LARGE CHARGE AT LARGE N

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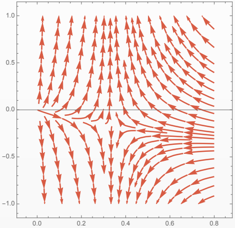
arXiv:1505.01537, arXiv:1610.04495, arXiv:2008.03308, arXiv:2102.12488, arXiv:2110.07616, arXiv:2110.07617 . . .

WHO'S WHO

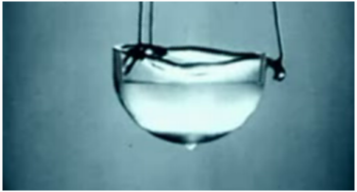
L. Álvarez Gaumé (SCGP and CERN);
D. Banerjee (Calcutta);
S. Chandrasekharan (Duke);
S. Hellerman (IPMU);
S. Reffert, N. Dondi, I. Kalogerakis , R. Moser, V. Pellizzani,
T. Schmidt (AEC Bern);
F. Sannino (CP3-Origins and Napoli);
M. Watanabe (Weizmann).

WHY ARE WE HERE? CONFORMAL FIELD THEORIES

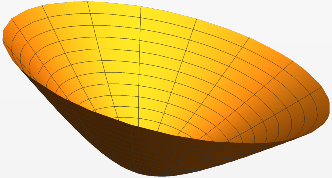
extrema of the RG flow



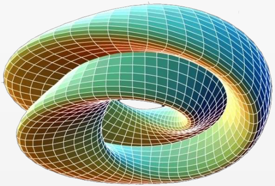
critical phenomena



quantum gravity



string theory



WHY ARE WE HERE? CONFORMAL FIELD THEORIES ARE HARD

Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

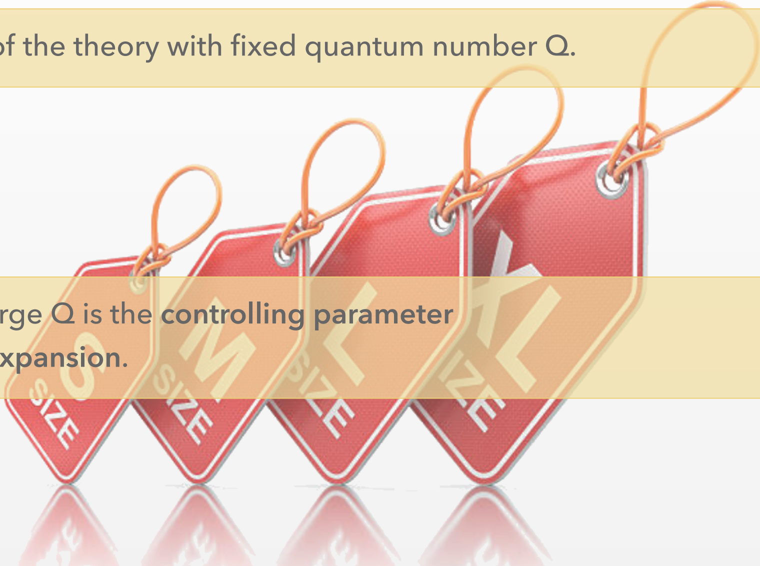
No parameter of the theory can be dialed to a simplifying limit.

WHY ARE WE HERE? CONFORMAL FIELD THEORIES ARE HARD

In presence of a **symmetry** there can be **sectors of the theory** where anomalous dimension and OPE coefficients simplify.

THE IDEA

Study **subsectors** of the theory with fixed quantum number Q.



In each sector, a large Q is the **controlling parameter** in a **perturbative expansion.**

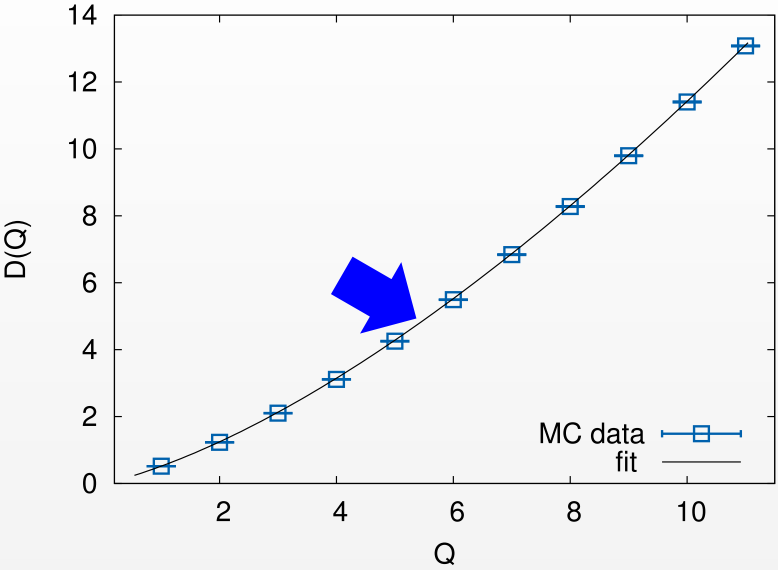
CONCRETE RESULTS

We consider the $O(N)$ **vector model in three dimensions**. In the IR it flows to a **conformal fixed point** [Wilson & Fisher].

