Non-Perturbative Defects in Tensor Models from Melonic Trees

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Kondo Effect

- Consider free fermions in 1D dimensions interacting to a spin impurity

$$H = \sum_{\vec{k}\alpha} \psi^{\dagger\alpha}_{\vec{k}} \psi_{\vec{k}\alpha} \epsilon(k) + \lambda \vec{S} \cdot \sum_{\vec{k}\vec{k}'} \psi^{\dagger}_{\vec{k}} \frac{\vec{\sigma}}{2} \psi_{\vec{k}'}$$

- If we compute resistivity, we will see that perturbation theory breaks down at small T.

$$\rho(T) \sim [\lambda + \nu\lambda^2 \ln \frac{D}{T} + ...]^2$$

- 1/D is a length of a sample. What happens at IR?
- As Nozieres put it “Theorists ‘diverged’ from experiment

RG and Kondo Effect

- Let us consider an interaction $$T\exp\left[-i\lambda\int \vec{S}(t)\cdot\psi^\dagger\frac{\vec{\sigma}}{2}\psi(\vec{0},t)\right]$$
- At the second order of perturbation theory we have

$$-\frac{1}{2}\lambda^2\int dt\ dt'\psi^\dagger\left[\frac{\sigma^a}{2},\frac{\sigma^b}{2}\right]\psi T\langle\psi(t)\psi^\dagger(t')\rangle(\theta(t-t')S^aS^b+\theta(t'-t)S^bS^a)$$

$$=\frac{\lambda^2}{2}\int dt\ dt'\psi^\dagger\frac{\vec{\sigma}}{2}\psi\cdot\vec{S}\ \mathrm{sn}(t-t')\langle\psi(t)\psi^\dagger(t')\rangle,$$

$$\frac{d\lambda}{d\ln l}=\nu\lambda^2.$$

- It is a renormalization of the coupling constant
- The electrons align with the impurity and make it stronger

Weak vs Strong coupling constant

- Just let us try to understand what's happening?
- At zero coupling we don't have any defect
- At any small coupling the effective interaction becomes strong
- It means that wave function must be set to zero at the origin
- Effective boundary condition

Large k appraoch

- Let us consider multi channel Kondo model

$$H = \sum_{\vec{k},\alpha,i=1,2,\ldots k} \epsilon_{\vec{k}} \psi^{\dagger\alpha i}_{\vec{k}} \psi_{\vec{k}\alpha i} + \lambda \vec{S} \cdot \sum_{\vec{k},\vec{k}'\alpha,\beta i} \psi^{\dagger\alpha i}_{\vec{k}} \vec{\sigma}^{\beta}_{\alpha} \psi_{\vec{k}'\beta i}.$$

Then doing the same type of computation we get

$$\frac{d\lambda}{d\, lnD} = -\nu\lambda^2 + \frac{k}{2}\nu^2\lambda^3 + O[ks(s+1)\lambda^4]$$

So we clearly see that at large k we have a fixed point

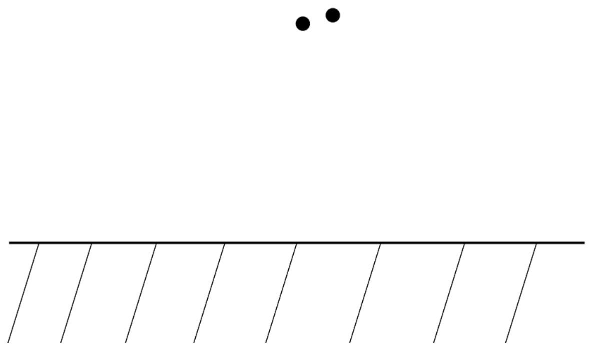
The existence of such strongly coupled point could be proved via bosonization

What does such a point describe?

- Very far from impurity we are getting a usual bulk CFT
- Very close to the impurity we are definitely not even a scale invariant behavior
- At the intermediate regime we have some influence, but it forgets about the boundary initial details. It could be just a boundary condition, or some additional degrees of freedom.
- We can talk about the boundary universality class

BCFT

- In 2d we can map a boundary to Im z = 0.
- Still there is a large number of CT.
- In higher dimensions the symmetry is



$$SO(p+1,1) \times SO(d-p) \subset SO(d+1,1)$$

- That is still powerful to help us solve or bootstrap a theory.
- So CFTs could host some additional and interesting dynamical effects connected to extended operators.
- Do Tensor Models have such interesting dynamical effects?

