# Phase transitions in Group field theory:

# Towards the phase structure of the complete Lorentzian Barrett-Crane model

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in collaboration with
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mostly **based on** arXiv:
**1904.00598** (PRD 98, 126006 (2018)),
**2110.15336** (JHEP 2021, 201 (2021)),
**2112.00091** (*JCAP* 01 (2022) 01, 050),
**2206.15442** (PRD 106, 066019, 2022),
**2209.04297** &
**wip**

**October 19, 2022**

**VTJC - Virtual Tensor Journal Club 2022**

Funded by Deutsche Forschungsgemeinschaft
German Research Foundation

# Outline

- General motivation: Landau-Ginzburg mean-field method
- Group Field Theory
- (complete) Barrett-Crane model
- LG theory applied to the Barrett-Crane Group Field Theory model
- Conclusions

# What is LG theory? How does it relate to the RG?

[Kopietz, Bartosch, Schütz; Sachs, Sen, Sexton; Zinn-Justin; Goldenfeld]

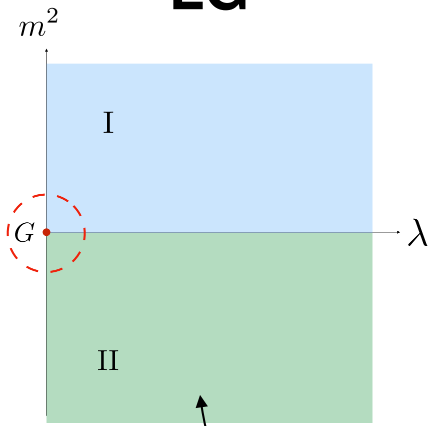
- statistical field theory method to describe **1st and 2nd order phase transitions at mean-field level**

- LG mean-field analysis **clarifies phase structure** of local field theories (coarse account)

- transition to condensate phase with **non-trivial VEV** (non-perturbative vacuum) $\langle\varphi\rangle \neq 0$

e.g. $$S[\varphi] = \frac{1}{2}\int \mathrm{d}^3x\varphi(x)(-\Delta + m^2)\varphi(x) + \frac{\lambda}{4!}\int \mathrm{d}^3x\varphi(x)^4$$

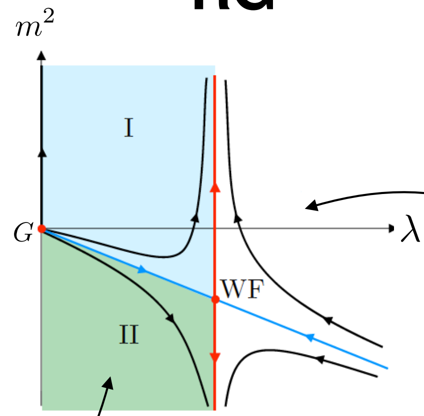
## LG




LG theory studies
fluctuations around
the Gaussian fixed point
(free theory)

quasi-Gaussian distribution

## RG




involved RG studies go
beyond small fluctuations

generally non-Gaussian distribution

$\langle\varphi\rangle \neq 0$

→ coarse account of the phase diagram

(good enough to start with…)

→ detailed account of the phase diagram

(more difficult…)

# What is LG theory? Setup (I)

[Kopietz, Bartosch, Schütz; Sachs, Sen, Sexton; Zinn-Justin; Goldenfeld]

- start with **free energy functional** as an expansion in terms of even + odd powers of the local field (order parameter) and its gradient

- consider **truncation** of this functional assumed to be valid from mesoscale to macroscale

- details on microphysics encoded in couplings and order parameter

- **order parameter** features only **universal properties** of the system (dimension of space, symmetries of the order parameter)

- **model/theory** global $\mathbb{Z}_2$-symmetry

$$S[\varphi] = \frac{1}{2}\int \mathrm{d}^d x \varphi(x)(-\Delta + m^2)\varphi(x) + \frac{\lambda}{4!}\int \mathrm{d}^d x \varphi(x)^4$$

free energy functional ($S[\varphi]$); order parameter ($\varphi(x)$); dimension of underlying space ($d$)

➡ **goal: approximately evaluate** (solve theory)

$$Z = \int \mathcal{D}\varphi \mathrm{e}^{-S[\varphi]}$$

partition function (sums all configs)

- allows to **control thermodynamic phases** of the system by studying long-range correlations of order parameter fluctuations over the distance $\xi$ (**correlation length**)

- beyond $\xi$ correlations decay exponentially; it diverges at criticality

# What is LG theory? Setup (II)

[Kopietz, Bartosch, Schütz; Sachs, Sen, Sexton; Zinn-Justin; Goldenfeld]

## The how to:

1) determine **uniform field configurations** which are minimizers of the free energy functional

$$\varphi_0 = 0 \text{ if } m^2 > 0 \text{ and } \varphi_0 = \pm\sqrt{-\frac{m^2}{\lambda/3!}} \text{ if } m^2 < 0$$

2) study **correlations of fluctuations** around this uniform background (aka Gaussian approximation)

2. a) **linearize classical equations of motion** using fluctuations over the background $\varphi(\vec{x}) \to \varphi_0 + \delta\varphi(\vec{x})$

$$\longrightarrow \quad \left(-\Delta + m^2\right)\delta\varphi(\vec{x}) + \frac{\lambda\varphi_0^2}{2}\delta\varphi(\vec{x}) = 0$$

2. b) solve for **correlation function** $\left(-\Delta + m^2 + \frac{\lambda\varphi_0^2}{2}\right) C(\vec{x}) = \delta(\vec{x})$ (go to Fourier representation)

2. c) correlator is exponentially decaying function $\longrightarrow$ determine **correlation length** $\xi^2 = \frac{1}{-2m^2}, \quad m^2 < 0$

3) determine **domain of validity**

$\longrightarrow$ fluctuations and coupling should remain small then mean-field theory self-consistent