We find an explicit formula for the **dimension of the lowest primary at fixed charge**:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

SUMMARY OF THE RESULTS: O(2)



SCALES

We want to write a **Wilsonian effective action.**

Choose a cutoff Λ, separate the fields into high and low frequency φ_H, φ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\varphi_L)} = \int \mathcal{D}\varphi_H \, e^{iS(\varphi_H, \varphi_L)}$$

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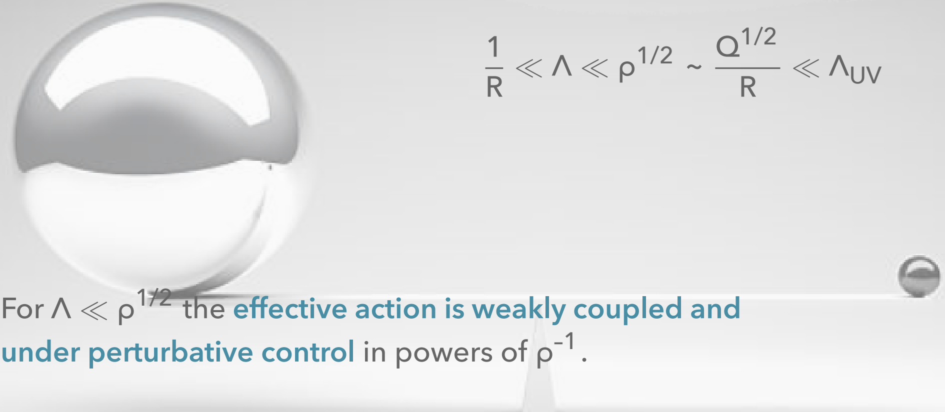
too hard

SCALES

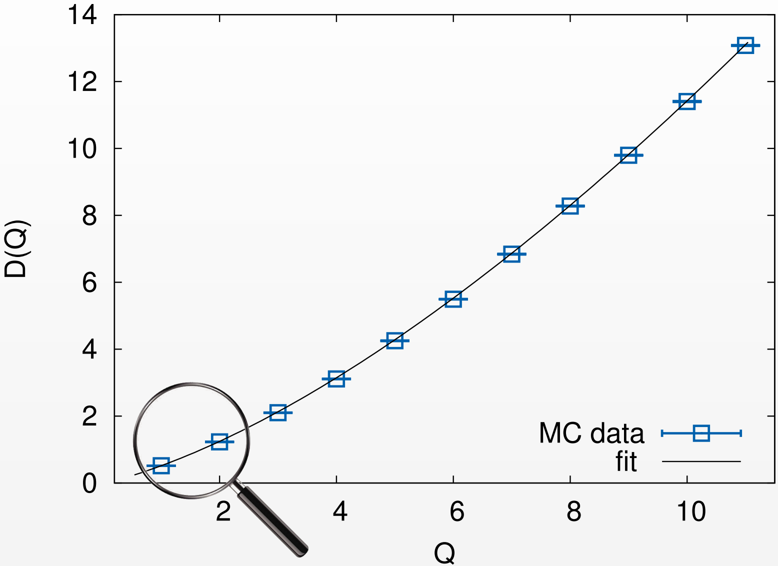
- We look at a finite box of typical **length R**
- The U(1) charge Q fixes a **second scale** $\rho^{1/2} \sim Q^{1/2}/R$

$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$



For $\Lambda \ll \rho^{1/2}$ the **effective action is weakly coupled and under perturbative control** in powers of ρ^{-1}.

TOO GOOD TO BE TRUE?



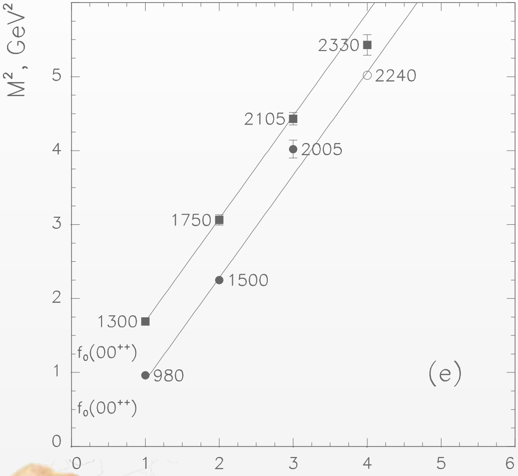
TOO GOOD TO BE TRUE?

Think of **Regge trajectories**.

The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but experimentally everything works so well at small J that String Theory was invented.



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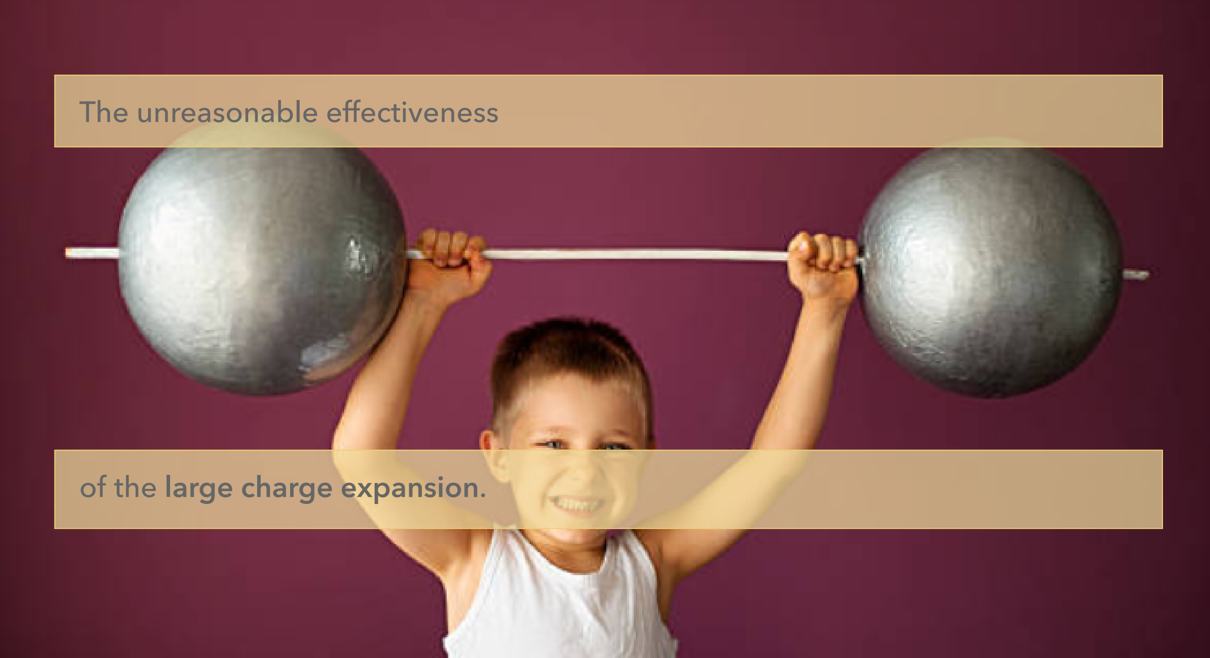

The unreasonable effectiveness

of the large charge expansion.

