Local-Potential Approximation in Tensor-invariant Theories

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What's the phase structure of tensorial theories?

- Understanding of large-N limits for tensors
- allows to identify renormalizable tensorial field theories
- some of which are asymptotically free
- functional renormalization group (FRG) techniques can be applied

Still, the phase structure is poorly understood (new phases/fixed points??)

Local potential approximation (LPA)

- results from ϕ^n truncation need stability check for larger n
- LPA approximates to any order, even analytic results (O(N) theory)
- hints for an LPA of melons [Carrozza,Lahoche1612] or necklaces [CaLaOriti1703]
- NGFPs and dimensional flow in cyclic-melonic LPA [Pithis/JT2010]
- But is it actually valid? What are the limits?

Result: LPA only for restricted regimes, but such exist at large N

Outline

1. Local potential approximation
   - FRG equation
   - Example: $\mathrm{O}(N)$ theory
2. Combinatorial point of view
   - Power series expansions
   - Diagramatics and large N
3. Scaling and dimension
   - Equivalence to vector theory
   - Effective dimension

Theory at scale k given by generating function with k-dep. IR-regulator $\mathcal{R}_k$

$$e^{W_k[J]} = \int D\phi \, e^{-S[\phi]-(\phi,\mathcal{R}_k\phi)+(J,\phi)}$$

Scale-dependent effective action via Legendre transform w.r.t. $\varphi = \frac{\delta W_k[J]}{\delta J}$

$$\Gamma_k[\varphi] = \sup_J\{(J,\varphi) - W_k[J]\} - (\varphi,\mathcal{R}_k\varphi)$$

RG flow determined by functional equation [Wetterich'93, Morris'94]

$$\boxed{k\partial_k\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\,\frac{k\partial_k\mathcal{R}_k}{\Gamma_k^{(2)}[\varphi]+\mathcal{R}_k}}$$

Interpolates between microscopic theory $k\to\infty$ and full quantum effective action $\Gamma = \lim_{k\to 0}\Gamma_k$

Explicit FRG equation needs approximations

- regulator $\mathcal{R}_k$ implies approximation, can be optimized
- eff. action Γ_k can be truncated in various ways:
  - vertex expansion: expand Γ_k in powers φ^n
  - derivative expansion: expand in powers of derivatives

$$\Gamma_k[\varphi] = \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x} \left[U_k[\varphi(\boldsymbol{x})] + \frac{1}{2} Z_k[\varphi(\boldsymbol{x})](\partial\varphi)^2(\boldsymbol{x}) + \frac{1}{4} Y_k[\varphi(\boldsymbol{x})](\partial\varphi^2)^2(\boldsymbol{x}) + \ldots \right]$$

(coefficients U_k, Z_k, $Y_k, \ldots$ still functionals of φ)

LPA is the 0'th order of the derivative expansion

Local QFT (point-like interactions):
Potential U_k completely determined evaluating on constant $\varphi(\boldsymbol{x}) = \chi$

$$\Gamma_k[\chi] = U_k[\chi] \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x} = a_{\mathbb{R}}^d U_k[\chi]$$

(still true in LPA$'$ with Z_k constant but k-dependent)

Paradigmatic example: $\mathrm{O}(N)$-symmetric scalar field theory

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2} Z_k \partial\varphi^a \partial\varphi_a + U(\varphi^a \varphi_a) \right)$$

Projection of the flow on constant average field $\rho = \frac{1}{2}\varphi^a\varphi_a$:

$$k\partial_k U_k(\rho) = c_d Z_k k^{d+1} \left(\frac{N-1}{Z_k k^2 + U'(\rho)} + \frac{1}{Z_k k^2 + U'(\rho) + 2\rho U''(\rho)} \right)$$

Rescaling $u = U_k/c_d Z_k k^d$ and $\rho = \frac{1}{2} Z_k k^{2-d} \varphi^a \varphi_a$ by canonical dimension:

$$\boxed{k\partial_k u + du - (d-2)\rho u' = \frac{N-1}{1+u'} + \frac{1}{1+u'+2\rho\, u''}}$$