**Ginzburg parameter** $Q = \dfrac{\int_\xi \mathrm{d}^d x C(\vec{x})}{\int_\xi \mathrm{d}^d x \varphi_0^2}$

(measures strength of fluctuations)

$$\boxed{Q \sim \lambda\xi^{4-d} \longrightarrow d_c = 4}$$

**critical dimension (on *flat space*) below which MFT seizes to be accurate; accounts for coarse picture of phase diagram (good enough)**

$$\boxed{Q \sim \lambda\xi^2 \mathrm{e}^{-(d-1)\xi/a} \longrightarrow d_c \to \infty}$$

**on the d-hyperboloid MFT is sufficient** [Benedetti 1403.6712]

# Why bother in Group Field Theory? Applicable?

**YES!**

**What is GFT?**

# Motivation via Matrix models for 2d gravity

[Di Francesco, Ginsparg, Zinn-Justin 9306153]

- Matrix models generate 2d random lattices
- at criticality they give rise to continuum geometries of dimension $d \leq 2$
- phase diagram of simple matrix models obtainable via diagonalization of matrices, computation of the partition function for large matrixes and then checking the (non-) analyticity of the free energy
- alternative route: phase structure via functional renormalization group
- continuum limit of simple matrix models agrees with that of 2d Liouville gravity

for example

$$S(M) = \frac{1}{2}\mathrm{tr}(M^2) + \frac{\lambda}{3!}\mathrm{tr}(M^3)$$

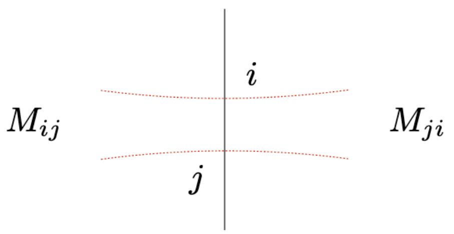
rank 2

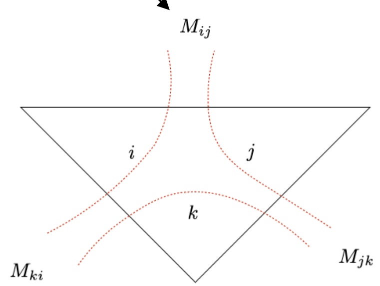
combinatorics of a 2-simplex

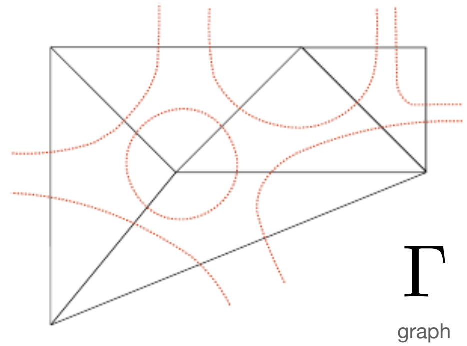
# of vertices

amplitude

$$Z_{\mathrm{MM}} = \int \mathrm{d}M \mathrm{e}^{-S(M)} = \sum_{\Gamma} \frac{\lambda^{n(\Gamma)}}{\mathrm{sym}(\Gamma)} \mathcal{A}_{\Gamma}$$



$\Gamma$

graph

snapshot of a triangulation

➡ **a way to generalize to higher dimensions: Tensor Models and GFTs**

# Group Field Theory

[Boulatov, Ooguri, Oriti, Freidel, Rovelli, Livine, Gurau, Baratin,...]

classical theory (kinematics):

rank $r$

group field $\quad \varphi(g_1, ..., g_r) : G^r \to \mathbb{R}, \mathbb{C}, \quad \varphi \in L^2(G^r)$

Lie group $G$

parallel transport $\quad g_I = \mathcal{P} e^{\int_{e_I} A} \quad$ for $\quad I = 1, ..., r, \quad$ link $e_I$, connection $A$

typically work with $r = 4$ and $G = \mathrm{SL}(2, \mathbb{C})$

$\longrightarrow$ to model 4-dimensional Lorentzian quantum geometries

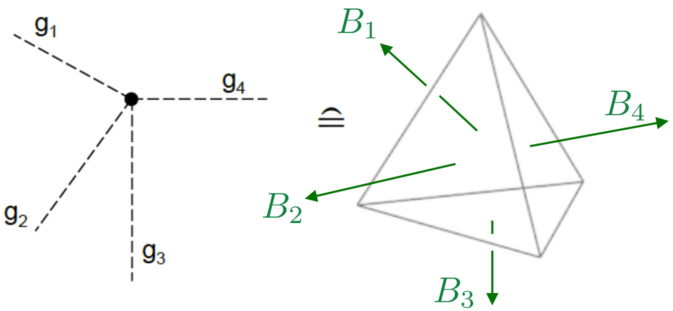
supplement field with invariance property: $\quad \varphi(g_1, ..., g_r) = \varphi(g_1 h^{-1}, ..., g_r h^{-1}), \quad \forall h \in G$

**(known as closure constraint)**

$\longrightarrow$ phase space: $\quad T^* G^d \cong G^d \times \mathfrak{g}^d$

$\longrightarrow$ dual formulation: $\quad \tilde{\varphi}(B_1, ..., B_4) = \int (\mathrm{d}g)^4 \varphi(g_1, g_2, g_3, g_4) \prod_{I=1}^{4} e_{g_I}(B_I)$

e.g. for **r=4** invariant field corresponds to a 3-simplex/tetrahedron




$$\sum_{t=1}^{4} B_t = 0$$

(closure)

# Group Field Theory

classical theory (dynamics):

$$S_{\rm GFT} = \int (\mathrm{d}g)^r \bar\varphi(g_I)\mathcal{K}\varphi(g_I) + \mathcal{V}[\bar\varphi(g_I), \varphi(g_I)]$$

$\mathcal{K}:$ kinetic operator, $\mathcal{V}:$ non-linear and non-local interaction term

model specified by: $G$, dimension $d$, $\mathcal{K}$, $\mathcal{V}$ and symmetries of $\varphi$