SELECTED TOPICS IN THE LQNE

- O(2) model [Hellerman, DO, Reffert, Watanabe] [Monin, Pirtskhalava, Rattazzi, Seibold]
- O(N) model [Álvarez-Gaumé, Loukas, DO, Reffert]
- holography [Loukas, DO, Reffert, Sarkar] [de la Fuente] [Guo, Liu, Lu, Pang] [Giombi, Komatsu, Offertaler]
- large N [Álvarez-Gaumé, DO, Reffert] [Giombi, Hyman]
- ε double-scaling [Badel, Cuomo, Monin, Rattazzi] [Arias-Tamargo, Rodriguez-Gomez, Russo] [Antipin, Bersini, Sannino, Wang, Zhang] [Jack, Jones]
- non-relativistic CFTs [Kravec, Pal] [Hellerman, Swanson] [Favrod, DO, Reffert] [DO, Reffert, Pellizzani] [Hellerman, DO, Reffert, Pellizzani, Swanson]
- $\mathcal{N} = 2$ [Hellerman, Maeda] [Hellerman, Maeda, DO, Reffert, Watanabe] [Bourget, Rodriguez-Gomez, Russo] [Grassi, Komargodski, Tizzano] [Cremonesi, Lanza, Martucci]
- bootstrap [Jafferis, Zhiboedov]
- resurgence [Dondi, Kalogerakis, DO, Reffert] [Antipin, Bersini, Sannino, Torres] [Watanabe]

WHAT HAPPENED?

We started from a CFT.
There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.
In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple effective field theory (EFT)**.
We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

would you like to know more?

LARGE N VS. LARGE CHARGE

THE MODEL

φ^4 model on $\mathbb{R} \times \Sigma$ for N complex fields

$$S_\theta[\varphi_i] = \sum_{i=1}^{N} \int dt\, d\Sigma \left[g^{\mu\nu} \left(\partial_\mu \varphi_i \right)^* \left(\partial_\nu \varphi_i \right) + r \varphi_i^* \varphi_i + \frac{u}{2} \left(\varphi_i^* \varphi_i \right)^2 \right]$$

It flows to the WF in the IR limit $u \to \infty$ when r is fine-tuned.

We compute the partition function at fixed charge

$$Z(Q_1, \ldots, Q_N) = \mathrm{Tr} \left[e^{-\beta H} \prod_{i=1}^{N} \delta(\hat{Q}_i - Q_i) \right]$$

where

$$\hat{Q}_i = \int d\Sigma\, j_i^0 = i \int d\Sigma \left[\dot{\varphi}_i^* \varphi_i - \varphi_i^* \dot{\varphi}_i \right].$$

FIX THE CHARGE

Explicitly

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{i=1}^{N} e^{i\theta_i Q_i} \, \mathrm{Tr}\left[e^{-\beta H} \prod_{i=1}^{N} e^{-i\theta_i \hat{Q}_i} \right].$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$Z_{\Sigma}(Q) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=e^{i\theta}\varphi(0)} D\varphi_i \, e^{-S[\varphi]}$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta)=\varphi(0)} D\varphi_i \, e^{-S^{\theta}[\varphi]}$$

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EFFECTIVE ACTION: COVARIANT DERIVATIVE

$$S^{\theta}[\varphi] = \sum_{i=1}^{N} \int dt\, d\Sigma \left((D_\mu \varphi_i)^* (D^\mu \varphi_i) + \frac{R}{8} \varphi_i^* \varphi_i + 2u (\varphi_i^* \varphi_i)^2 \right)$$

$$\begin{cases} D_0 \varphi = \partial_0 \varphi + i \dfrac{\theta}{\beta} \varphi \\ D_i \varphi = \partial_i \varphi \end{cases}$$

Stratonovich transformation

$$S_Q = \sum_{i=1}^{N} \left[-i\theta_i Q_i + \int dt\, d\Sigma \left[\left(D^i_\mu \varphi_i \right)^* \left(D^i_\mu \varphi_i \right) + (r + \lambda) \varphi_i^* \varphi_i \right] \right]$$

Expand around the VEV

$$\varphi_i = \frac{1}{\sqrt{2}} A_i + u_i, \qquad \lambda = \left(\mu^2 - r \right) + \hat{\lambda}$$

EFFECTIVE ACTION FOR $\hat{\lambda}$

We can now integrate out the u_i and get an effective action for $\hat{\lambda}$ alone

$$S_\theta[\hat{\lambda}] = \sum_{i=1}^{N} \left[V\beta \left(\frac{\theta_i^2}{\beta^2} + m^2 \right) \frac{A_i^2}{2} + \mathrm{Tr}\left[\log\left(-D_\mu^i D_\mu^i + m^2 + \hat{\lambda} \right) \right] - \frac{A_i^2}{2} \mathrm{Tr}\left(\hat{\lambda} \Delta \hat{\lambda} \right) \right].$$

Non-local action for $\hat{\lambda}$.

To be expanded order-by-order in 1/N.

SADDLE POINT EQUATIONS

$$\begin{cases} \dfrac{\partial S_Q}{\partial m^2} = \displaystyle\sum_{i=1}^{N} \left[\frac{V\beta}{2} A_i^2 + \zeta(1|\theta_i, \Sigma, m) \right] = 0, \\ \dfrac{\partial S_Q}{\partial \theta_i} = -iQ + \dfrac{\theta_i}{\beta} V A_i^2 + \dfrac{1}{s} \dfrac{\partial}{\partial \theta_i} \zeta(s|\theta_i, \Sigma, m) \Big|_{s=0} = 0 \\ \dfrac{\partial S_Q}{\partial A_i} = V\beta \left(\dfrac{\theta_i^2}{\beta^2} + m^2 \right) A_i = 0. \end{cases}$$

where

$$\zeta(s|\theta, \Sigma, m) = \sum_{n \in \mathbb{Z}} \sum_{p} \left(\left(\frac{2\pi n}{\beta} + \frac{\theta}{\beta} \right)^2 + E(p)^2 + m^2 \right)^{-s}.$$

SADDLE POINT EQUATIONS

With some massaging, we find the final equations

$$\begin{cases} F_{\Sigma}(Q) = mQ + N\zeta(-\frac{1}{2}|\Sigma, m), \\ m\zeta(\frac{1}{2}|\Sigma, m) = -\frac{Q}{N}. \end{cases}$$

The control parameter is actually Q/N.