Can be solved exactly at large N, otherwise expansion in powers ρ^n

Taylor expansion of $u(\rho) = \tilde{\mu}\rho + \sum \frac{\tilde{\lambda}_n}{n!}\rho^n$ given by Faà-di-Bruno formula for

- $f \circ g = \frac{1}{\bullet} \circ (1 + u')$, yields expansion in partial Bell polynomials $B_{n,l}$

$$b_n^{\mathrm{v1}}(\tilde{\mu}, \tilde{\lambda}_j) = \sum_{l=1}^{n} \frac{(-1)^l l!}{(1+\tilde{\mu})^{l+1}} B_{n,l}(\tilde{\lambda}_2, \tilde{\lambda}_3, ..., \tilde{\lambda}_{n-l+2})$$

- and for $g(\rho) = 1 + u'(\rho) + 2\rho u''(\rho)$ shift $b_n^{\mathrm{v2}}(\tilde{\mu}, \tilde{\lambda}_j) = b_n^{\mathrm{v1}}(\tilde{\mu}, (2j-1)\tilde{\lambda}_j)$

$\rightarrow$ infinite tower of coupled equations (depend on $\tilde{\lambda}_{n+1}$ at order n)

$$\boxed{k\partial_k \tilde{\lambda}_n = -d\tilde{\lambda}_n + n(d-2)\tilde{\lambda}_n + (N-1)b_n^{\mathrm{v1}}(\tilde{\lambda}_i) + b_n^{\mathrm{v2}}(\tilde{\lambda}_i)}$$

*** anybody seen these Bell polynomials in the FRG literature? References??

Large-N solutions

In the large-N limit, the u'' term vanishes:

$$k\partial_k u + du - (d-2)\rho u' = \frac{1}{1+u'}$$

Flow equations for vacuum expansion $u(\rho) = \sum_{n\geq 2} \frac{g_n}{n!}(\rho - \kappa)^n$ decouple:

$$\partial_t \kappa + (d-2)\kappa = 1$$
$$\partial_t g_n + d g_n + (d-2) n g_n = b_n^{\mathrm{V1}}(1, g_2, ..., g_n, g_{n+1} = 0)$$

$\rightarrow$ exact recursive fixed point solution ($\partial_t g_n = 0$)

$$\kappa^* = \frac{1}{d-2}, \quad g_2^* = \frac{4-d}{2}, \quad g_3^* = \frac{3}{4}\frac{(d-4)^3}{d-6}, \ldots$$

this is the Wilson-Fisher fixed point

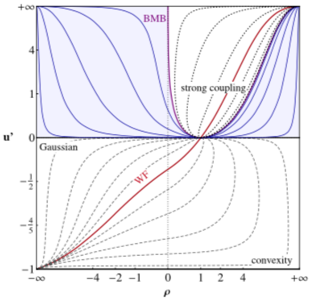
- non-vanishing vacuum (κ local minimum) for $2 < d < 4$
- converges to the Gaussian fixed point $u = 0$ for $d \to 4$
- scaling exponents $\theta_n = d - 2n$, only $\theta_1 > 0$

Possible to solve the flow for u' exactly with method of characteristics:
[Busiello/DeCesare/Rabuffo '81]...[Litim/Tetradis 9501]

$$k\partial_k u + 2u' - (d-2)\rho u'' = \frac{u''}{(1+u')^2}$$

has implicit 1-parameter fixed-point solutions

$$\rho = \frac{1}{d-2}\,{}_2F_1\left(2, \frac{2-d}{2}, \frac{4-d}{2}, -u'\right) + c(u')^{\frac{d-2}{2}}$$

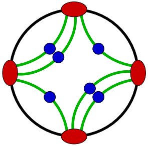
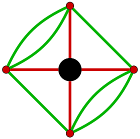


[Litim/Marchais/Mati 1702]

LPA in non-local field theories??

