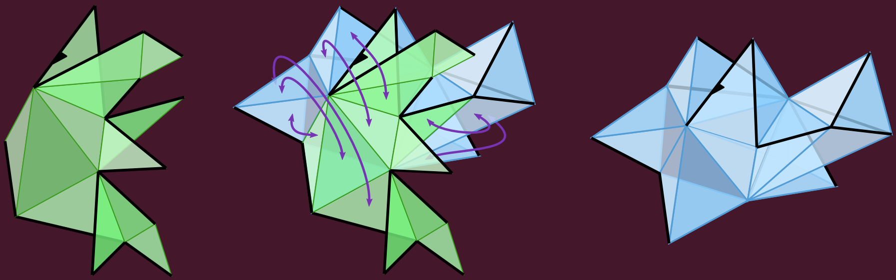
26-10-2022, Tensor Journal Club

A family of triangulated 3-spheres constructed from trees

Timothy Budd



Based on arXiv:2203.16105 with Luca Lionni

Radboud University

Motivation: The search for universality classes in quantum gravity

- Asymptotic safety hypothesis in quantum gravity: gravity is described by a QFT of the (pseudo)Riemannian metric $g_{\mu\nu}$ on spacetime that at microscopic scales

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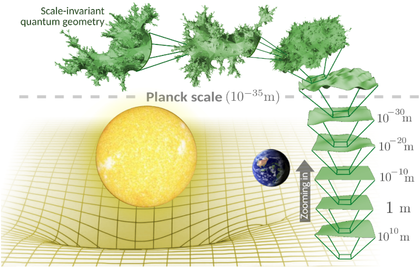

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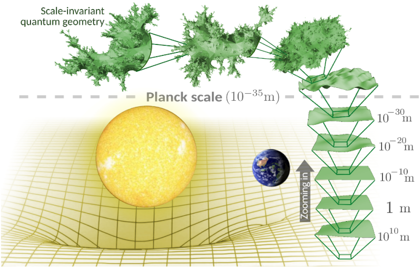

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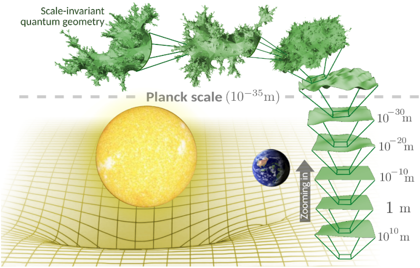
- It requires the existence of scale-invariant quantum geometry modeling the spacetime geometry on sub-Planckian length scales.
- In the (wick-rotated) Euclidean setting, it amounts to the existence of scale-invariant random geometry:

$$Z = \int_{\text{geometries}} \frac{\mathcal{D}g_{ab}}{\text{Diff}}\, e^{-S[g]} \quad \rightsquigarrow \quad \text{probabilistic interpretation?}$$



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Scale-invariant random geometry

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random geometry

(simulations of
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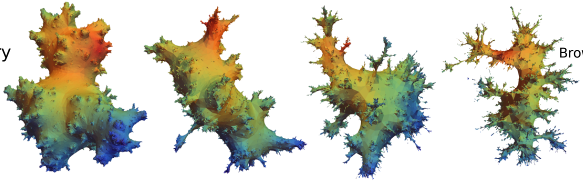
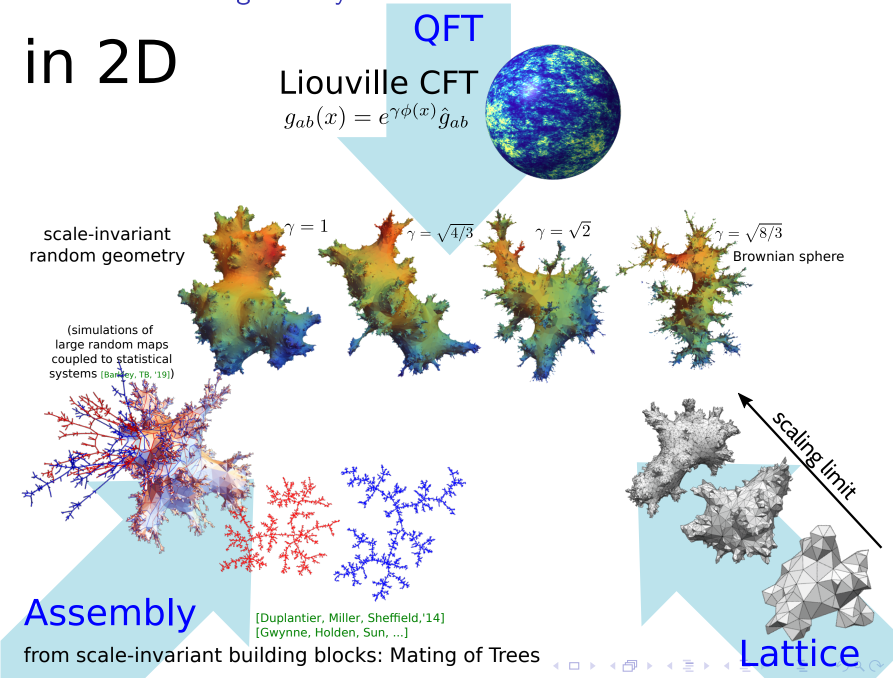