CTKT Tensor Models

- Let's try to make a similar analysis in tensor models

$$S_{\rm TM} = \int d^d x \left[\frac{1}{2} \left(\partial_\mu \phi_{abc} \right)^2 + \frac{\lambda_{\rm T}}{4N^{\frac{3}{2}}} \phi_{abc} \phi_{ab'c'} \phi_{a'bc'} \phi_{a'b'c} \right]$$

- Posses quite interesting large N limit, that allows to solve the theory in IR.
- Thus we believe that in the IR it is a conformal field theory and therefore we can ask a question whether BCFT or DCFT arise in this model?

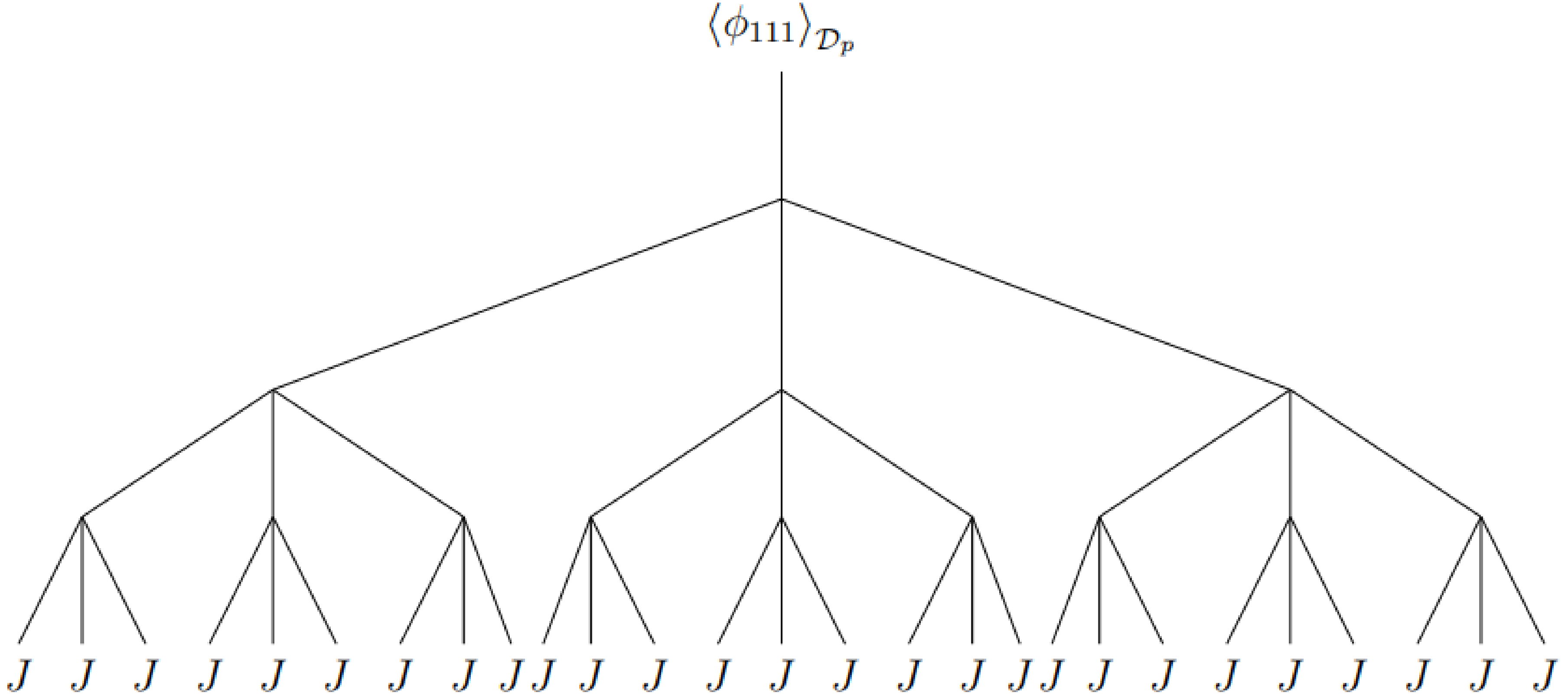
Localized Magnetic Field

- So let us consider the following action

$$S_{\mathrm{DTM}} = S_{\mathrm{TM}} - \bar{J}_{abc} \int d^p x_{\|} \, \phi_{abc}(x_{\|}, x_{\perp} = 0)$$