Dimensions of operators of fixed charge Q on $\mathbb{R}^3$ (state/operator):

$$\Delta(Q) = -\frac{1}{\beta}\log Z_{S^2}(Q).$$

SMALL Q/N

The zeta function can be expanded in perturbatively in small Q/N.

Result:

$$\frac{\Delta(Q)}{Q} = \frac{1}{2} + \frac{4}{\pi^2}\frac{Q}{N} + \frac{16\left(\pi^2 - 12\right)Q^2}{3\pi^4 N^2} + \ldots$$

- Expansion of a closed expression
- Start with the engineering dimension 1/2
- Reproduce an infinite number of diagrams from a fixed-charge one-loop calculation

LARGE Q/N

If $Q/N \gg 1$ we can use Weyl's asymptotic expansion.

$$\mathrm{Tr}\left(e^{\Delta_\Sigma t}\right) = \sum_{n=0}^{\infty} K_n t^{n/2-1} .$$

The zeta function is written in terms of the geometry of Σ (heat kernel coefficients)

$$m_\Sigma = \sqrt{\frac{4\pi}{V}} \left(\frac{Q}{2N}\right)^{1/2} + \frac{R}{24}\sqrt{\frac{V}{4\pi}}\left(\frac{Q}{2N}\right)^{-1/2} + \dots$$

$$\frac{F_\Sigma}{2N} = \frac{2}{3}\sqrt{\frac{4\pi}{V}} \left(\frac{Q}{2N}\right)^{3/2} + \frac{R}{12}\sqrt{\frac{V}{4\pi}}\left(\frac{Q}{2N}\right)^{1/2} + \dots$$

ORDER N

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

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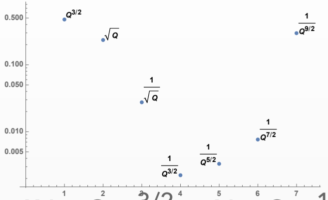

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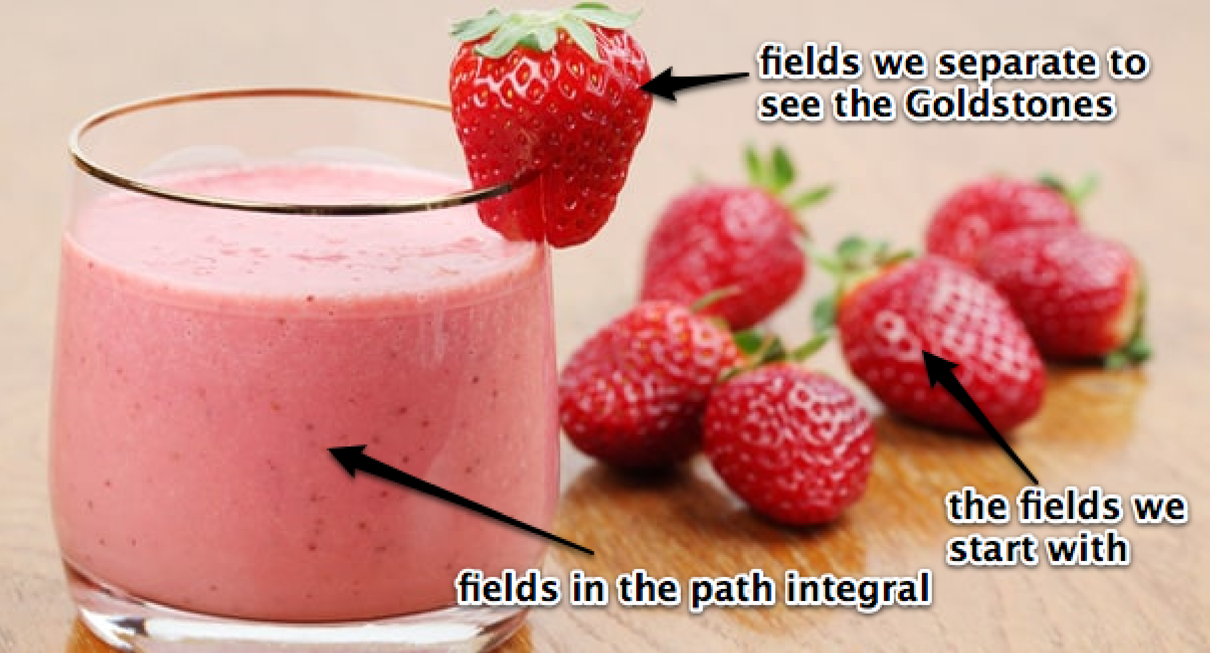
UNIVERSAL TERM: INTEGRATE ALL BUT ONE



ORDER N°

The order N^0 terms are

$$S^{\theta}[\hat{\sigma},\hat{\lambda}] = \int dt\, d\Sigma \left((D_\mu \hat{\sigma})^* (D^\mu \hat{\sigma}) + (\mu^2 + \hat{\lambda})\hat{\sigma}^* \hat{\sigma} + \frac{\hat{\lambda} v(\hat{\sigma} + \hat{\sigma}^*)}{(N-1)^{1/2}} \right)$$

$$+ \frac{1}{2} \int dx_1\, dx_2\, \hat{\lambda}(x_1)\hat{\lambda}(x_2) D(x_1 - x_2)^2$$

where $D(x-y)$ is the propagator $(D_\mu D^\mu + m^2)^{-1}$.

At low energies we can approximate the non-local term as

$$\int dt\, d\Sigma\, \hat{\lambda}(x)^2 \zeta(2|\theta,\Sigma,\mu) \approx \frac{V}{2\mu} \int dt\, d\Sigma\, \hat{\lambda}(x)^2$$

and we can integrate $\hat{\lambda}$ out.