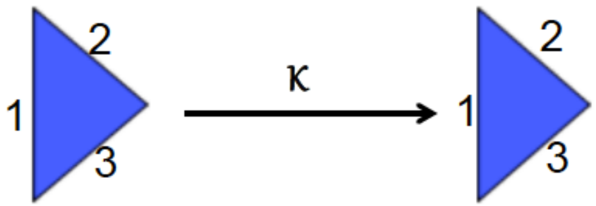
crucial feature of GFT models: combinatorially non-local interaction

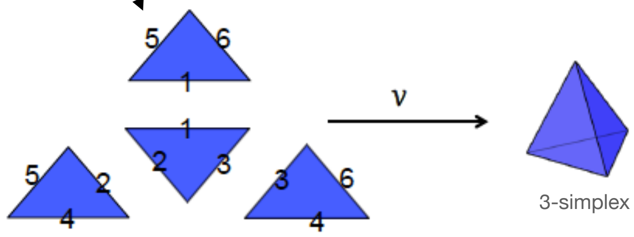
example in 3d: (Boulatov)

[Boulatov 9202074]

$$S = \int (\mathrm{d}g)^3 |\varphi_{123}|^2 + \frac{\lambda}{4!}\int (\mathrm{d}g)^6 \varphi_{123}\varphi_{145}\varphi_{256}\varphi_{364} + \text{c.c.}, \qquad \varphi_{123} \equiv \varphi(g_1, g_2, g_3)$$

combinatorics of a 3-simplex



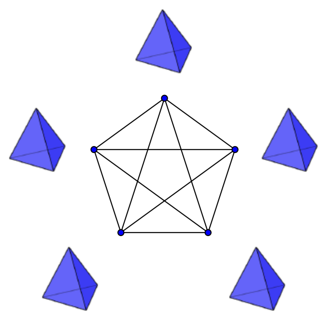




example in 4d: (Ooguri)

[Ooguri 9205090]

$$S = \int (\mathrm{d}g)^4 |\varphi_{1234}|^2 + \frac{\lambda}{5!}\int (\mathrm{d}g)^{10} \varphi_{1234}\varphi_{4567}\varphi_{7389}\varphi_{962(10)}\varphi_{(10)851} + \text{c.c.}$$

combinatorics of a 4-simplex



“quantum theory” (dynamics):

$$Z_{\mathrm{GFT}} = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} e^{-S_{\mathrm{GFT}}[\varphi,\bar{\varphi}]} = \sum_{\Gamma} \frac{\lambda^{V_\Gamma}}{\mathrm{Sym}(\Gamma)} \mathcal{A}_\Gamma$$

GFT Feynman amplitude $\mathcal{A}_\Gamma$

graph corresponds to discrete geometries

**Curiously:** Boulatov and Ooguri model provide GFT quantizations of BF-theory in 3d & 4d

$\longrightarrow$ **build models starting with BF-theory (TFT)**

## BF-theory:

$$S[\omega, B] = \int B_{IJ} \wedge F^{IJ}(\omega)$$

BF-action

2-form

curvature/field strength

connection 1-form

$$Z = \int \mathcal{D}\omega \mathcal{D}B e^{iS[\omega,B]} = \int \mathcal{D}\omega \delta(F(\omega))$$

**(integral over flat connections, i.e. no local dof)**

(=volume of space of flat connection, infinitely large?!)

**ill-defined in the continuum** $\longrightarrow$ **quantization on a regulating lattice structure**

- **quantisation of BF-theory on a lattice: “GFT does the job”** [Ooguri 9205090]
- **Rather: GFT prescription has become better in getting the job done (e.g. color dof)** [Gurau et al.]

# But: need to link with theory for gravity

- **How to yield a model for Lorentzian gravitational degrees of freedom?**

➡**Impose so-called simplicity constraints (s. next slides)**

➡**How to? Classically and at GFT (quantum) level?**

➡in Lorentzian setting: what info for the lattice is needed be in accordance with microcausality?

➡ want a lattice model for Lorentzian quantum gravity: difficulty to map Lorentzian structure faithfully

➡ does that impact continuum limit? (see e.g. CDT)

**Summary: How to properly generate/decorate the lattice such that causality, topology etc nice?**

⟶ **Attempt: Lorentzian Barrett-Crane model**

**To this aim first more on:**

**1) Link with (classical) gravity**

**2) disambiguation what we mean here by causal structure**

# Link with Einstein gravity

➡ **via constrained BF-theory:**

$$S[\omega, B, \mu] = \int \left[ B_{IJ} \wedge F^{IJ}(\omega) + \frac{1}{2}\mu_{IJKL} B^{IJ} \wedge B^{KL} \right]$$

Lagrange multiplier

$\mathfrak{sl}(2,\mathbb{C})$ – valued 1-form

$\mathfrak{sl}(2,\mathbb{C})$ – valued 2-form

field strength: $F^{IJ}(\omega) = \mathrm{d}\omega^{IJ} + \omega^{I}_{J} \wedge \omega^{KJ}$

variation wrt $\mu$ $\longrightarrow$ "simplicity constraint" on B:

$$B^{IJ} \wedge B^{KL} = e\epsilon^{IJKL}, \quad e = \frac{1}{4!}\epsilon_{IJKL} B^{IJ} \wedge B^{KL}$$

solve for B $\longrightarrow$ solutions in two sectors: (1) topological sector vs. (2) gravitational sector (Palatini)

first-order formulation

$$\longrightarrow S_{\text{Palatini}}[e, \omega] = \frac{1}{2} \int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}$$

tetrad field

$\delta_e S = 0 \rightarrow$ Einstein field eqns. $\quad \delta_\omega S = 0 \rightarrow$ 1st Cartan: $\mathrm{d}_\omega e^I + \omega^I_J \wedge e^J = 0$

spin connection

# Lorentzian structure

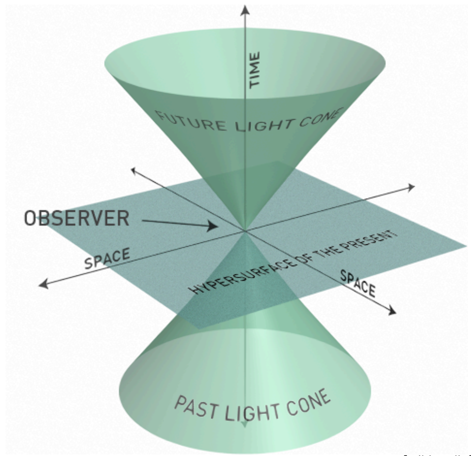
**classical level:** Lorentzian structure/causality plays an important role in continuum spacetime physics

What do we mean **here** by causal structure?