- We want to check that such deformation would flow to some interesting point.
- Due to the simple large N structure, we can hope that we can solve this problem using large N techniques.
- Thus if we consider the following limit

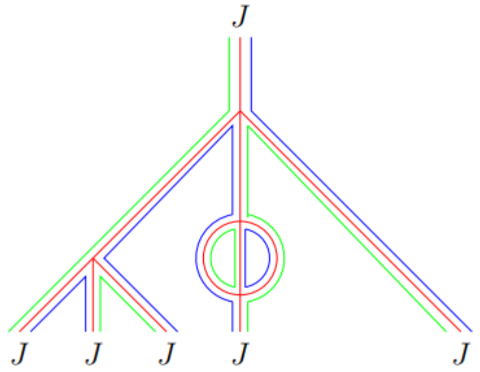
$$N \to \infty \quad \text{with} \quad J \equiv \frac{\bar{J}_{111}}{N^{3/4}} \ \text{fixed}\,,$$



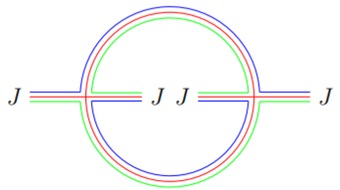
$$\langle\phi_{111}(x)\rangle_{\mathcal{D}_p} = N^{\frac{3}{4}} f(\lambda_{\mathrm{T}}, J) + \mathcal{O}(N^{-\frac{5}{4}})\,, \quad f(\lambda_{\mathrm{T}}, J) = (\text{melonic trees})\,.$$

Large N limit

- Such diagrams we will call melonic trees.
- We will prove that this diagrams dominate in the large N limit
- First we consider a general feynman diagram. We can use again a trick that we can forget about some of the colors and we would get a fat graph.
- We can get the following relation



$$(2 - n_{ij}) - 2g_{ij} = V + S - E + F_{ij} \quad \text{with } g_{ij} \geq 0\,,\ n_{ij} \geq 1$$



Large N limit

- Since any Edge terminates on a Source or Vertex we have

$$4V + S = 2E$$

- Combing together with Euler formula we get

$$3 - \frac{1}{2}n - g + \frac{3}{2}V - \frac{3}{4}S = F \;\Rightarrow\; F - \frac{3}{2}V + \frac{3}{4}S \leq \frac{3}{2}$$

- From that we are getting a usual coupling scaling and source scaling

Maximal graphs

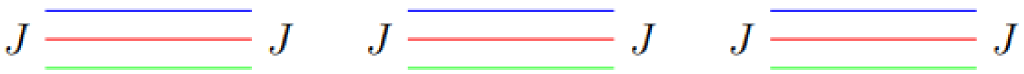
- Now let us consider a maximal graph. From the above relations we see that graph must be flat and connected.

$$F = \sum_{m\geq 1} P_m = \frac{3}{2} + \frac{3}{2}V - \frac{3}{4}S \qquad \sum_{m\geq 1} m(P_m + L_m) = 6V\,, \quad \sum_{m\geq 0} L_m = \frac{3}{2}S$$

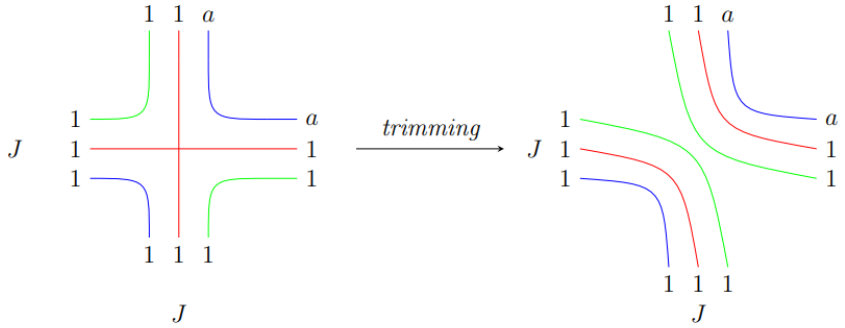
$$2P_2 + L_1 + 2L_0 = 6 + \sum_{m\geq 5}(m-4)P_m + \sum_{m\geq 3}(m-2)L_m \geq 6$$