ORDER N°

The inverse propagator for σ is

$$\begin{pmatrix} 1/2(\omega^2 + p^2 + 4\mu^2) & \mu\omega \\ -\mu\omega & 1/2(\omega^2 + p^2) \end{pmatrix}$$

It describes a massive mode and a massless mode with dispersion

$$\omega^2 + \frac{1}{2}p^2 + \ldots = 0 \qquad \omega^2 + 8\mu^2 + \frac{3}{2}p^2 + \ldots = 0$$

This is the conformal Goldstone that we have seen in the EFT.

Its contribution to the partition function is

$$E_G = \frac{1}{2}\frac{1}{\sqrt{2}}\zeta(1/2|S^2) = -0.0937\ldots$$

This is universal. Does not depend on N or Q.

WAS IT WORTH IT?

FINAL RESULT

$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}\left(N^0\right)\right)\left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}\left(N^0\right)\right)\left(\frac{Q}{2N}\right)^{1/2} + \ldots - 0.0937\ldots$$

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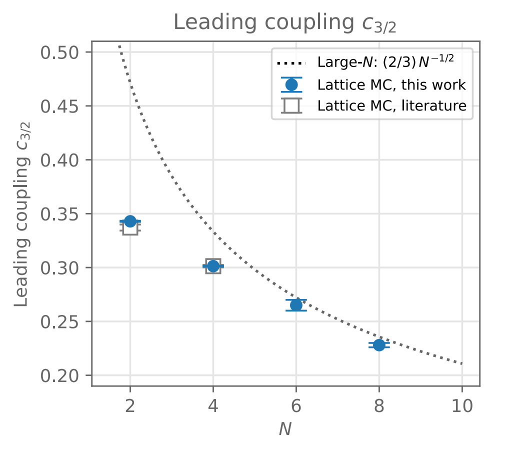

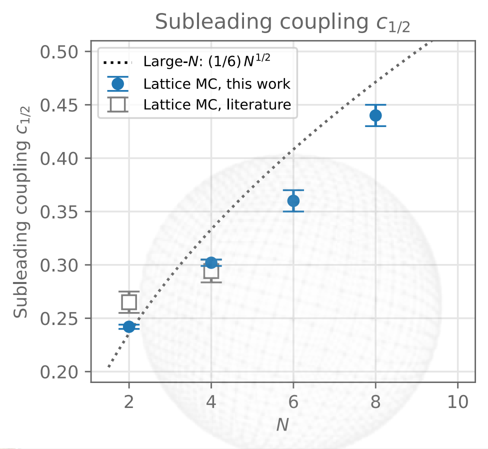

RESURGENCE AND THE LARGE CHARGE

RESULTS FROM LARGE N

In the double-scaling limit $N \to \infty$, $Q \to \infty$ with $\hat{q} = Q/(2N)$ fixed.

$$
\begin{cases}
F_{\Sigma}(Q) = \mu Q + N \zeta(-\frac{1}{2}|\Sigma, \mu), \\
\mu \zeta(\frac{1}{2}|\Sigma, \mu) = -\frac{Q}{N}.
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RESULTS FROM LARGE N

In the **double-scaling** limit $N \to \infty$, $Q \to \infty$ with $\hat{q} = Q/(2N)$ fixed.
The free energy per DOF $f(\hat{q}) = F/(2N)$ is a Legendre transform

$$f(\hat{q}) = \sup_{\mu}(\mu\hat{q} - \omega(\mu)), \qquad \hat{q} = \frac{d\omega(\mu)}{d\mu}, \qquad \omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|\Sigma, \mu),$$

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$\zeta(s|\Sigma, \mu)$ is the zeta function for the operator $-\triangle + \mu^2$.

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Large $\hat{q}$ is large μ and is small t. The classical Seeley-de Witt problem:

$$\mathrm{Tr}\left(e^{\triangle t}\right) \sim \frac{V}{4\pi t}\left(1 + \frac{R}{12} t + \dots\right).$$

WARM UP: THE TORUS

$$\mathrm{spec}(\triangle) = \left\{ -\frac{4\pi^2}{L^2}\left(k_1^2 + k_2^2\right) | k_1, k_2 \in \mathbb{Z} \right\}.$$

The heat kernel trace is the square of a theta function:

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \sum_{k_1, k_2 \in \mathbb{Z}} e^{-\frac{4\pi^2}{L^2}(k_1^2 + k_2^2)t} = \left[\theta_3(0, e^{-\frac{4\pi^2 t}{L^2}})\right]^2.$$

For the small-t limit we Poisson-resum the series:

$$\sum_{n \in \mathbb{Z}} h(n) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} h(\rho) e^{2\pi i k \rho} \, d\rho$$

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \left[\frac{L}{\sqrt{4\pi t}}\left(1 + \sum_{k \in \mathbb{Z}} e^{-\frac{k^2 L^2}{4t}}\right)\right]^2 = \frac{L^2}{4\pi t}\left(1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{\|k\|^2 L^2}{4t}}\right)$$

THE TORUS

Grand potential

$$\omega(\mu) = -\frac{1}{2}\zeta(-\tfrac{1}{2}|T^2, \mu) = \frac{L^2\mu^3}{12\pi}\left(1 + \sum_{\mathbf{k}} \frac{e^{-\|\mathbf{k}\|\mu L}}{\|\mathbf{k}\|^2\mu^2L^2}\left(1 + \frac{1}{\|\mathbf{k}\|\mu L}\right)\right).$$

Free energy

$$f(\hat{q}) = \sup_{\mu}(\mu\hat{q} - \omega(\mu)) = \frac{4\sqrt{\pi}}{3L}\hat{q}^{3/2}\left(1 - \sum_{\mathbf{k}} \frac{e^{-\|\mathbf{k}\|\sqrt{4\pi\hat{q}}}}{8\|\mathbf{k}\|^2\pi\hat{q}} + \dots\right).$$

- perturbative expansion in μ (here a single term) plus exponentially suppressed terms controlled by the dimensionless parameter μL
- the free energy is written as a double expansion in the two parameters $1/\hat{q}$ and $e^{-\sqrt{4\pi\hat{q}}}$.
- more important than the "usual" instantons $\mathcal{O}\left(e^{-\hat{q}}\right)$