Brownian sphere

Scale-invariant random geometry



Scale-invariant random geometry

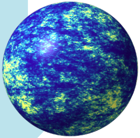
in ~~2D~~ 3D (or higher)

Not a single explicit example known with 3D topology!

QFT

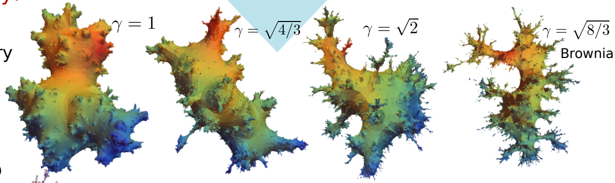
Liouville CFT

$$g_{ab}(x) = e^{\gamma\phi(x)}\hat{g}_{ab}$$



Proposals:
Random Feuilletages
[Lionni, Marckert, '19]
Mating of Trees generalized
[TB, Castro, '22]
Topology = ??

scale-invariant random geometry



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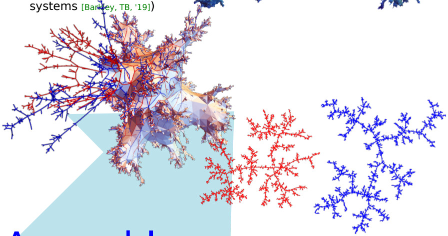
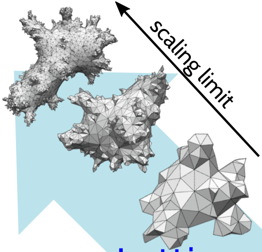




scaling limit

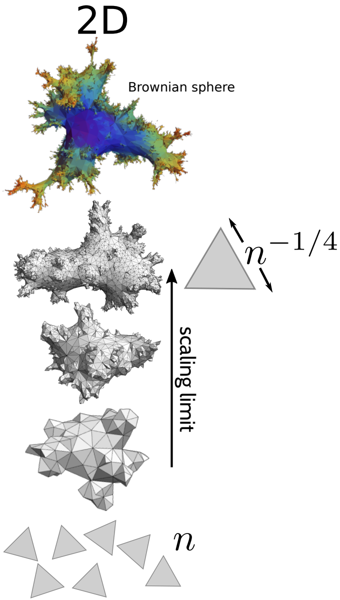
Assembly

[Duplantier, Miller, Sheffield,'14]
[Gwynne, Holden, Sun, ...]

from scale-invariant building blocks: Mating of Trees

Lattice

Dynamical Triangulations (DT) [Ambjorn, Boulatov, Krzywicki, Varsted, Caterall, . . .]



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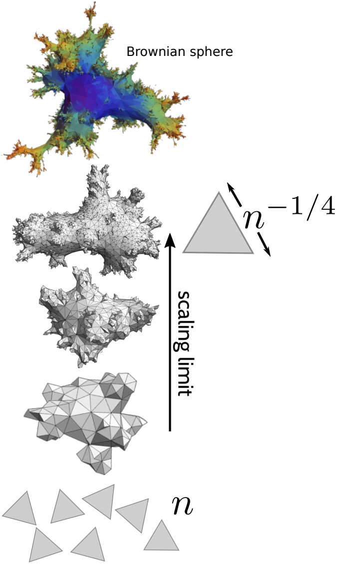
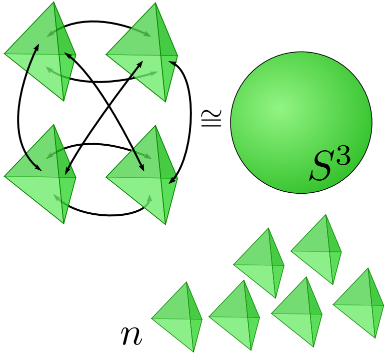