Reaping and Trimming

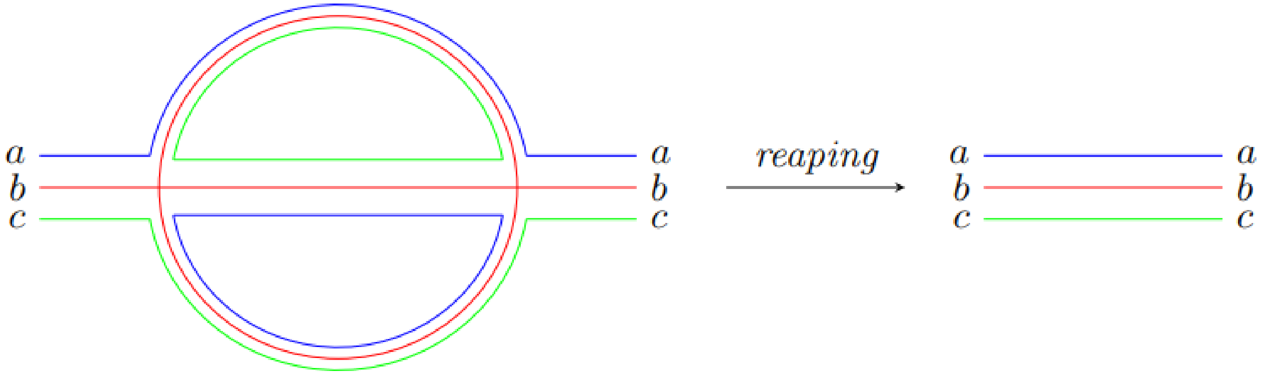
L0 > 0



L1 >0



P2>0



Dyson Schwinger equation

Analogous with sourceless theory the bulk propagator still satisfy the same equation

$$\langle \phi_{abc}(p)\phi_{a'b'c}(-p)\rangle = \delta_{aa'}\delta_{bb'}\delta_{cc'}G(p)\,,$$
$$G^{-1}(p) = p^2 - \lambda_{\mathrm{T}}^2 \int \frac{d^dk}{(2\pi)^d}\frac{d^dq}{(2\pi)^d} G(p-k-q)G(k)G(q)\,,$$

But now we have to take into account that the field could acquire some non-zero vev.

We are getting the following equation:

$$F(x) \equiv \frac{1}{N^{\frac{3}{4}}}\langle \phi_{111}(x)\rangle \qquad F(x) = \int d^dy\, G(x-y)\,\left[J(y) - \lambda_{\mathrm{T}} F^3(y)\right]$$

If we consider small J we can replace the exact propagator with the effective IR propagator that would give

$$\sqrt{\lambda_{\mathrm{T}}} B_d(-\Delta_\perp)^{\frac{d}{4}} F + \lambda_{\mathrm{T}} F^3 = J\delta^{d-p}(x_\perp)\,, \qquad (-\Delta)^\alpha f(x) \equiv \int \frac{d^dk}{(2\pi)^d}\int d^dy\, e^{ik(x-y)}|k|^{2\alpha} f(y)$$

Irrelevant deformations

- Let us first consider a situation when we couple to an irrelevant deforamation

$$-J_{\text{irrel}} N^{\frac{3}{4}} \int d^p x_{\parallel} \, (\Delta_\perp)^n \phi_{111}(x_\parallel, x_\perp = 0) \, ,$$

- Equations are the same

$$F(x) = J_{\text{irrel}} \Delta_\perp^n G(x_\perp) - \lambda_{\text{T}} \int d^{d-p} y_\perp G_\perp(x_\perp - y_\perp) F^3(y) \, ,$$

$$G_\perp(x) \equiv \int d^p x_\parallel G(x_\perp, x_\parallel) = \pi^{p/2} A_d \frac{\Gamma\left(\frac{d-2p}{4}\right)}{\Gamma\left(\frac{d}{4}\right)} \frac{1}{x_\perp^{\frac{d}{2}-p}}$$

- But it is easy to see that the solution will be just $F(x) \to \frac{J_{\text{irrel}}}{x_\perp^{\frac{d}{2}+2n-p}}$

- It corresponds to a trivial defect. Notice an explicit dependence on the parameters of the defect. It is not conformal!