THE SPHERE

On the two sphere $\mathrm{spec}(\triangle) = \{-\ell(\ell+1) \mid \ell \in \mathbb{N}_0\}$ with multiplicity $2\ell + 1$.
Again, we use Poisson resummation

$$\sum_{n\in\mathbb{Z}} h(n) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} h(\rho) e^{2\pi i k\rho}\, d\rho$$

to rewrite the heat kernel in terms of the imaginary error function

$$\mathrm{Tr}\left(e^{\triangle t}\right) e^{-t/4} = \sum_{\ell\geq 0} (2\ell+1) e^{-(\ell+1/2)^2 t} = \frac{r^2}{t} + 2\sum_{k\in\mathbb{Z}} (-1)^k \left[\frac{r^2}{t} - \frac{2k\pi r^3}{t^{3/2}} F\left(\frac{\pi r k}{t^{1/2}}\right)\right]$$

where

$$F(z) = e^{-z^2} \int_0^z dt\, e^{-t^2} = \frac{\sqrt{\pi}}{2} e^{-z^2}\, \mathrm{erfi}(z)$$

SPHERE: ASYMPTOTIC EXPANSION

For small t

$$F(z) \sim \sum_{n=0}^{\infty} \frac{(2n+1)!!}{2^{n+1}} \left(\frac{1}{z}\right)^{2n+1}$$

and

$$\mathrm{Tr}\left(e^{(\triangle - \frac{1}{4})t}\right) \sim \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(1-2^{1-2n})}{n!} B_{2n} t^n$$

The series is asymptotic: the Seeley-de Witt coefficients diverge like n!:

$$a_n = \frac{(-1)^{n+1}(1-2^{1-2n})}{n!} B_{2n} \sim \frac{2n^{1/2}}{\pi^{5/2+2n}} n!.$$

the divergence implies the existence of non-perturbative corrections.

RESURGENCE

The key idea is that we should think in terms of transseries

$$H(t) = t^{-b_0} \sum_{n\geq 0} a_n^{(0)} t^n + \sum_{k\geq 1} C_k e^{-A_k/t} t^{-b_k} \sum_{n\geq 0} a_n^{(k)} t^n,$$

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The coefficients of the non-perturbative part are encoded in the large-n behavior of the perturbative piece:

$$a_n^{(0)} \sim \sum_{k\geq 1} \frac{C_k}{2\pi i} \frac{1}{A_k^{n/\beta+b_k}} \left(a_0^{(k)} \Gamma(\beta n + b_k) + a_1^{(k)} A_k \Gamma(\beta n + b_k - 1) + \ldots \right)$$

RESURGENCE

In our case, the a_n are

$$a_n = 4\sqrt{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(n+\frac{1}{2})}{(\pi k)^{2n}}.$$

Comparing the two expressions we find that for the trace of the heat kernel:

$$\beta = 1, \qquad b_k = \frac{1}{2}, \qquad A_k = (\pi k)^2, \qquad \frac{C_k}{2\pi i}a_0^{(k)} = 4(-)^k k\pi^{3/2}, \qquad a_{>0}^{(k)} = 0.$$

The series around each exponential are truncated to only one term and the non-perturbative correction to the heat kernel is

$$4i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k=1}^{\infty}(-)^k k\, e^{-(\pi k)^2/t}.$$

BOREL RESUMMATION

BOREL TRANSFORM

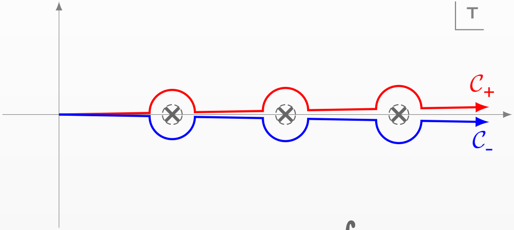
We need to make sense of the divergent series and the imaginary terms.

$$H(t) = \sum_{n \geq 0} a_n t^n \longrightarrow \hat{H}(\tau) = \sum_{n \geq 0} \frac{a_n}{\Gamma(\beta n + b)} \tau^n$$

$$s(H)(t) = \int_0^\infty w^b e^{-w} \hat{H}(t w^\beta) \frac{dw}{w}$$

LATERAL TRANSFORM

If there are poles on the real positive axis there is an ambiguity



$$s_\pm(H)(t) = s(H)(t) = \int_{\mathcal{C}_\pm} w^b e^{-w} \hat{H}(t w^\beta) \frac{dw}{w}$$

$$s_+(H) - s_-(H) = (2\pi i) \sum_k \text{residue}$$

We need an independent definition of the non-perturbative effects to cancel the imaginary ambiguity.

BOREL TRANSFORM FOR THE HEAT KERNEL ON S²

The Borel transform can be summed in terms of elementary functions

$$H(\tau) = \frac{1}{\tau} \sum_{n \geq 0} \frac{a_n}{\Gamma(n + 3/2)} \tau^n = \frac{1}{\sqrt{\pi \tau} \sin\left(\sqrt{\tau}\right)}$$

and if we Laplace transform [Perrin, 1928]

$$s(H)(t) = \frac{2}{\sqrt{\pi} t^{3/2}} \int_0^\infty dy \, y \frac{e^{-y^2/t}}{\sin(y)}$$

there are simple poles for $y = k\pi$, $k = 1, 2, \ldots$. The residues are

$$(2\pi i) \operatorname{Res}\left(\frac{2}{\sqrt{\pi} t^{3/2}} y \frac{e^{-y^2/t}}{\sin(y)}, k\pi \right) = (-)^{k+1} 4i|k| \left(\frac{\pi}{t}\right)^{3/2} e^{-\frac{k^2 \pi^2}{t}} .$$

MORE INGREDIENTS

WORLDLINE INTERPRETATION

We need a **non-perturbative interpretation** of these exponential terms.

We read the heat kernel as the partition function of a particle at inverse temperature t and Hamiltonian $H = -\partial_0^2 - \triangle$, i.e. a **free quantum particle moving on $\mathbb{R} \times \Sigma$.**

We can write the partition function as a **path integral**

$$\mathrm{Tr}\left(e^{(\partial_0^2+\triangle)t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]}$$

where the action is

$$S[X] = \frac{1}{4t}\int_0^1 d\tau\, g_{\mu\nu}\dot{X}^{\mu}(\tau)\dot{X}^{\nu}(\tau)$$

A TRANSSERIES FROM GEODESICS

In the limit $t \to 0$ the path integral localizes on a sum over all the closed geodesics γ.

