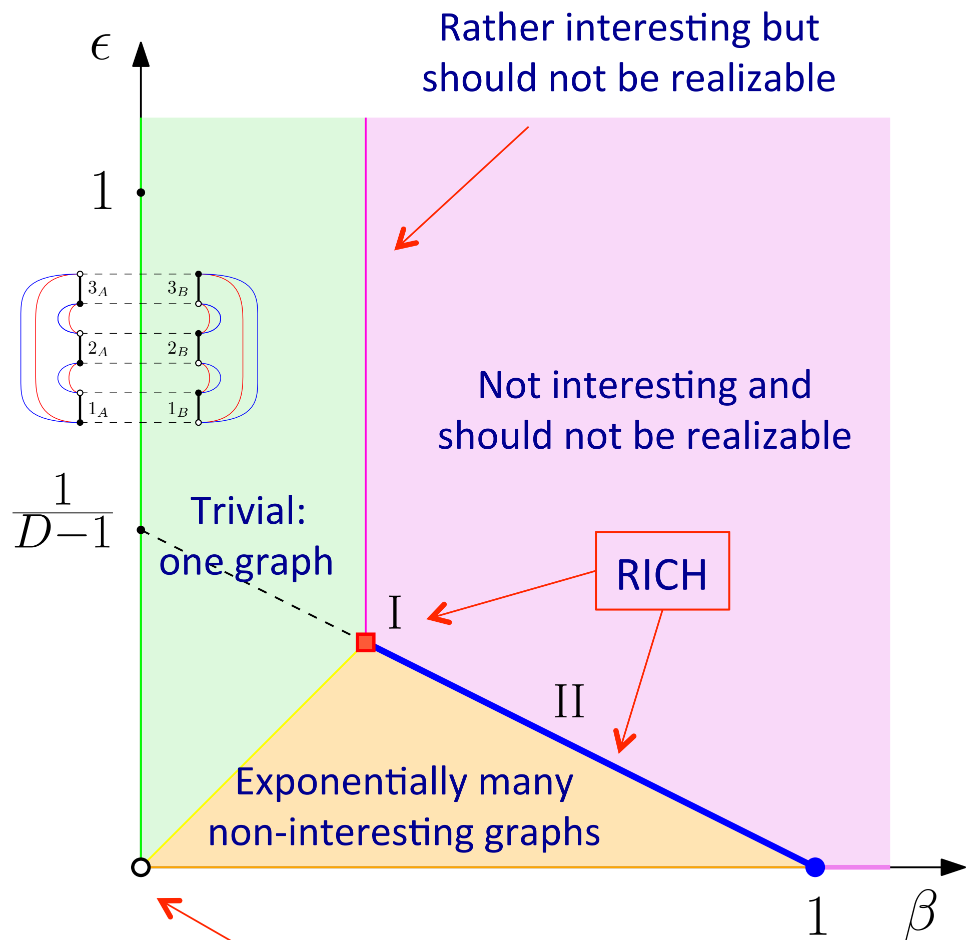


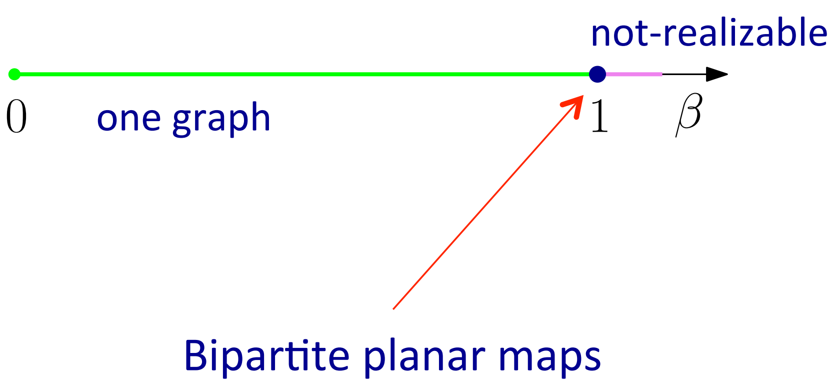
Super-exponentially many non-interesting graphs