➡ bare causality + time orientation

e.g. [Bianchi, Martin-Dussaud 2109.00986; Jercher, Oriti, Pithis 2206.15442]

| **Causal structure** | **bare causality** | **time orientation** |
|---|---|---|
| **local level** | tangent vectors: timelike, lightlike spacelike | timelike tangent vectors either past or future pointing |
| **global level** | two points (events) either have spacelike, timelike or lightlike separation | causal ordering of timelike separated points |




[wikipedia]

- Encoded by the Lorentz group $G = \mathrm{SL}(2, \mathbb{C})$

# Lorentzian structure

**quantum level:** either encode causal structure directly or demonstrate how it emerges in the continuum

here: intend to impose causal structure at quantum and discrete geometric level

- more refined imposition may be required (see for instance Lorentzian Regge Calculus and locally CDT)
- **here** focus on bare causal aspects: $G = \mathrm{SL}(2, \mathbb{C})$
- time orientation aspect would require: $G = \mathrm{Pin}(1, 3)$

[Livine, Oriti 0210064; Bianchi, Martin-Dussaud 2109.00986]

**Rough status of models:**

- standard formulation of Lorentzian spin foam and GFT models (BC and EPRL models) focusses mostly on the glueing of spacelike building blocks [Barrett, Crane 9904025; Engle, Pereira, Rovelli, Livine 0711.0146]

**problems:** 1) how does continuum bare causal structure emerge?

2) how to deal with lightlike and timelike boundaries?

- notable exceptions (EPRL-CH extension with spacelike *and* timelike tetrahedra; BC-PR with spacelike *or* timelike tetrahedra)

[Conrady, Hnybida 1002.1959 & 1003.5652; Perez, Rovelli 0009021 & 0011037]

**goal:** formulate spin foam and GFT model which treats spacelike, timelike and lightlike tetrahedra with all possible interactions (simplicial) [Jercher, Oriti, Pithis 2206.15442]

# complete Barrett-Crane GFT model

[Barrett, Crane 9904025; Perez, Rovelli 0009021 & 0011037; Oriti, Baratin 1108.1178; Jercher, Oriti, Pithis 2112.00091 & 2206.15442]

[Jercher ILQGS talk 09/22]

**a model for Lorentzian quantum gravity in 4d**

- start with Ooguri model: GFT model for BF-theory in 4d (topological field theory)
- impose so-called simplicity constraints to turn it into a theory of gravity (first-order Palatini)

➡ **for this add non-dynamical timelike, spacelike and lightlike normal vector X to domain**

⟶ **allows to impose closure and simplicity covariantly and commutatively**

$$\varphi(g_1, ..., g_4; X_\alpha) : \mathrm{SL}(2,\mathbb{C})^4 \times \mathrm{SL}(2,\mathbb{C})/U^{(\alpha)} \to \mathbb{C} \qquad \text{with} \qquad \alpha \in \{+, 0, -\}$$

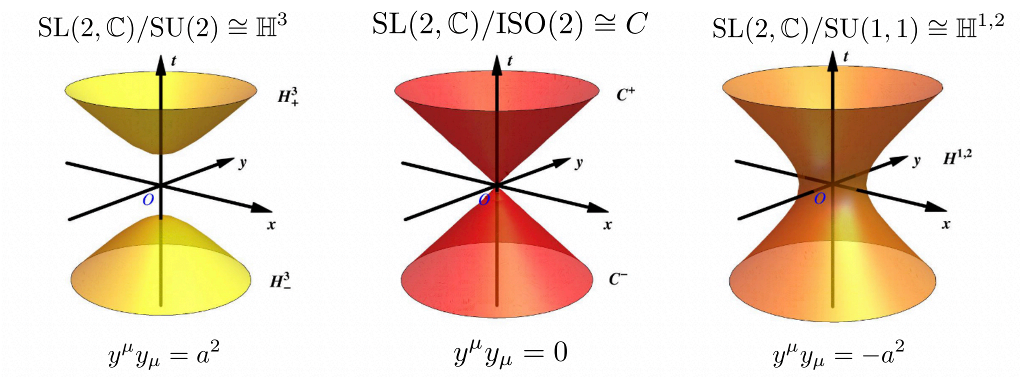
$U^{(+)} = \mathrm{SU}(2),\ U^{(0)} = \mathrm{ISO}(2),\ U^{(-)} = \mathrm{SU}(1,1)$ **stabilizers of** $X_+ = (1,0,0,0)$ (timelike), $X_0 = \frac{1}{\sqrt{2}}(1,0,0,1)$ (lightlike), $X_- = (0,0,0,1)$ (spacelike)

**distinguished hypersurfaces in Minkowski space**

$y \in \mathbb{R}^{1,3}$

$a:$ skirt radius

z-axis suppressed




$X_+ = (1,0,0,0) \in \mathbb{H}^3$ $\quad X_0 = \frac{1}{\sqrt{2}}(1,0,0,1) \in C$ $\quad X_- = (0,0,0,1) \in \mathbb{H}^{1,2}$

**symmetries:**

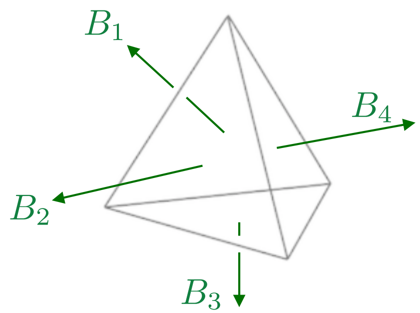
$$1)\ \varphi(g_1, ..., g_4; X_\alpha) = \varphi(g_1 h^{-1}, ..., g_4 h^{-1}; h \cdot X_\alpha), \quad \forall h \in \mathrm{SL}(2, \mathbb{C})$$ **(closure)**

$$2)\ \varphi(g_1, ..., g_4; X_\alpha) = \varphi(g_1 u_1, ..., g_4 u_4; X_\alpha), \quad \forall u_1, ..., u_4 \in \mathrm{U}_{X_\alpha}$$ **(simplicity)**