For each geodesic a perturbative series in t, weighted by $e^{-\ell(\gamma)^2/(4t)}$

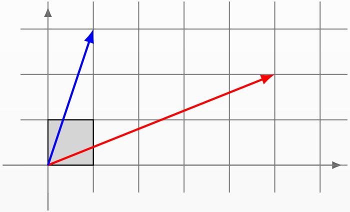
$$
\begin{aligned}
\mathrm{Tr}\left(e^{(\partial_0^2+\triangle)t}\right) &= \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]} \\
&= t^{-b_0} \sum_{n=0}^{\infty} a_n^{(0)} t^n + \sum_{\gamma \,\in\, \text{closed geodesics}} e^{-\frac{\ell(\gamma)^2}{4t}}\, t^{-b_\gamma} \sum_{n=0}^{\infty} a_n^{(\gamma)} t^n,
\end{aligned}
$$

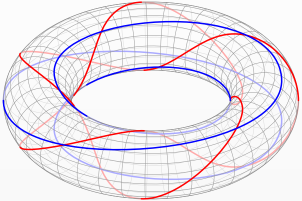
the b_γ depend on the geometry. This is precisely the same structure predicted by resurgence.

Now we have a geometric interpretation.

THE TORUS

Closed geodesics on the torus are labelled by two integers (k_1, k_2)





The length of the geodesic is $\ell(k_1, k_2) = L\sqrt{k_1^2 + k_2^2}$.

The integral is quadratic and the fluctuations around each geodesic give the usual

$$\mathcal{N} \int_{h(1)=h(0)=0} \mathcal{D}h\, e^{-\frac{1}{4t}\int_0^1 d\tau (\dot{h}^1)^2 + (\dot{h}^2)^2} = \mathcal{N} \det\left(\frac{1}{4t}\partial_\tau^2\right)^{-1} = \frac{1}{4\pi t}.$$

THE TORUS

Now we can write the result of the path integral

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \mathcal{N} \int_{X(1)=X(0)} \mathcal{D}X\, e^{-S[X]} = \mathcal{N} L^2 \sum_{X_{cl}} \int_{h(1)=h(0)=0} e^{-S[X_{cl}]-S[h]}$$

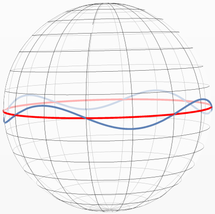
$$= \mathcal{N} L^2 \sum_{k \in \mathbb{Z}^2} e^{-\frac{L^2 (k_1^2+k_2^2)}{4t}} \int_{h(1)=h(0)=0} \mathcal{D}h\, e^{-S[h]},$$

$$= \frac{L^2}{4\pi t}\left[1 + \sum_{k \in \mathbb{Z}^2} e^{-\frac{L^2 \|k\|^2}{4t}}\right]$$

This is exactly what we had found before just by looking at the spectrum. Now we can understand the non-perturbative effects in terms of closed geodesics.

THE SPHERE

Closed geodesics on the sphere go around the equator k times



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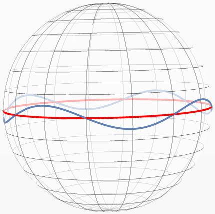

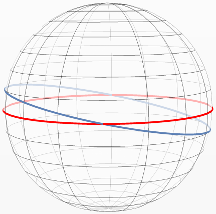

There is a zero mode because we can rotate the equator

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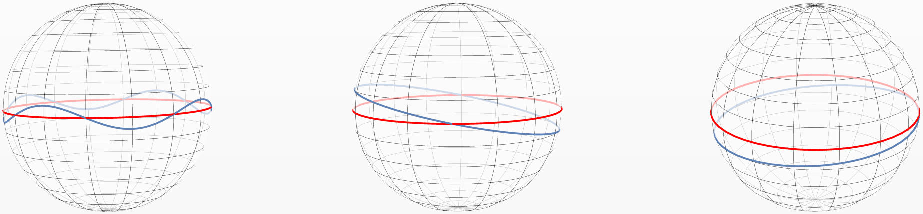

There is a zero mode because we can rotate the equator

And an instability because we can slide off

THE SPHERE PATH INTEGRAL

At leading order we can just pick a coordinate system and expand the action

$$L = \dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2$$

around the geodesic

$$\theta = \frac{\pi}{2} \qquad\qquad \varphi(\tau) = 2\pi k \tau$$

so that the fluctuations give a massless and a massive mode

$$\mathrm{Tr}\left(e^{\triangle t}\right) = \sum_{k\in\mathbb{Z}} e^{-\frac{(2\pi k)^2}{4t}} \int \mathcal{D}h_\theta \mathcal{D}h_\varphi \exp\left[-\frac{1}{4t}\int_0^1 d\tau \left(\dot{h}_\varphi^2 + \dot{h}_\theta^2 - (2\pi k)^2 h_\theta^2\right)\right]$$

THE SPHERE PATH INTEGRAL

The h_φ fluctuation is massless and gives

$$\int \mathcal{D}h_\varphi \, \exp\left[-\frac{1}{4t}\int_0^1 d\tau\, \dot{h}_\varphi^2\right] = \frac{1}{(4\pi t)^{1/2}}$$

THE SPHERE PATH INTEGRAL

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For h_θ we need to work a bit more. Decompose in modes:

$$h_\theta = \sqrt{2}\sin(\pi n \tau) \qquad\qquad \lambda_n = \frac{\pi^2}{2}\left(n^2 - 4k^2\right)$$

- a zero mode for $n = 2k$
- $2n - 1$ unstable modes

Once we regularize the determinant we get

$$\int \mathcal{D}h_\theta \exp\left[-\frac{1}{4t}\int_0^1 d\tau\left(\dot{h}_\theta^2 - (2\pi k)^2 h_\theta^2\right)\right] = \pm i\frac{\pi}{2\sqrt{2}}\frac{k}{t}$$