Large N limits of the cumulants in D=1

Symmetric

$$\mathrm{Tr}_\sigma(A) \sim N^{\beta\#(\sigma)}\ \mathrm{tr}_\sigma(a),$$

$$\mathrm{Tr}_\sigma(B) \sim N^{\beta\#(\sigma)}\ \mathrm{tr}_\sigma(b),$$



→ 2d random geometry

Microscopic A

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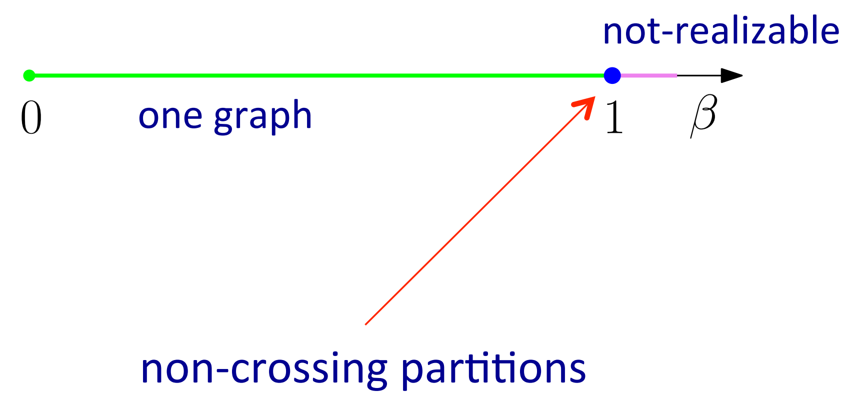
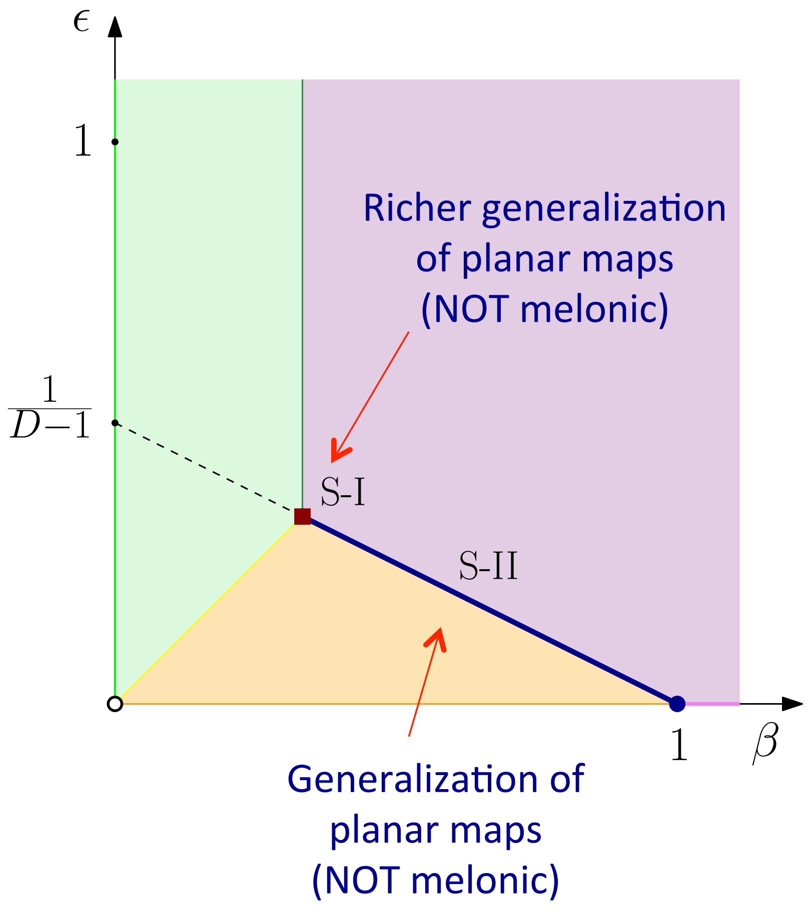