3D

$$\mathbb{P} \propto x^{\#\text{vertices}} = e^{k_0 \#\text{vertices}}$$



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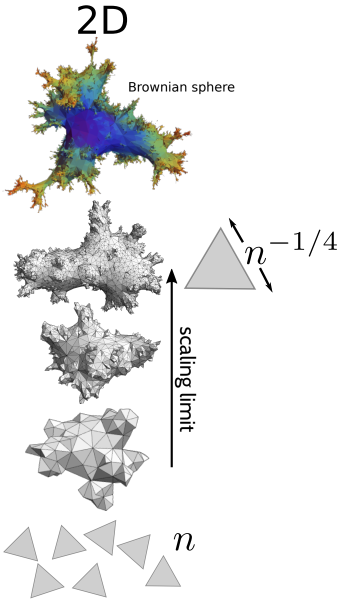

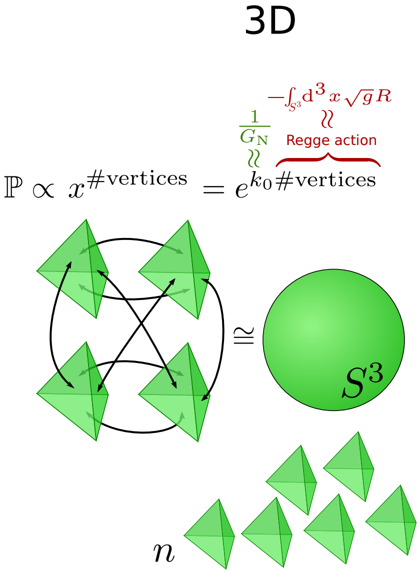

Challenges faced by 3D DT & guiding principles

3D topology is hard.

- No simple topological invariants.
- No polynomial-time 3-sphere recognition algorithm known (recognition ∈ P**??**)...
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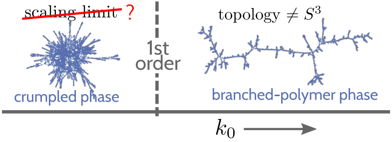

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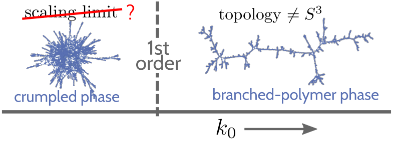

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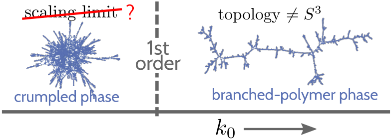

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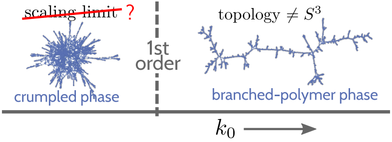

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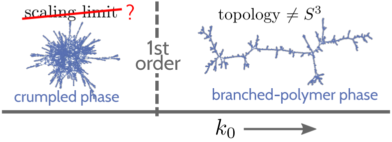

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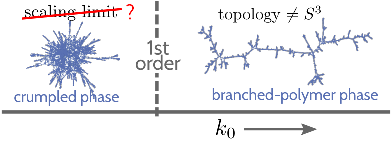

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Note: 3D Causal Dynamical Triangulations (CDT) satisfies $2\frac{1}{2}$ of these! [Ambjørn, Loll,. . .]

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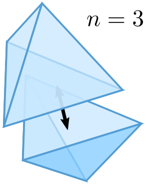

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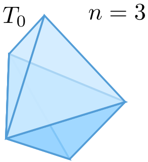

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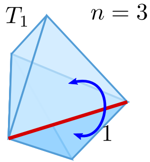

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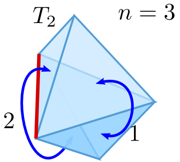

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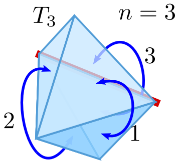

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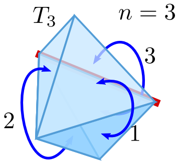

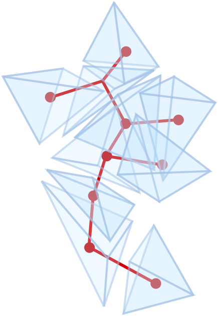

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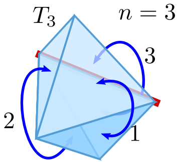