**Geometric interpretation:**

➡**Go to bi-vector representation**

$$1) \Leftrightarrow \sum_{t=1}^{4} B_t = 0$$




**bi-vectors close to form a tetrahedron**

$$2) \Leftrightarrow X_A (*B)^{AB} = 0$$

Lorentz index $A \in \{0,1,2,3\}$

**geometric information in bi-vectors is "orthogonal" to respective normal**

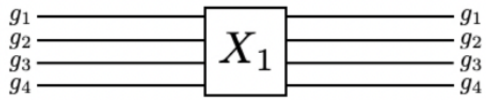
→ **fields correspond to spacelike, timelike and lightlike tetrahedra**

# imposition in the dynamics:

⟶ **unique model** (no ambiguous amplitudes as in previous formulations, X's drop from dynamical amplitudes)

$$S[\varphi,\bar\varphi] = K[\varphi,\bar\varphi] + V[\varphi,\bar\varphi]$$

$$K[\varphi,\bar\varphi] = \sum_{\substack{\alpha \\ \alpha\in\{+,0,-\}}} \int_{\mathrm{SL}(2,\mathbb{C})^4} (\mathrm{d}g)^4 \int_{\mathrm{SL}(2,\mathbb{C}/\mathrm{U}^{(\alpha)}} \mathrm{d}X_\alpha \bar\varphi(g_1,...g_4;X_\alpha)\varphi(g_1,...g_4;X_\alpha)$$

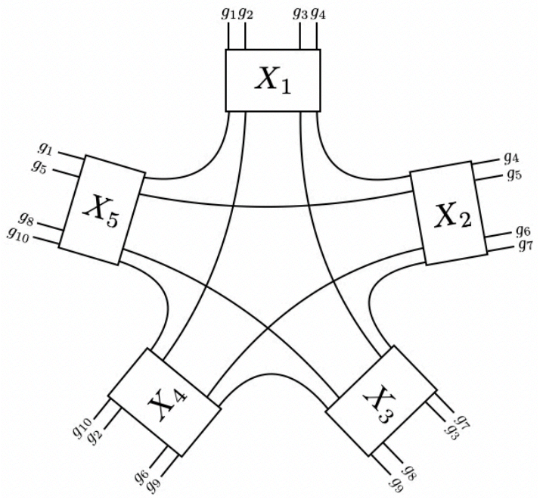



$$V[\varphi,\bar\varphi] = \int (\mathrm{d}g)^{10} \underbrace{\sum_{\alpha_1...\alpha_5}} \int \mathrm{d}X_{\alpha_1}... \int \mathrm{d}X_{\alpha_5} \varphi_{1234}(X_{\alpha_1})\varphi_{4567}(X_{\alpha_2})\varphi_{7389}(X_{\alpha_3})\varphi_{962(10)}(X_{\alpha_4})\varphi_{(10)851}(X_{\alpha_5}) + \mathrm{c.c}$$

**21 vertices (!)** e.g.:

- 5 spacelike tets (standard BC and EPRL models)
- 1 spacelike tet, 4 timelike tets & 5 timelike tets (CDT-like)

⟶ more refined causality imposition may be required
(e.g. locally CDT/Lorentzian Regge Calculus: vertex causality) [Jordan, Loll 1305.4582; Asante, Dittrich, Padua-Argüelles 2112.15387]




kernels computed using integral geometry methods in
[Jercher, Oriti, Pithis 2112.00091 & 2206.15442]

- can be straightforwardly formulated as a **colored** model [Gurau 1006.0714, 1011.2726, 1102.5759, 1105.6072; Gurau, Rivasseau 1101.4182; Gurau, Ryan 1109.4812; Bonzom, Gurau, Rivasseau 1202.3637]
  - introduction of r+1=5 colored fields reduces combinatorial complexity
  - generated coloured graphs bijective to 4-dimensional simplicial pseudo-manifolds
  - **keep it simple here:** work with ordinary simplicial (and tensor-invariant interactions)
- based on colored model reduction to **causal tensor model** can be given: 2 CDT vertices + dual weighting

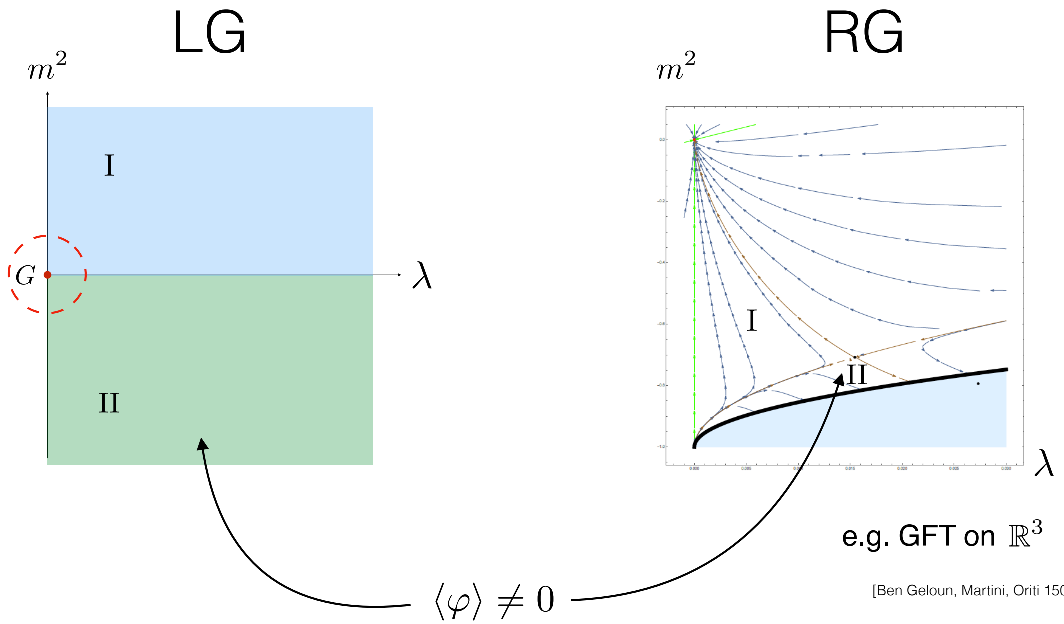
➡generates causal dynamical triangulations in 3+1d [Jercher, Oriti, Pithis 2206.15442]

# Back to Landau-Ginzburg method

**Applicable in GFT?**

# Why bother in GFT? Applicable?

- transition to condensate phase in GFT with non-trivial VEV?!