Relevant deformation

- Let's come back to the previous problem for d<4 the defect is relevant
- We can solve with the following ansatz

$$F(x) = \int d^d y\, G(x-y)\, \left[J(y) - \lambda_{\mathrm{T}} F^3(y)\right] \qquad F(x) = -\lambda_{\mathrm{T}} \int d^d y G(x-y) F^3(y) \qquad \langle \phi_{111}(x) \rangle = \frac{N^{\frac{3}{4}}}{\lambda_{\mathrm{T}}^{\frac{1}{4}}} \frac{C_{d,p}}{|x_\perp|^{\frac{d}{4}}}$$

- It is easy to check that it satisfy the equation

$$C_{d,p}^2 = \frac{\Gamma\left(\frac{3d}{8}\right)\Gamma\left(\frac{3d-4p}{8}\right)\left(-\Gamma\left(-\frac{d}{4}\right)\right)^{\frac{1}{4}}}{\pi^{\frac{d}{4}}\Gamma\left(\frac{d}{8}\right)\left(-\Gamma\left(\frac{d-4p}{8}\right)\right)\Gamma\left(\frac{3d}{4}\right)^{\frac{1}{4}}}$$

- Notice that the solution forgot about the details of the defect
- The first sentence of Leo Tolstoy's novel Anna Karenina is: "Happy conformal defects are all alike; every unhappy non-conformal defect is unhappy in its own way."

Exact solution in the epsilon expansion

- In 4-d at first order of epsilon we can solve this equation

$$\tilde{F} = \lambda_{\mathrm{T}}^{\frac{1}{4}} F\,, \quad \tilde{J} = \lambda_{\mathrm{T}}^{-\frac{1}{4}} J \qquad\qquad B_d(-\Delta_\perp)^{\frac{d}{4}}\tilde{F} + \tilde{F}^3 = \tilde{J}\delta^{d-1}(x_\perp)\,.$$

- We will use the following ansatz to solve the equation

$$\tilde{F}(x_\perp) = \frac{1}{x_\perp^{\frac{d}{2}-1}} \sum_{n=0}^{\infty} \tilde{a}_n \tilde{J}^{1+2n} (B_d)^{3n+2} x_\perp^{\left(2-\frac{d}{2}\right)n}\,, \quad F(k_\perp) = \frac{1}{k_\perp^{\frac{d}{2}}} \sum_{n=0}^{\infty} \frac{a_n}{k_\perp^{\left(2-\frac{d}{2}\right)n}} \tilde{J}^{1+2n} (B_d)^{3n+2}\,, \quad P_{d,n}\tilde{a}_n = -\sum_{n_1+n_2+n_3+1=n} \tilde{a}_{n_1}\tilde{a}_{n_2}\tilde{a}_{n_3}\,, \quad P_{d,n} = \frac{2^{\frac{d}{2}}\Gamma\left(\frac{n(4-d)+d}{4}\right)\Gamma\left(\frac{n(d-4)+2(d-1)}{4}\right)}{\Gamma\left(\frac{n(4-d)}{4}\right)\Gamma\left(\frac{n(d-4)+d-2}{4}\right)}\,.$$

- This equation could be solved and at the end we are getting the following solution

$$\tilde{F}(x_\perp) = \frac{2^{-\frac{1}{4}}\epsilon^{\frac{1}{4}}\tilde{J}}{x_\perp^{\frac{2-\epsilon}{2}}} \frac{1}{\sqrt{8\cdot 2^{\frac{1}{4}}\pi \tilde{J}^2 \epsilon^{-\frac{1}{4}} x_\perp^{\frac{\epsilon}{2}} + 1}} \qquad\qquad x_\perp \ll \tilde{J}^{-4/\epsilon}:\ \tilde{F}(x_\perp) \to \frac{2^{-\frac{1}{4}}\epsilon^{\frac{1}{4}}\tilde{J}}{x_\perp^{\frac{d}{2}-1}}\,, \quad x_\perp \gg \tilde{J}^{-4/\epsilon}: \tilde{F}(x_\perp) \to \mathrm{sgn}(\tilde{J})\frac{\epsilon^{\frac{3}{8}}}{2^{\frac{15}{8}}\sqrt{\pi}\, x_\perp^{\frac{d}{4}}}\,.$$