Two ways beyond locality (point-like interactions):

- derivatives $\partial\varphi$
- combinatorially non-local interactions encoded by graphs γ, e.g.

$\mathrm{tr}_\gamma(\varphi_{abc}(x)) \quad \cong$  $\cong$ 

Then even for single scalar field various interactions at given order φ^{2n}!

- $\rightarrow$ projection to constant field $\varphi(\boldsymbol{x}) = \chi$ does not determine potential U_k

$$\Gamma_k[\chi] = \int_{\mathbb{R}^d} U_k(\chi) = \int_{\mathbb{R}^d} \mathrm{d}x \sum_\gamma \lambda_{\gamma;k} \mathrm{tr}_\gamma(\chi) = a_{\mathbb{R}}^d \sum_{n=0}^{\infty} \left(\sum_{\gamma; V_\gamma = 2n} \lambda_{\gamma;k} \right) (a^p \chi^2)^n$$

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Another way to Bell polynomials

- PF expansion: split Gaussian part C in Γ_k such that with $P_{\mathrm{R}}^{-1} = C + \mathcal{R}_k$

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\frac{\partial_k \mathcal{R}_k}{\Gamma_k^{(2)}[\varphi] + \mathcal{R}_k} = \frac{1}{2}\mathrm{Tr}\frac{\partial_k \mathcal{R}_k}{P_{\mathrm{R}}^{-1} + F[\varphi]} \; ,$$

- assume formal power series $F[\varphi] = \sum_{j\geq 1} a_j \varphi^{2j}$ and use geometric series

$$\frac{\partial_k \mathcal{R}_k}{P_{\mathrm{R}}^{-1} + F[\varphi]} = P_{\mathrm{R}} \frac{\partial_k \mathcal{R}_k}{1 + P_{\mathrm{R}} F[\varphi]} = \sum_{l\geq 0} P_{\mathrm{R}} (-P_{\mathrm{R}} F[\varphi])^l$$

- then multinomial expansion

$$(a_1 \varphi^2 + a_2 \varphi^4 + \ldots)^l = \sum_{s_1+s_2+\ldots=l} \binom{l}{s_1, s_2, \ldots} a_1^{s_1} a_2^{s_2} \cdots \varphi^{2\sum_{j\geq 1} j s_j}$$

- $\rightarrow$ at ϕ^{2n}, sum over partitions $\sigma \vdash n$ of length $|\sigma| = l$

$$\sum_{l\geq 1} P_{\mathrm{R}} (-P_{\mathrm{R}} F[\varphi])^l |_{\phi^{2n}} = \sum_{l\geq 1} (-1)^l P_{\mathrm{R}}^{l+1} \sum_{\substack{\sigma\vdash n \\ |\sigma|=l}} \binom{l}{s_1, s_2, \ldots, s_n} a_1^{s_1} a_2^{s_2} \cdots$$

Another way to Bell polynomials II

These sums over partitions are (up to factor $l!/n!$)

- partial ordinary Bell polynomials

$$\widehat{B}_{n,l}\left(a_1, a_2, \ldots, a_{n-l+1}\right) = \sum_{\substack{\sigma \vdash n \\ |\sigma| = l}} \binom{n}{s_1, s_2, \ldots, s_{n-l+1}} \prod_{j=1}^{n-l+1} \left(a_j\right)^{s_j}$$

- or for an exponential series $a_j = b_j / j!$ partial exponential Bell polynomials

$$B_{n,l}\left(b_1, b_2, \ldots, b_{n-l+1}\right) = \sum_{\sigma} \binom{n}{s_1, s_2, \ldots, s_{n-l+1}} \prod_{j=1}^{n-l+1} \left(\frac{b_j}{j!}\right)^{s_j}$$

- coefficients also found in [Carozza/Lahoche 1610], but there convention λ_j / j

General PF expansion for F a power series

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr}\left[P_{\mathrm{R}} + \sum_{n \geq 1} \frac{\varphi^{2n}}{n!} \sum_{l \geq 1} (-1)^l l! P_{\mathrm{R}}^{l+1} \widehat{B}_{n,l}\left(a_1, a_2, \ldots, a_{n-l+1}\right)\right]$$

Question: Conditions to treat $F[\varphi]$ as such a power series?