[Ben Geloun, Martini, Oriti 1508.01855 & 1601.08211]

- problem of the continuum limit in GFT/spin foam models
- mapping phases/phase structure of such models
- to this aim: exploit field theory character of GFTs

- condensate remains **hypothesis** for realistic models (but getting there); **test with LG theory applied to GFT**

- important for group field theory condensate cosmology: condensate phase is important pillar

e.g. [Gielen, Oriti, Sindoni 1303.3576 & 1311.1238; Gielen, Sindoni 1602.08104; Oriti 1612.09521; Pithis, Sakellariadou 1904.00598]

- upshot: LG MFT applicable to GFT in spite of non-locality of its interactions, gauge invariance and simplicity

# Landau-Ginzburg mean-field theory of GFTs

(goal: determine ingredients to realize phase transition)

[Thürigen, Pithis 1808.09765; Marchetti, Oriti, Thürigen, Pithis 2110.15336 & 2209.04297]

local scalar field theory: **LG theory gives coarse picture of phase structure** thus sufficient to point to the formation of a condensate phase; fully accurate only above critical dimension

$\longrightarrow$ **method works also for GFTs** (non-local)

(shown for simplified models on Abelian compact/non-compact group with/out closure constraint with/out additional local dof; **wip** on Lorentz group and simplicity constraints imposed)

mean-field analysis for $\varphi(\boldsymbol{g}) : G^r \to \mathbb{R}, \mathbb{C}$ take $G = \mathbb{R}$

- devise regularisation scheme due to non-locality together with projection onto uniform fields: $G \to \mathrm{U}(1)$

- extract critical dimension via Ginzburg quantity

$$Q \sim \lambda_\gamma \xi^{4-(r-\underbrace{s_0}_{\text{non-local contribution}})}$$

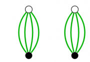
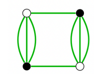
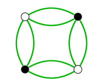
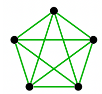
coupling

rank of group field

specifies combinatorics

correlation length

quartic interaction

| | |
|---|---|
| double-trace melon |  |
| quartic melonic |  |
| quartic necklace |  |
| simplicial |  |

- various interactions (power and combinatorics) checked and result generalized

- impose closure constraint $r \to r-1$ (or equally $s_0 \to s_0+1$)

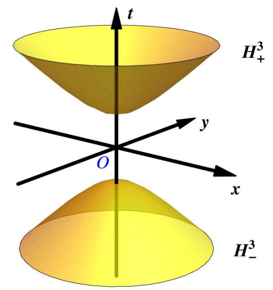
# More realistic scenario - kinematics

[Marchetti, Oriti, Thürigen, Pithis 2209.04297]

- work within context of the complete Barrett-Crane model
- **caveat here:**
  - **restrict to BF-quantization of first-order Palatini gravity with spacelike hypersurfaces**
  - **BC model restricted to spacelike tetrahedra/timelike normals**

$$\varphi(\boldsymbol{g}, X) = \varphi(g_1, g_2, g_3, g_4, X) = \mathrm{SL}(2,\mathbb{C})^4 \times \mathbb{H}^3 \to \mathbb{R}, \mathbb{C}$$

$$\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) \cong \mathbb{H}^3$$




- decomposition of the field in terms of irreducible representations

$$\varphi(\boldsymbol{g}, X) = \prod_{i=1}^{4} \left( \int \mathrm{d}\rho_i \rho_i^2 \sum_{j_i, m_i} D^{\rho_i, 0}_{j_i m_i 00}(g_i X) \right) \varphi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}$$

Wigner matrices of $\mathrm{SL}(2,\mathbb{C})$ in the so-called unitary principal series $D^{(\rho,\nu)}_{jmln}(g)$ $\quad (\rho,\nu) \in \mathbb{R} \times \mathbb{Z}/2$, $\; j,l \in \{|\nu|, |\nu|+1, \ldots\}, \quad m \in \{-j, \ldots, j\}, \quad n \in \{-l, \ldots, l\}$

- integration over normal to get rid of irrelevant information on embedding

$$\varphi(\boldsymbol{g}) = \int_{\mathbb{H}^3} \mathrm{d}X \varphi(\boldsymbol{g}, X) = \prod_{i=1}^{4} \left( \int \mathrm{d}\rho_i \rho_i^2 \sum_{j_i, m_i, l_i, n_i} D^{\rho_i, 0}_{j_i m_i 00}(g_i) \right) B^{\rho_1\rho_2\rho_3\rho_4}_{l_1 n_1 l_2 n_2 l_3 n_3 l_4 n_4} \varphi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}$$

Barrett-Crane intertwiner $\quad B^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4} \equiv \int \mathrm{d}X \prod_{i=1}^{4} D^{(\rho_i, 0)}_{j_i m_i 00}(X)$