BACK TO RESURGENCE

Putting it **all together**, the non-trivial geodesics give

$$\pm 2i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k\in\mathbb{Z}}|k|e^{-\frac{k^2\pi^2}{t}}$$

The one-loop result **perfectly cancels** the imaginary ambiguity of the Borel sum!

$$\mathrm{Tr}\left(e^{(\Delta-\frac{1}{4})t}\right) = s_{\pm}(H)(t) \mp 2i\left(\frac{\pi}{t}\right)^{3/2}\sum_{k\geq 1}(-1)^k k e^{-\frac{k^2\pi^2}{t}} = \mathrm{Re}[s_{\pm}(H)(t)]$$

BACK TO RESURGENCE

We can write the **exact expression** for the grand potential ($m^2 = \mu^2 + 1/4$):

$$\omega(\mu) = \mathrm{Re}\left[\frac{2rm^2}{\pi}\int_0^\infty dy\,\frac{K_2(2mry)}{y\sin(y)}\right] = \frac{r^2}{3}m^3 - \frac{m}{24} + \cdots - \frac{2ir^{1/2}m^{3/2}}{(4\pi)^{3/2}}e^{-2\pi rm} + \ldots$$

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As a numerical test, we can compare with the convergent small-charge expansion ($\hat{q} \approx 0.6$)

$$r\omega(mr = 0.4)\Big|_{\text{small charge}} = 0.012\,777\,296\,63\ldots$$

$$r\omega(mr = 0.4)\Big|_{\text{resurgence}} = 0.012\,777\,297\,69\ldots$$

OPTIMAL TRUNCATION

LESSONS FROM LARGE N

Let's go back to the EFT.
The effective action is identified with the asymptotic expansion: the **grand potential** is the value of the **action at the minimum** $\chi = \mu t$:

$$\omega(\mu) = L_{EFT}\Big|_{\chi=\mu t}$$

where

$$L_{EFT} = \omega_0 \left(\partial_\mu \chi \, \partial^\mu \chi\right)^{3/2} + \omega_1 \left(\partial_\mu \chi \, \partial^\mu \chi\right)^{1/2} + \ldots,$$

In general the **coefficients are unknown**

BUT

Now we have a **geometric understanding** of the non-perturbative effects

LESSONS FROM LARGE N

Assume:

1. the large-charge expansion is **asymptotic**;
2. the leading pole in the Borel plane is **a particle of mass μ going around the equator**.

A CFT has no intrinsic scales.
The only dimensionful parameter is due to the fixed charge density.

The conformal dimension is a transseries

$$\Delta(Q) = Q^{3/2} \sum_{n \geq 0} f_n^{(0)} \frac{1}{Q^n} + C_1 Q^{b_1} e^{-3\pi\kappa f_0^{(0)} \sqrt{Q}} \sum_{n \geq 0} f_n^{(1)} \frac{1}{Q^{n/2}} + \dots$$

(we used $\mu = 3 f_0^{(0)} \sqrt{Q}/2 + \dots$.)

LESSONS FROM LARGE N

- The controlling parameter for the non-perturbative effects $e^{-3\pi\kappa f_0 \sqrt{Q}}$ is fixed by the leading term in the 1/Q expansion.
- The non-perturbative coefficient $e^{-3\pi\kappa f_0^{(0)} \sqrt{Q}}$ fixes the large-n behavior of the perturbative series $f_n^{(0)}$.

$$f_n^{(0)} \sim (2n)!(3\pi\kappa f_0^{(0)})^{-n}$$

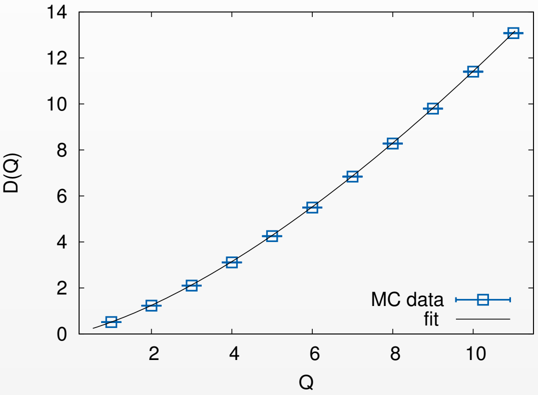
We don't know enough for a Borel resummation, but we can estimate an optimal trucation (the value of n where $f_n^{(0)} Q^{-n}$ is minimal)

$$N^* \approx \frac{3\pi\kappa f_0^{(0)}}{2} Q^{1/2}$$

corresponding to an error of order $\varepsilon(Q) = \mathcal{O}\left(e^{-\sqrt{Q}}\right)$

CAN WE UNDERSTAND THE LATTICE RESULTS NOW?

In O(2), $f_0^0 \approx 0.301(3)$, so $N^* = \mathcal{O}\left(\sqrt{Q}\right)$ and $\varepsilon(Q) = \mathcal{O}\left(e^{-\sqrt{Q}}\right)$.



This fit was obtaind with N = 3 terms.

For Q = 1 we get an error $\approx 6 \times 10^{-2}$ and for Q = 11 the error is $\approx 5 \times 10^{-5}$

(Compared to $e^{-\pi} \approx 4 \times 10^{-2}$ and $e^{-\pi\sqrt{11}} = 3 \times 10^{-5}$).

WHAT HAS HAPPENED?

- The large-charge expansion of the Wilson–Fisher point is **asymptotic**
- In the **double-scaling** limit $Q \to \infty$, $N \to \infty$ we control the perturbative expansion
- We can **Borel**-resum the expansion
- We have a **geometric interpretation for the non-perturbative effects**
- We can use this geometric interpretation also in the **finite-N** case
- We obtain an **optimal truncation** and estimate of the error
- The results are **consistent with lattice simulations**

CONCLUSIONS

- With the large-charge approach we can study **strongly-coupled systems perturbatively**.
- Select a sector and we write a **controllable effective theory**.
- The strongly-coupled physics is (for the most part) subsumed in a **semiclassical state**.
- Qual(nt)itative control of the **non-pertubative** effects.
- Compute the CFT data.
- Very good agreement with **lattice** (supersymmetry, large N).
- Precise and **testable predictions**.