Naive answer: "$\mathrm{Tr}[\varphi^{2n}]$" must have unique meaning.

- this is a rephrasing of the LPA condition
- if unique, then one can set "$\mathrm{Tr}[\varphi^{2n}]$" $\propto \rho^n$
- obviously true for a local scalar field

$$\mathrm{Tr}[\varphi^{2n}] = \mathrm{Tr}[\delta(x-y)\varphi^{2n}(x)] = \int \mathrm{d}x\, \varphi^{2n}(x)$$

- but also true for cyclic case: if $\mathrm{Tr}[\varphi^{2n}(x)]$ is an actual trace

Corollary: LPA expands in Bell polynomials in general

Could be an inverse strategy: finding Bell polynomials in PF expansion may hint at possibility for constant-field projection method...

Example O(N) theory

Instead of projection $\rho = \frac{\varphi(x)^2}{2}$, power series treatment yields same result:

- Take second derivatives

$$\frac{\delta^2 (\varphi_c\varphi_c)^j(x)}{\delta\varphi_a(x)\delta\varphi_b(y)} = 2j\left[\delta_{ab}(\varphi\varphi)^{j-1}(x) + 2(j-1)\varphi_a(x)\varphi_b(x)(\varphi\varphi)^{j-2}(x)\right]\delta(x-y)$$

- thus possible to express F in terms of potential U

$$\mu + F[\varphi] = \delta_{ab}U'[\varphi(x)^2/2] + \varphi_a(x)\varphi_b(x)U''[\varphi(x)^2/2]$$

- Wetterich trace yields the same FRG equation as before

$$\begin{aligned}\mathrm{Tr}[(\delta_{ab}U' + \varphi_a\varphi_b U'')^n] &= \mathrm{Tr}\sum_{k=0}^{n}\binom{n}{k}(\delta_{ab}U')^k(\varphi_b\varphi_c U'')^{n-k}\\ &= (N-1)\mathrm{Tr}[U'^n] + \mathrm{Tr}[(U' + \varphi^2 U'')^n]\end{aligned}$$

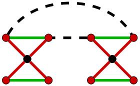
- both $\mathrm{Tr}[\delta_{ab}(\varphi\varphi)^n]$ and $\mathrm{Tr}[\varphi_a\varphi_b(\varphi\varphi)^{n-1}]$ yield again $\int \mathrm{d}x\,(\varphi\varphi)^n(x)$, but at different powers in N

O(N) theory diagramatically

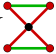
Index structure in the trace easier to grasp diagramatically:

- green lines for index contractions at vertices
- terms $\propto N$ have green line going through all "Wick contractions":

$$\mathrm{Tr}[(\delta_{ab}\varphi^2)^2] \cong$$

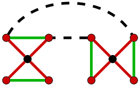


$$\propto N$$

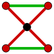


- in all other terms, all green lines stop at some external vertex

$$\mathrm{Tr}[(\delta_{ab}\varphi^2)(\varphi_b\varphi_a)] \cong$$



$$\propto$$



- same for higher order interactions φ^{2j}, and for higher order in $(P_{\mathrm{R}}F)^n$

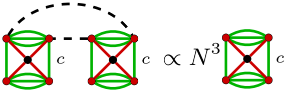
Vector theory: all diagrams yield again vector interactions $(\varphi_c\varphi_c)^j$

thus LPA works for the complete theory at finite N

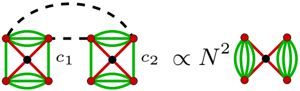
General non-local diagrams

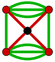
For tensors (more indices) usually new interactions are generated