Beta-Function

- We can cross check the previous results using conventional epsilon-expansion

$$S = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi_i)^2 - \frac{1}{4!} Y_{ijkl} \phi_i \phi_j \phi_k \phi_l \right] - h \int d\tau \phi_1(\tau, \vec{x} = 0)$$

$$\beta_{ijkl} = -\epsilon Y_{ijkl} + \frac{3}{(4\pi)^2} Y_{mn(ij} Y_{kl)mn} + \frac{1}{3(4\pi)^4} Y_{mncd} Y_{d(ijk} Y_{i)mnp} - \frac{6}{(4\pi)^4} Y_{(i|mpq} Y_{|j|npq} Y_{kl)mn} ,$$

$$\beta_h = -\frac{\epsilon}{2} h + \frac{1}{(4\pi)^2} h^3 Y_{1111} + \frac{h}{12(4\pi)^4} Y_{1mnc}^2 + \frac{h^3}{4(4\pi)^4} Y_{11mn}^2 - \frac{h^5}{12(4\pi)^4} Y_{111m}^2 ,$$

$$S = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi_{abc})^2 + \frac{1}{4} g_{\rm T} O_{\rm T} + \frac{1}{4} g_{\rm P} O_{\rm P} + \frac{1}{4} g_{\rm dt} O_{\rm dt} \right] - h \int d\tau \phi_1(\tau, \vec{x} = 0)$$

$$O_{\rm T} = \phi_{abc} \phi_{ab'c'} \phi_{a'bc'} \phi_{a'b'c} ,$$

$$O_{\rm P} = \frac{1}{3} \left[\phi_{abc} \phi_{abc'} \phi_{a'b'c} \phi_{a'b'c'} + \phi_{abc} \phi_{ab'c} \phi_{a'bc'} \phi_{a'b'c'} + \phi_{abc} \phi_{a'bc} \phi_{a'b'c'} \phi_{ab'c'} \right]$$

$$O_{\rm dt} = \phi_{abc} \phi_{abc} \phi_{a'b'c'} \phi_{a'b'c'} ,$$

Beta Functions at Large N

- Using usual large N assumptions we arrive at

$$h \to J N^{\frac{3}{4}}, \quad g_{\mathrm{T}} \to \frac{\lambda_{\mathrm{T}}}{N^{\frac{3}{2}}}, \quad g_{\mathrm{P}} \to \frac{\lambda_{\mathrm{P}}}{N^2}, \quad g_{\mathrm{dt}} \to \frac{\lambda_{\mathrm{dt}}}{N^3},$$

$$\begin{aligned}
\beta_J &= -\frac{\epsilon}{2} J + \frac{J^3 \lambda_{\mathrm{T}}}{16\pi^2} + \frac{J \lambda_{\mathrm{T}}^2}{512\pi^4} - \frac{3 J^5 \lambda_{\mathrm{T}}^2}{256\pi^4}, \\
\beta_{\mathrm{T}} &= -\epsilon \lambda_{\mathrm{T}} + \frac{\lambda_{\mathrm{T}}^3}{128\pi^4}, \\
\beta_{\mathrm{P}} &= -\epsilon \lambda_{\mathrm{P}} + \frac{\lambda_{\mathrm{P}}^2}{24\pi^2} - \frac{\lambda_{\mathrm{P}} \lambda_{\mathrm{T}}^2}{128\pi^4} + \frac{3\lambda_{\mathrm{T}}^2}{8\pi^2}, \\
\beta_{\mathrm{dt}} &= -\epsilon \lambda_{\mathrm{dt}} + \frac{\lambda_{\mathrm{dt}}^2}{8\pi^2} + \frac{\lambda_{\mathrm{dt}} \lambda_{\mathrm{P}}}{4\pi^2} - \frac{5 \lambda_{\mathrm{dt}} \lambda_{\mathrm{T}}^2}{128\pi^4} + \frac{\lambda_{\mathrm{P}}^2}{12\pi^2} - \frac{\lambda_{\mathrm{P}} \lambda_{\mathrm{T}}^2}{32\pi^4}
\end{aligned}$$