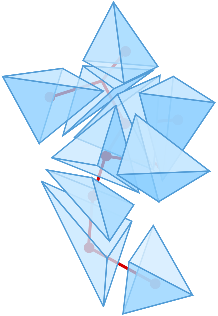

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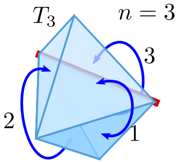

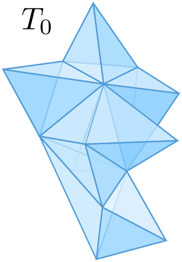

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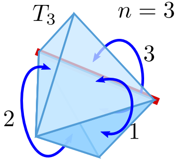

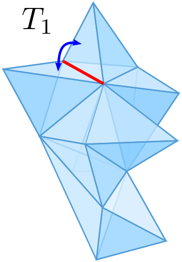

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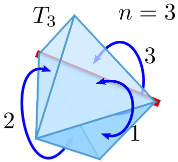

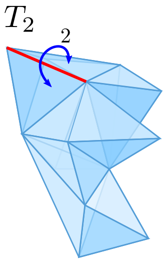

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Locally constructible triangulations [Durhuus, Jonnson, '95]

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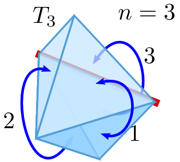

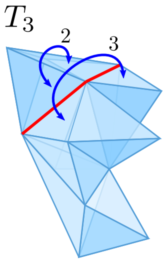

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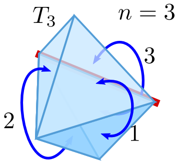

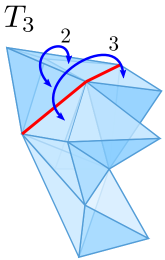

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- Some triangulated 3-spheres are not locally constructible. [Benedetti, Ziegler, '11]

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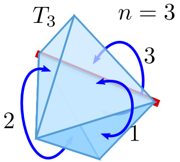

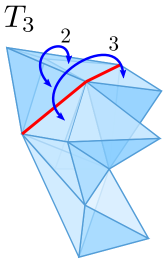

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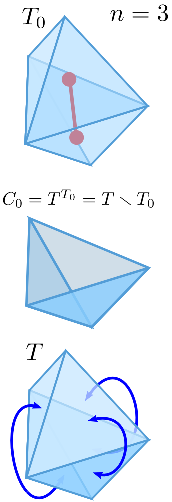

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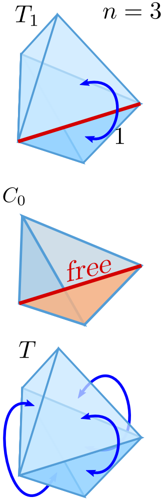

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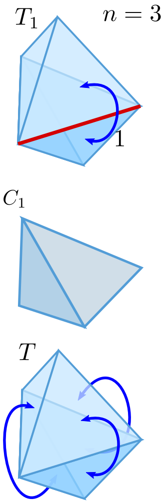

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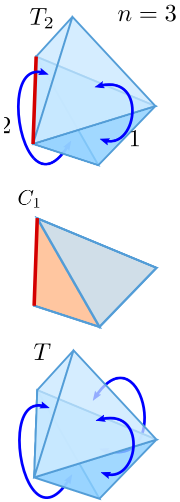

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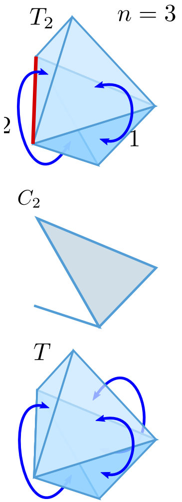

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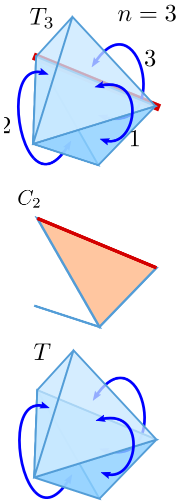

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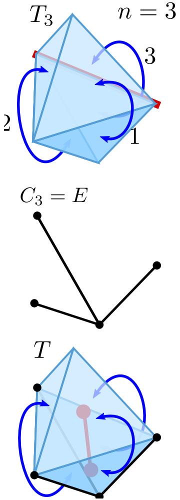