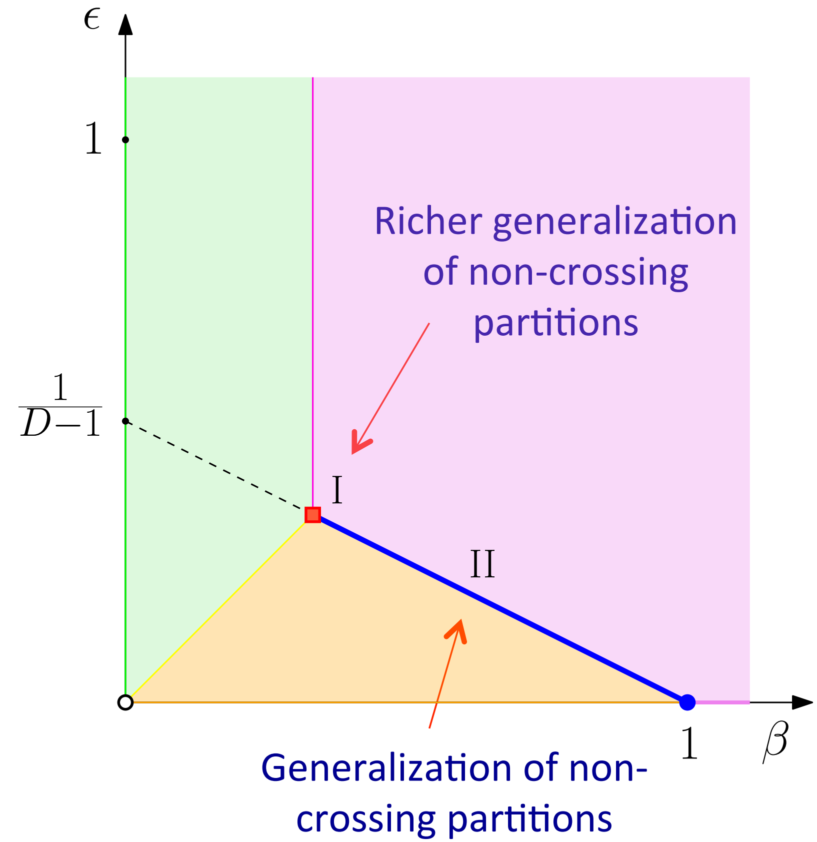
→ The values of the cumulants of the HCIZ integral at this point allow defining free-cumulants for random matrices and deriving their additivity for sums of independent random matrices

Large N limits of the cumulants - 3

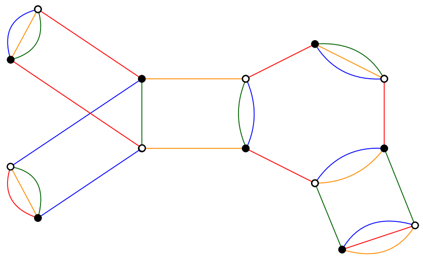
Symmetric



Microscopic A



Graph invariants



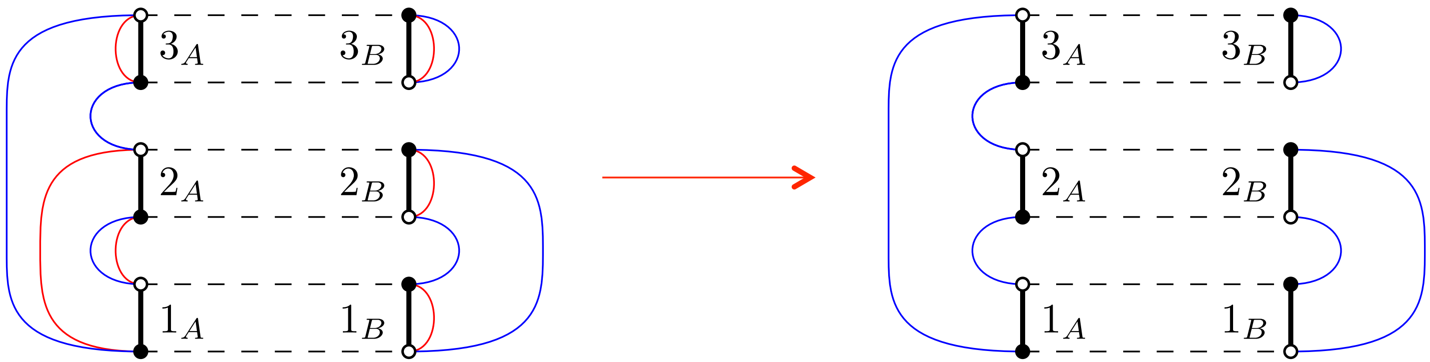
Degree: $\omega(\vec{\sigma}) = D\,\mathrm{cc}(\vec{\sigma}) + n\frac{D(D-1)}{2} - F(\vec{\sigma}) \geq 0$