- cyclic melonic diagrams yield again cyclic melonic interactions



- but melons with different colours already yield others



In fact, already  c for different c generate any bipartite coloured graph

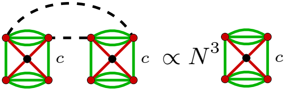
$\rightarrow$ no LPA in general

Stable regimes at large N

Still, at large N a given class of graphs may generate only themselves

Example: Cyclic-melonic potential approximation

- advantage of RG flow: pick regime by UV boundary condition $\Gamma_{k=\Lambda}$
- observation in [Carrozza/Lahoche1612]: cyclic melons dominate flow at large N
- indeed easy to prove: N^{p-1} diagrams like



  yield always again cyclic melonic interactions
- thus at large N this is an LPA (as $\mathrm{Tr}(\varphi^{2n})$ is unique, i.e. cyclic-melonic)

Strategy towards new phases: New LPA regimes from simple diagramatics

Cyclic necklaces already identified [CaLaOriti1703], there should be many more

Meaning of large N

A) standard case: N size of internal indices

- array of local fields $\varphi_{a_1 a_2 \ldots a_p}(x)$ propagating on domain $\mathbb{R}^d \ni x$
- possibly some symmetry like $\mathrm{O}(N)^p$ (not necessary)
- examples: usual vector theories, SYK related tensor field theories

B) dynamic case: $N = ak$ cutoff of propagating d.o.f.

- fields $\varphi(\boldsymbol{g})$ on compact domain $G^p \ni \boldsymbol{g}$ of volume a
- thus discrete (countable) spectrum of momenta $\varphi_{a_1 a_2 \ldots a_p}$
- combined fields $\varphi(x, \boldsymbol{g})$ on $\mathbb{R}^d \times G^p$ possible, then again $\varphi_{a_1 a_2 \ldots a_p}(x)$
- RG scale k leads then to momentum cutoff $N = a \cdot k$
- large N means either i) UV $k \to \infty$ or ii) "thermodynamic" limit $a \to \infty$

Main difference (in LPAs) only in scaling! Qualitatively similar results!

Ex.: FRG equation in cyclic-melonic LPA

Full FRG equation in cyclic melonic (isotropic) LPA on $\mathbb{R}^d \times U(1)^p$:

$$\boxed{k\partial_k U = \frac{I_\eta^{(d,0)}}{k^2 Z_k + U' \ + 2\rho U''} + \frac{p I_\eta^{(d,1)}}{k^2 Z_k + U'} + \sum_{s=2}^{p} \binom{p}{s} \frac{I_\eta^{(d,s)}}{k^2 Z_k + M_k^{(s)}}}$$

- again potential $U(\rho) = \mu_k \rho + \sum_{n\geq 2} \frac{\lambda_n}{n!} \rho^n = \mu_k \rho + V_k(\rho)$
- at higher orders, comb. factors in eff. mass $M_k^{(s)}(\rho) := \mu_k + \frac{p-s}{p} V_k'(\rho)$
- two (A) or three (B) types of threshold functions in

  $$I_\eta^{(d,s)}(k) = k^2 Z_k \left(1 - \frac{\eta_k}{2}\right) I_0^{(d,s)} + Z_k \frac{\eta_k}{2} \left(I_1^{(d,s)} + I_2^{(d,s)}\right)$$

  which all behave qualitatively like

  $$I_0^{(d,s)} \propto I_1^{(d,s)}/k^2 \propto I_2^{(d,s)}/N^2 \propto k^d \cdot N^s$$

- therefore in case B) the FRGE is non-autonomous but in A) autonomous

Outline

Terms still depend on U' and $U' + 2\rho U''$, thus same exp. coefficients $b_n^{\mathrm{V1}}, b_n^{\mathrm{V2}}$:

$$\beta_{n,k}^{\mathrm{CM}}(\mu, \lambda_i) = b_n^{\mathrm{V2}}(\mu_k, \lambda_i) I_\eta^{(d,0)}(N) + b_n^{\mathrm{V1}}(\mu_k, \lambda_i) I_\eta^{(d,1)}(N) + \sum_{s=2}^{p-1} \binom{p}{s} b_n^{\mathrm{V1}} \left(\mu_k, \frac{p-s}{p} \lambda_i \right) I_\eta^{(d,s)}(N)$$