$$\lambda_{\mathrm{T}} = 8\pi^2 \sqrt{2\epsilon}, \quad \lambda_{\mathrm{P}} = 24\pi^2 \sqrt{-2\epsilon}, \quad \lambda_{\mathrm{dt}} = -8\pi^2 \sqrt{2}(3+\sqrt{3})\sqrt{-\epsilon}, \quad J = \pm \frac{\epsilon^{\frac{1}{4}}}{2^{\frac{3}{4}}}$$

- So it is indeed a conformal point and we can apply a lot of interesting approaches to solve the theory but it is outside of my talk

G-theorem

- We can prove an analog of the c-theorem for defects
- It is g-theorem and relates that the entropy of the defects decreases

$$S_{\text{imp}}(T) \equiv lim_{l\to\infty}[S(l,T) - S_0(l,T)],$$

- There are two pieces: regularization dependent and proportional to the defect length and independent.
- The independent one decreases
- More general proof by Komargodski et al.

Defect Entropy of Tensor Line Defects

- We make a conformal transformation to a circle

$$\langle\phi_{111}(x)\rangle = \frac{N^{\frac{3}{4}}C_{d,1}}{\lambda_{\mathrm{T}}^{\frac{1}{4}}}\left(\frac{4r^2}{\left(-R^2+x_1^2+x_2^2\right)^2+2\left(R^2+x_1^2+x_2^2\right)x_\perp^2+x_\perp^4}\right)^{\frac{d}{8}}, \quad x_\perp^2 = \sum_{i=3}^{d} x_i^2 .$$

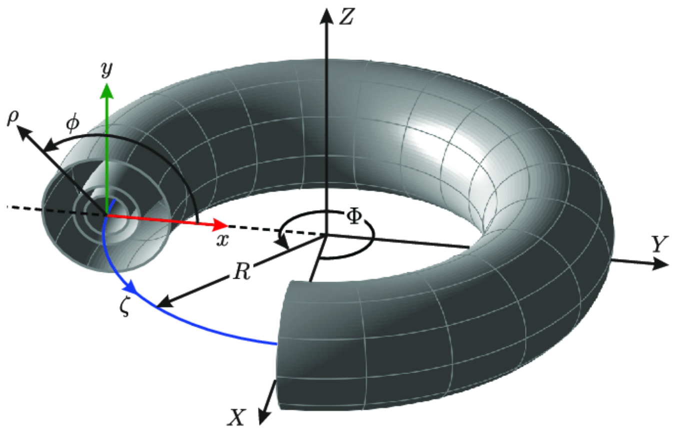
- We can notice that the defect entropy is

$$\frac{1}{N^{\frac{3}{2}}}\log\langle\mathcal{D}_1\rangle = \frac{1}{4}\lambda_{\mathrm{T}}\int d^dx\,\langle\phi_{111}(x)\rangle_{\mathcal{D}_1}^4$$

- Easy to compute in the toroidal coordinates

Defect Entropy of Tensor Line Defects

$$x_1 = R\frac{\sinh\tau}{\cosh\tau-\cos\sigma}\cos\varphi\,,$$
$$x_2 = R\frac{\sinh\tau}{\cosh\tau-\cos\sigma}\sin\varphi\,,$$
$$x_{i+2} = \hat{n}_i\frac{\sin\sigma}{\cosh\tau-\cos\sigma}\,,\quad i=1,2,\dots,d-2\,,$$



$$\langle\phi_{111}(\sigma,\tau,\varphi,\vec{n}_i)\rangle_{\mathcal{D}_1} = \frac{N^{\frac{3}{4}}C_{d,1}}{\lambda_{\mathrm{T}}^{\frac{1}{4}}}\frac{|\cos\sigma-\cosh\tau|^{\frac{d}{4}}}{R^{\frac{d}{4}}}\,.$$