- For D=1, automatically 0
- For D=2, twice the genus of the graph
- For D>2, graphs of vanishing degree are well-known (melonic), and have a tree-like structure.

Large *N* limits of "colored" random tensor models usually selects melonic graphs

New invariants:

$$(\sigma_c, \tau_c)$$



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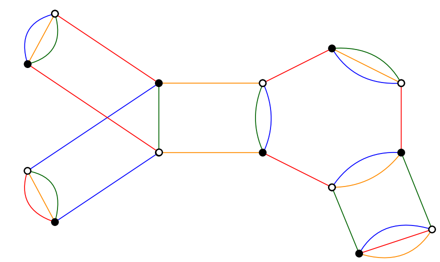

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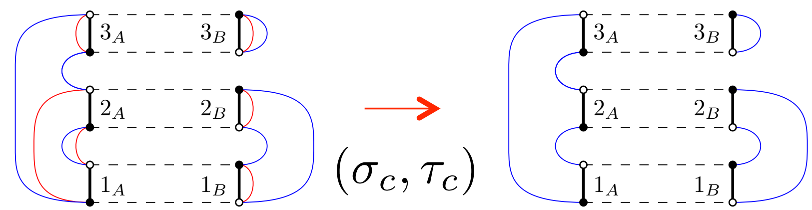


New invariants:

Delta: $\Delta(\vec{\sigma},\vec{\tau}) = n(D-1) + \text{cc}(\vec{\sigma},\vec{\tau}) - \sum_{c=1}^{D} \text{cc}(\vec{\sigma}_c,\vec{\tau}_c) \geq 0$

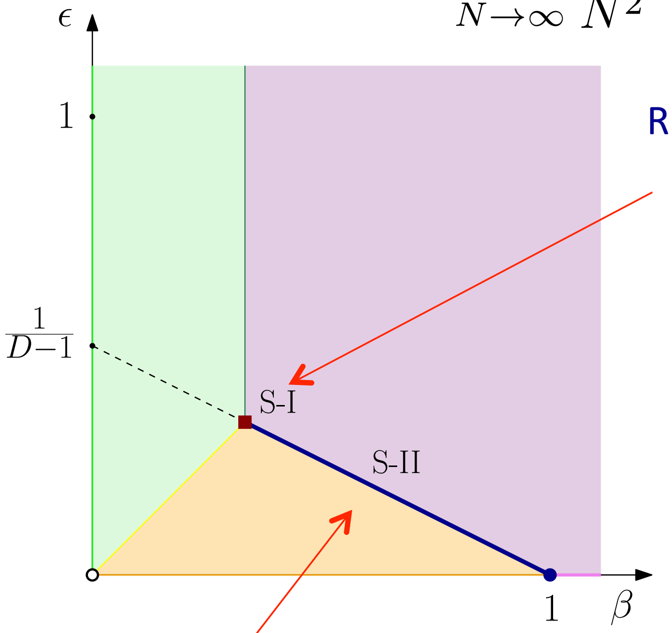
Box: $\square(\vec{\sigma},\vec{\tau}) = 2\text{cc}(\vec{\sigma},\vec{\tau}) - \text{cc}(\vec{\sigma}) - \text{cc}(\vec{\tau}) - 2\sum_{c=1}^{D} \text{cc}(\vec{\sigma}_c,\vec{\tau}_c) + F_{D+1}(\vec{\sigma}) + F_{D+1}(\vec{\tau}) \geq 0$