- vector case included for $p = 1$
- for $p \geq 3$: relative factor $\frac{p-s}{p}$ between mass and couplings at order N^s
- at large N equivalence to large-N vector-model (only b_n^{V1} occurs):

$$\beta_{n,k}^{\mathrm{CM}}(\mu, \lambda_i) \sim p\, b_n^{\mathrm{V1}} \left(\mu_k, \frac{p-s}{p} \lambda_i \right) k^d N^{p-1}$$

- independent of whether theory class A) or B); difference only in scaling!

Rescaling and dimension

Dimensionless beta equations can be obtained:

- Case A): Scaling like local field theory, independent of order in N:

$$\lambda_n = Z_k^n k^{d-(d-2)n} \tilde{\lambda}_n$$

- Case B): at scale $k^d N^s = a^s k^{d+s}$ rescaling:

$$\lambda_n = Z_k^n k^{d+s-(d+s-2)n} a^{(1-n)s} \tilde{\lambda}_n$$

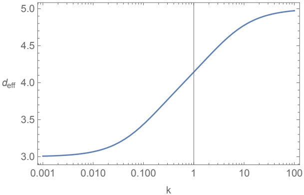
- thus, theory B) behaves there effectively like a $(d+s)$-dim. local field theory
- at large $N = ak$, this dimension is $d + p - 1$ in the melonic case
- in agreement with perturbative renormalization: divergence degree $(d = 0)$

$$\omega^{\text{s.d.}} = p - 1 - (p - 1 - 2)n - (\delta_{\text{Gurau}} + K_\partial - 1)$$

- different dimension in other regimes, e.g. $d + \frac{p}{2}$ in necklace LPA

Dimensional flow

Non-autonomy of equations B) at finite a: dimension changes continously!



- LPA is only valid at large N
- still, the scaling results apply also to the full FRG flow
- qualitative result of dimensional flow is valid in general

At each order l in expansion in Bell polynomials, k-dependence $F_l(k)$ factorizes:

$$\beta^{\mathrm{CM}}_{n,k}(\mu, \lambda_i) \cong \sum_{l=1}^{n} F_l^{(d,p)}(k) \frac{(-1)^l l!}{(Z_k k^2 + \mu_k)^{l+1}} B_{n,l}\left(\lambda_2, \lambda_3, ..., \lambda_{n-l+2}\right)$$

Continuous rescaling $\lambda_n = Z_k^n k^{2n} \left(F_1^{(d,p)}(k)\right)^{1-n} \tilde{\lambda}_n$ yields effective dimension

$$\boxed{d_{\mathrm{eff}}(k) := \frac{\partial \log F_1^{(d,p)}(k)}{\partial \log k}}$$

Summary

Insights:

- LPA not meaningful for combinatorially non-local theories in general,
- but possible at large N in some regimes
- then beta equations expand in Bell polynomials, "LPA Universality class"
- this explains equivalence of cyclic-melonic regime with vector theory
- crucial difference: scaling and thus effective dimension
- dynamic non-locality (B) has $N = a \cdot k$, therm. limit $\cong$ large-N limit
- in large-N cyclic-melonic regime, dimension $d + d_{\mathrm{G}}(p-1)$
- at finite a, dimensional flow from $d + d_{\mathrm{G}}(p-1)$ to d

Outlook

Cyclic-melonic LPA only first step to understanding of non-local phase spaces:

- proper argument for dimensional flow without reference to LPA
- systematic exploration of LPA regimes, proof for universality
- LPA regimes in the IR (stable theories at N^0)?
- better understanding of LPA$'$ (flowing anomalous dimension η)
- analytic methods for more precise statements about NGFPs
- non-compact groups, realistic models of quantum gravity
- ...

Thanks for your attention!