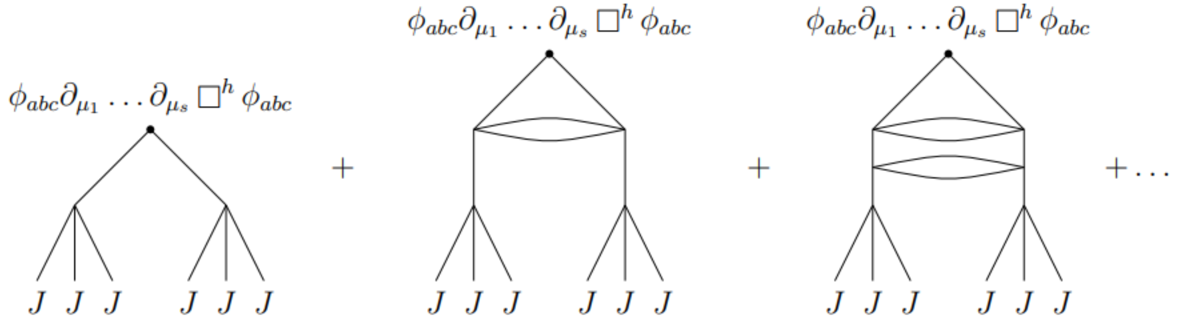
$$\log\langle\mathcal{D}_1\rangle = \frac{1}{4}\times 2\pi N^{\frac{3}{2}}\mathrm{vol}_{S^{d-3}}C_{d,1}^4\int d\sigma d\tau\sinh\tau|\sin\sigma|^{d-3} = C_{d,1}^4 N^{\frac{3}{2}}\frac{\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^\infty d\tau\sinh\tau\,,$$

$$e^{\tau_*} = R\Lambda$$

$$\log\langle\mathcal{D}_1\rangle_{\mathrm{reg}} = N^{\frac{3}{2}}C_{d,1}^4\frac{\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\left(\frac{1}{2}R\Lambda-1\right)+\mathcal{O}\left(\frac{1}{R\Lambda}\right)$$

$$s(\mathcal{D}_1) = -N^{\frac{3}{2}}\frac{2^{1-d}\sqrt{\pi}(d-4)^2\Gamma\left(\frac{3d}{4}\right)^{\frac{3}{2}}\Gamma\left(1-\frac{d}{4}\right)^{\frac{1}{2}}}{(4-3d)^2\sqrt{d}\,\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d}{4}\right)^2}\,,$$

One point functions of bilinear operators



$$\frac{1}{N^{\frac{3}{2}}}\langle\mathcal{O}(x)\rangle_{\mathcal{D}_p}=\frac{1}{2}\lambda_{\mathrm{T}}^2\int d^dx_1d^dx_2\,\langle\mathcal{O}(x)\phi_{111}(x_1)\phi_{111}(x_2)\rangle_{\text{bulk}}F^3(x_1)F^3(x_2)\,,$$

$$\langle T_{\alpha\beta}(y)\rangle_{\mathcal{D}_p}=-\frac{h_p\delta_{\alpha\beta}}{|y_\perp|^d}\,,\ \langle T_{ij}(y)\rangle_{\mathcal{D}_p}=\frac{(p+1)h_p}{(d-p-1)|y_\perp|^d}\left(\delta_{ij}-\frac{d}{p+1}\frac{(y_\perp)_i(y_\perp)_j}{|y_\perp|^2}\right)\,,\ \langle T_{\alpha i}\rangle_{\mathcal{D}_p}=0\,,$$

$$h_p=N^{\frac{3}{2}}\frac{A_dC_{d,p}^6d^2|y_\perp|^d}{8(d-1)}\int d^dx_1d^dx_2\frac{\hat{X}_1^{12}\hat{X}_2^{12}|x_1-x_2|^{\frac{d}{2}}}{|y-x_2|^d|y-x_1|^d|x_{1\perp}|^{\frac{3d}{4}}|x_{2\perp}|^{\frac{3d}{4}}}\,.$$

Conclusion

- Line Defects are interesting dynamical phenomenon that provide some additional tools for understanding the properties of Conformal Field Theories.
- Tensor Models also allow to have such defects
- They satisfy usual unitary properties of line defects
- Check ANEC?
- More sophisticated defects?
- GW, SYK or higher tensor models?

Thank you for your attention!