Both are automatically 0 for D=1, which is the matrix case (usual HCIZ integral)

Large N limits of the cumulants - 4

Symmetric

$$\lim_{N\to\infty} \frac{1}{N^2} C_n \left(N \mathrm{Tr}\left[AUBU^\dagger \right] \right) = \sum_{\substack{\vec\sigma,\vec\tau \\ \omega(\vec\sigma)=\omega(\vec\tau)=0 \\ \square(\vec\sigma,\vec\tau)=0 \\ \forall c,\ g(\sigma_c,\tau_c)=0}} \mathrm{tr}_{\vec\sigma}(a)\ \mathrm{tr}_{\vec\tau^{-1}}(b)\ f[\vec\sigma,\vec\tau]$$



Richer generalization of planar maps (NOT melonic)

Generalization of planar maps (NOT melonic)

$$\lim_{N\to\infty} \frac{1}{N^2} C_n \left(\frac{N}{N^{D(D-1)(\frac{1}{D}-\epsilon)}} \mathrm{Tr}\left[AUBU^\dagger \right] \right) = \sum_{\substack{\vec\sigma,\vec\tau \\ \Delta(\vec\sigma,\vec\tau)=0 \\ \forall c,\ g(\sigma_c,\tau_c)=0}} \mathrm{tr}_{\vec\sigma}(a)\ \mathrm{tr}_{\vec\tau^{-1}}(b)\ f[\vec\sigma,\vec\tau]$$

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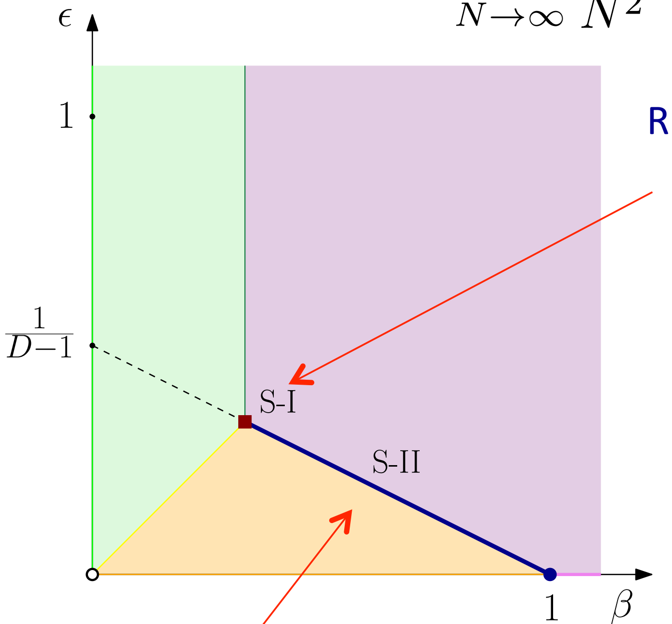


Richer generalization of planar maps (NOT melonic)

Generalization of planar maps (NOT melonic)

In both cases, planar graphs are recovered for the matrix case D=1 (i.e. the usual HCIZ integral)