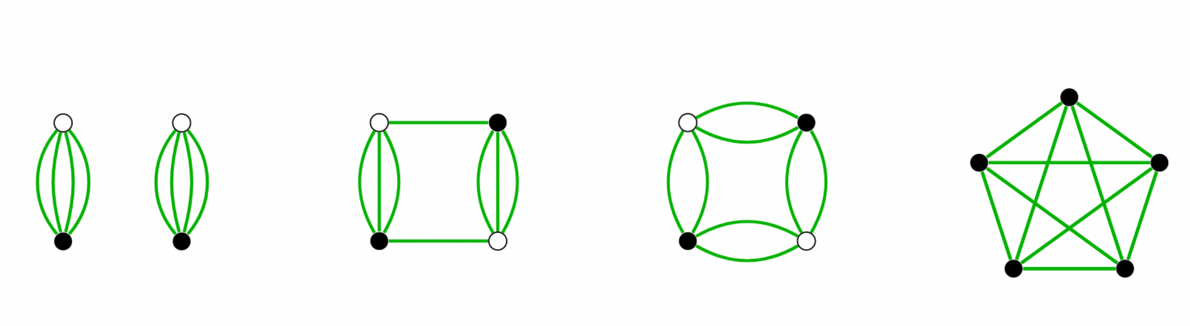
# More realistic scenario - dynamics

$$S[\varphi,\bar\varphi] = S_0[\varphi,\bar\varphi] + S_{\rm IA}[\varphi,\bar\varphi]$$

GFT action kinetic term interaction(s)

$$S_0[\varphi,\bar\varphi] = \int_{\mathrm{SL}(2,\mathbb{C})^4} \mathrm{d}\boldsymbol{g} \int_{\mathbb{H}^3} \mathrm{d}X\, \bar\varphi(\boldsymbol{g},X) \left(-\sum_{i=1}^{4} \Delta_i + \mu\right) \varphi(\boldsymbol{g},X)$$

consider interactions of type:



double-trace melon simple melon necklace simplicial

e.g.

$$S_{\rm IA,simplex}[\varphi,\bar\varphi] = \frac{\lambda}{5!} \int_{\mathrm{SL}(2,\mathbb{C})^{10}} [\mathrm{d}g]^{10} \int_{\mathbb{H}^{3\cdot 5}} [\mathrm{d}X]^5 \varphi_{1234}(X_1)\varphi_{4567}(X_2)\varphi_{7389}(X_3)\varphi_{9620}(X_4)\varphi_{0851}(X_5) + \text{c.c.}$$

# More realistic scenario - regularization

- due to closure constraint together with projection onto uniform fields $\varphi_0$ one has infinite volume factors as $\mathrm{SL}(2,\mathbb{C})$ is non-compact

$\longrightarrow$ have to regularize models: done by analytic continuation and compactification of $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{Spin}(4)$ [Dona, Gozzini, Nicotra 2106.14672]

concretely:

- at local level it amounts to **map** between corresponding **Lie algebras** $\mathfrak{spin}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \leftrightarrow \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$

- at global level it amounts to map between corresponding Lie groups via **mapping respective Cartan decompositions** into each other:

$\mathrm{SL}(2,\mathbb{C})$

$$\mathrm{SU}(2) \times \mathrm{A}^+ \times \mathrm{SU}(2) \to \mathrm{SL}(2,\mathbb{C})$$
$$(u, \mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3}, v) \mapsto u\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3}v^{-1}$$

$\mathrm{Spin}(4)$

$$\mathrm{SU}(2) \times \mathrm{T}^+ \times \mathrm{SU}(2) \to \mathrm{Spin}(4)$$
$$(u, \mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3}, v) \mapsto (u\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3}v^{-1}, u\mathrm{e}^{i\frac{1}{2}\frac{t}{a}\sigma_3}v^{-1})$$

$$\mathrm{A}^+ = \{\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3} | \eta \in \mathbb{R}_+\} \xrightarrow[\Lambda]{\text{introduce regulator}} \mathrm{A}^+_\Lambda = \{\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3} | \eta \in [0,\Lambda)\} \xrightarrow[\eta \to -it]{\text{Wick rotate}} \mathrm{T}^+_\Lambda = \{\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3} | t \in [0,\Lambda)\} \xrightarrow[\Lambda \to 2\pi a]{\text{compactify}} \mathrm{T}^+ = \{\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3} | t \in [0, 2\pi a)\}$$

- essentially amounts to mapping of respective homogeneous spaces into each other

$$\mathbb{H}^3 \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$$
$$\mathrm{d}H^2 = \underset{\substack{\uparrow \\ \text{skirt radius}}}{a^2} \left( \left(\frac{\mathrm{d}\eta}{a}\right)^2 + \sinh^2\left(\frac{\eta}{a}\right)\mathrm{d}\Omega_2 \right)$$

$$S^3 \cong \mathrm{Spin}(4)/\mathrm{SU}(2)$$
$$\mathrm{d}S^2 = a^2 \left( \left(\frac{\mathrm{d}t}{a}\right)^2 + \sin^2\left(\frac{t}{a}\right)\mathrm{d}\Omega_2 \right)$$

- map representation labels $\rho \to -ip$ $\longrightarrow$ $\boxed{\text{work with } \mathrm{Spin}(4)\text{-representation theory instead}}$

# More realistic scenario - correlation function and length

Starting from regularized action:

→ **linearize equations of motion** over non-trivial background $\varphi_0$

→ solve for regularized correlation function:

$$C(\boldsymbol{g}) = \prod_{i=1}^{4} \left( \sum_{p_i} \frac{p_i^2}{\mathrm{vol}(\mathrm{T}^+)} \sum_{\substack{j_i, m_i; \\ l_i, n_i}} D^{(p_i,0)}_{j_i m_i l_i n_i}(g_i) \right) B^{p_1 p_2 p_3 p_4}_{l_1 n_1 l_2 n_2 l_3 n_3 l_4 n_4} \hat{C}^{p_1 p_2 p_3 p_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}$$

$$\hat{C}^{p_1 p_2 p_3 p_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4} = \frac{1}{\frac{1}{a^2} \sum_i \left(-\mathrm{Cas}_{1,p_i}\right) + b_{\boldsymbol{p},\boldsymbol{j},\boldsymbol{m}}}$$

encapsulates remaining non-locality of interactions after projection onto $\Phi_0$