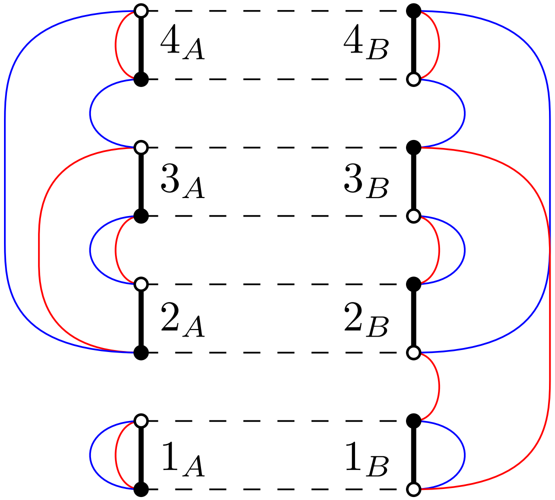
$$\lim_{N\to\infty} \frac{1}{N^2} C_n\left(\frac{N}{N^{D(D-1)(\frac{1}{D}-\epsilon)}} \mathrm{Tr}\left[AUBU^\dagger\right]\right) = \sum_{\substack{\vec\sigma,\vec\tau \\ \Delta(\vec\sigma,\vec\tau)=0 \\ \forall c,\ g(\sigma_c,\tau_c)=0}} \mathrm{tr}_{\vec\sigma}(a)\ \mathrm{tr}_{\vec\tau^{-1}}(b)\ f[\vec\sigma,\vec\tau]$$

Large *N* limits of the cumulants - 5

Symmetric

Regime S-II (vanishing Delta, etc): dominant graphs should be in the universality class of trees (numerical estimate of counting exponent by Guillaume Chapuy)

Regime S-I (vanishing Box, etc): … unknown.



Preliminary conclusions

Study of the large *N* asymptotics of the cumulants of the tensor HCIZ integral reveals rich ''phase-diagram''.

Application to (random) geometry in dimension >2 for symmetric scalings:
Interesting combinatorics in two regions, with two new generalizations of planar graphs for (D+2)-colored graphs (known to be dual to (D+1)-dimensional triangulations). Universality class of trees expected for one region, unknown for the ''triple-point'', where the richest combinatorics is found.

Application to free-probability for random tensors for microscopic A:
One may use these results to define a generalization of free-cumulants for random tensors, and prove their additivity for sums of independent tensors. This is the property needed in free-probability to compute the asymptotic spectrum of the sum of two independent random matrices, a consequence of the asymptotic freeness (Voiculescu).

Application to quantum information for microscopic A (randomized measurements):
→ Next slides

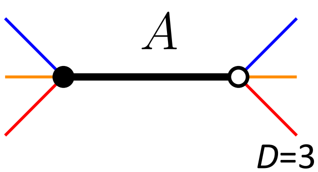
4 – Application to the detection of entanglement

The tensor HCIZ integral and randomized measurements

D-partite quantum system → tensor product of Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_D$

$$\dim(\mathcal{H}_c) = N$$

State *B (density matrix)* and observable *A* → operators on $\mathcal{H}$ … *tensors*



Entanglement → 'how far' is *B* from a convex sum of tensor products of the form:

$$B = \sum_{i=1}^{K} p_i \, \rho_1^{(i)} \otimes \ldots \otimes \rho_D^{(i)} \qquad \sum_{i=1}^{K} p_i = 1 \qquad \rho_c^{(i)} \in \mathcal{M}_{N_c}(\mathbb{C}) \qquad p_i, \rho_c^{(i)} \geq 0$$

→ Separable state

The tensor HCIZ integral and randomized measurements

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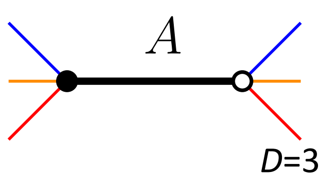

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→ Separable state

Expectation value of observable *A* on state *B* $\langle A \rangle_B = \mathrm{Tr}(AB)$

Randomized measurements

→ distribution $\langle A_U \rangle_B$ for $A_U = (U_1 \otimes \ldots \otimes U_D)\, A\, (U_1^\dagger \otimes \ldots \otimes U_D^\dagger)$, $U_c \in U(N)$ *Haar*

→ Accessible experimentally (and solves experimental issue of alignment of reference frames). Known that for small D, N, the knowledge of the first moments allows detecting even weak forms of entanglement.

→ The previous slides concern the large *N* moments and cumulants of this distribution!

Ansatz and interpretation of results

- Take: $A = A_1 \otimes \ldots \otimes A_D$ (local observable), where A_c is of small rank,

e.g. a projector on a state $A_c = |\Psi_c\rangle\langle\Psi_c|$

→ *A* is microscopic $\mathrm{Tr}_{\vec{\sigma}}(A) \sim \mathrm{tr}_{\vec{\sigma}}(a),$

- Assume that *B* is unknown but that:

$$s_B(\vec{\sigma}) = \beta \sum_{c=1}^{D} \#(\sigma_c) + \epsilon \sum_{c_1<c_2} \#(\sigma_{c_1}\sigma_{c_2}^{-1}) = \beta\, F_{D+1}(\vec{\sigma}) + \epsilon\, F(\widehat{\vec{\sigma}^{D+1}})$$

The parameters of the scaling are chosen to represent:

- N^β: common *"local rank" of the density matrix B (rank of B_c if B is a tensor product* $B = B_1 \otimes \ldots \otimes B_D$ *)*
- N^ϵ: *how much exchange there is between the different subsystems (pairwise)*

At this preliminary stage, it's not so clear what properties of the state B these parameters encode, but ϵ, β should respectively have something to do with how entangled or mixed the state *B* is…

Ansatz and interpretation of results

- N^{β}: common *"local rank" of the density matrix B (rank of B_c if B is a tensor product $B = B_1 \otimes \ldots \otimes B_D$)*
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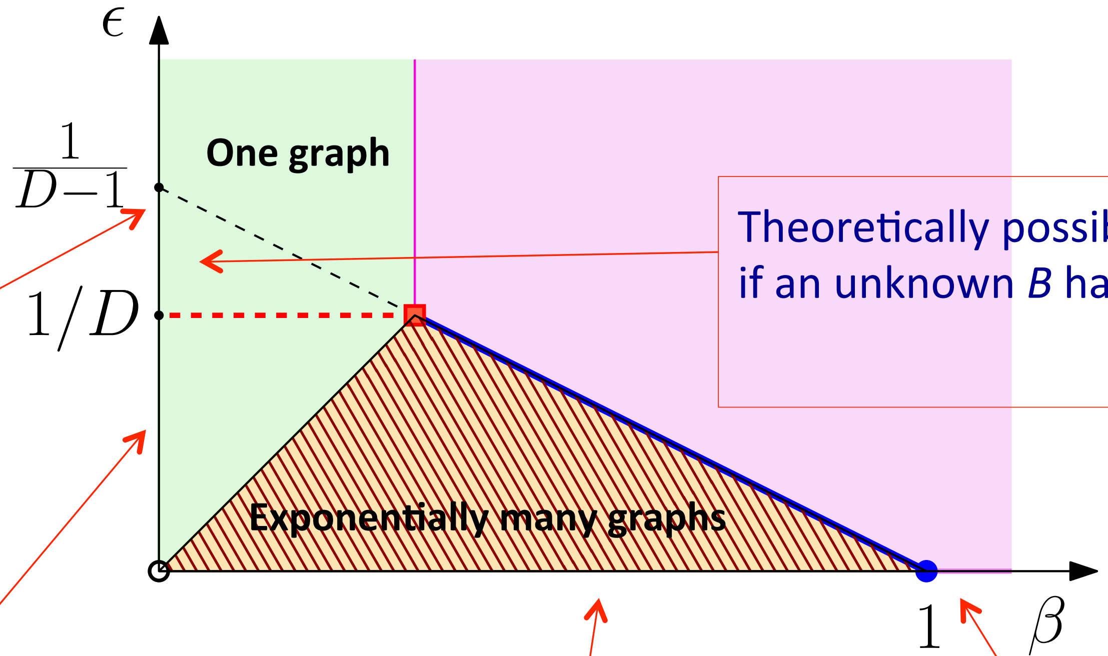

Theoretically possible to recognize if an unknown *B* has $\epsilon \geq \frac{1}{D}$

Some maximally entangled (pure) states such as GHZ

Axis $\beta = 0$: contains pure states $B = |\Psi\rangle\langle\Psi|$

Axis $\epsilon = 0$: product states and separable states satisfying certain assumptions

Maximally mixed state (the identity)

Thank you for your attention!