→ analyze correlation function **mode-by-mode**

→ turns out that only the zero-mode behaviour of the correlator is important for us; there we can Wick rotate back and decompactify to $\mathrm{SL}(2,\mathbb{C})$

→ **only** these **zero-modes contribute to** the **correlation length** and **determine** the behaviour of the **Ginzburg Q**-parameter

→ **result for correlation length** (via asymptotic analysis or second-moment-method):

$$\xi^2 \sim \frac{1}{a^2 \mu^2} + \frac{1}{\mu}$$

flat limit: $a \to \infty$ $\quad \xi^2 \sim \frac{1}{\mu}$

modification due to hyperbolicity of domain

see also [Benedetti 1403.6712]

# More realistic scenario - Ginzburg Q

- results for local scalar field theory on one 3-hyperboloid

$$Q \sim \lambda_\gamma \xi^2 \boxed{\mathrm{e}^{-2\cdot 1\frac{\xi}{a}}}$$

[Benedetti 1403.6712]

coupling

exponential suppression due to hyperbolicity of domain

rank of the group field

for finite skirt radius $a$ :

$$Q \sim \lambda_\gamma \xi^2 \boxed{\mathrm{e}^{-2(4-s_0)\frac{\xi}{a}}}$$

combinatorics of interaction $s_0 \leq 4$

dimension of 3-hyperboloid

flat limit: $a \to \infty$

$$Q \sim \lambda_\gamma \xi^{4-3(4-s_0)}$$

[agrees with our results 2110.15336]

impact of closure constraint via the BC intertwiner:

$\cancel{r \to r - 1}$ (or equally $s_0 \to s_0 + 1$)

extended formalism: domain has one more slot

- can be generalized to arbitrary interactions

$$Q \sim \lambda_\gamma^{\frac{2}{V_\gamma - 2}} \xi^{\frac{V_\gamma}{V_\gamma - 2}} \mathrm{e}^{-2(4-s_0)\frac{\xi}{a}}$$

flat limit: $a \to \infty$

$$Q \sim \lambda_\gamma^{\frac{2}{V_\gamma - 2}} \xi^{\frac{2V_\gamma}{V_\gamma - 2} - 3(4-s_0)}$$

⟶ **Ginzburg Q always very small**

⟶ **LG MFT can self-consistently describe phase transition** $(\varphi_0 = 0 \text{ vs. } \varphi_0 \neq 0)$

# Summary & Conclusions

- LG MFT theory is also applicable to GFT models in spite of their non-local interactions
- it informs us about the coarse phase structure of different models
- here applied to:
  - BC model for Lorentzian (first order Palatini) quantum gravity
  - restricted to spacelike hypersurfaces ➡ spacelike tetrahedra
- results:
  - MFT description gives accurate account of phase structure
    ➡full phase structure inferable from GFP **wip:** [Marchetti, Oriti, Pithis, Thürigen]
  - non-compactness of Lorentz group prerequisite for non-perturbative vacuum
    ➡important for continuum limit

# Extensions

- local dof (free massless scalar matter) can be added on the lattice structure [Marchetti, Oriti, Pithis, Thürigen 2209.04297]
- consider all the bare causal structure (spacelike, timelike and lightlike tetrahedra) **wip:** [Jercher, Pithis]
- consider local causality conditions (like in locally CDT)
- extension to other relevant models (EPRL)
- complement by full-fledged (functional) **RG and 1/N analyses**
- devise **observables** & tools to characterize different phases wrt their geometric properties

**Thank you for your attention!**

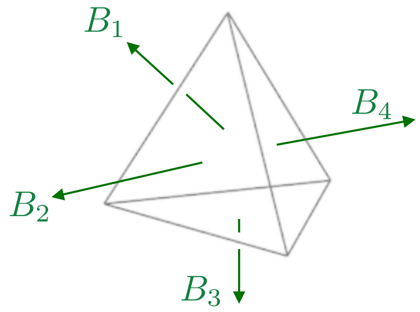
# Backup slides

# Recovery of metric information from tetrahedron

$G = \mathrm{SL}(2, \mathbb{C})$

➡ **relate metric and bivector variables**

$$(B_1, B_2, B_3) \longmapsto g_{ij} = e_{iA} e_j^A = \frac{1}{8\mathrm{tr}(B_1 B_2 B_3)} \varepsilon_i^{kl} \varepsilon_{jmn} \left(B_k^{AB} B_{AB}^m\right) \left(B_l^{CD} B_{CD}^n\right)$$




bivectors $B_i^{AB} = \varepsilon_i{}^{jk} e_j^A e_k^B$ $\quad$ $e_i^A$ tetrads

Lorentz index $A \in \{0, 1, 2, 3\}$

$B_i \in \mathfrak{sl}(2, \mathbb{C}),\ i \in \{1, 2, 3\}$

# Criticism against the BC model and alleviations

- BC vertex does not yield tensorial structure of lattice graviton propagator [Alesci, Rovelli 0708.0883]
  - Obvious mismatch of LQG boundary states and BC boundary states [Baratin, Oriti 1108.1178]
- Area-length constraints are missing [Alexandrov 0802.3389]
  - Recently it was shown (on a hypercubical lattice) that the BC model is still viable and potentially lies in the same universality class as the EPRL model in an effective continuum limit [Dittrich 2105.10808]
- What is the role of degenerate geometries in the BC model? [Barrett, Steele 0209023]
  - Need further analysis including timelike and lightlike configurations.
- Constraints are "too strongly" imposed [Engle, Pereira, Rovelli 0705.2388]
- Closure and simplicity are imposed in a non-covariant and non-commuting manner [Baratin,Oriti1002.4723]
  - Problems resolved in extended BC model [Baratin, Oriti 1108.1178]
- EPRL model favored since boundary states are closer to canonical LQG, the Barbero-Immirzi parameter is incorporated
  - Absence of BI parameter does not rule out the BC model. At the same time, questions wrt the precise value and running of the BI parameter and parity violation issues should be addressed [Charles 1705.10984; Benedetti, Speziale 1111.0884]

**For now, criticisms are not conclusive and the BC model deserves further attention.**