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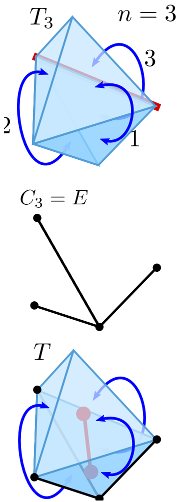

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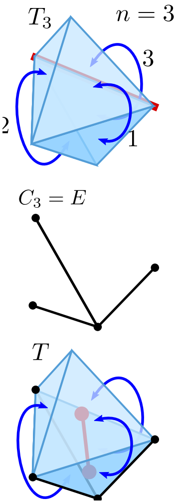

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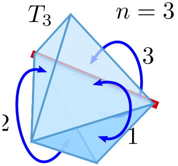

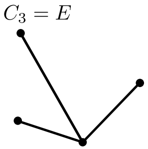

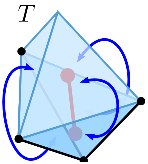

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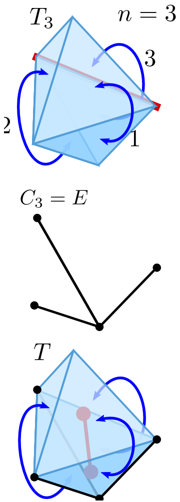
- Simplest choice, sampling (T, T_0, E) uniformly, seems to not satisfy last two criteria (departure from DT phase diagram, encoding in trees).



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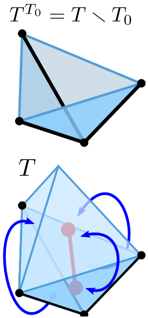

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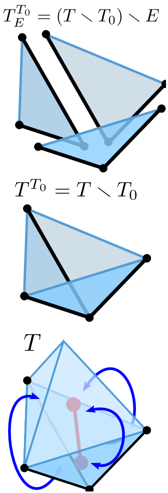

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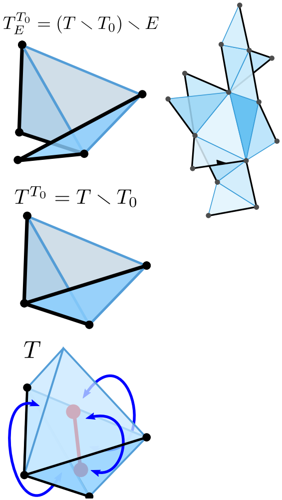

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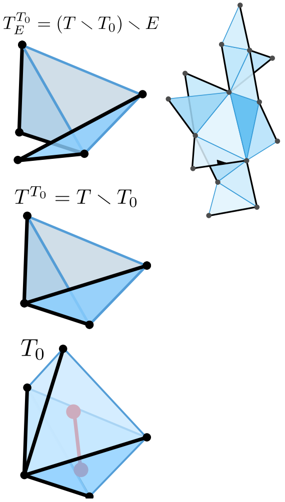

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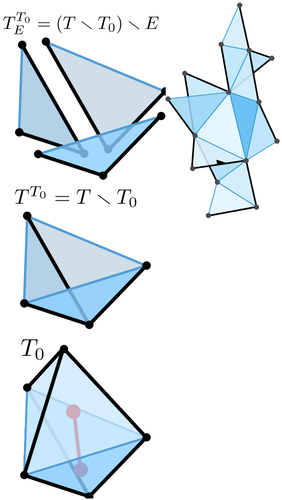

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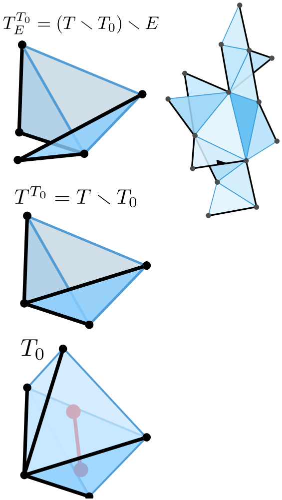

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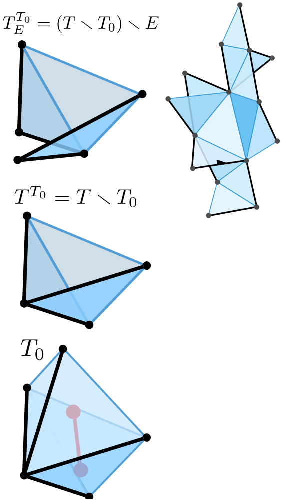

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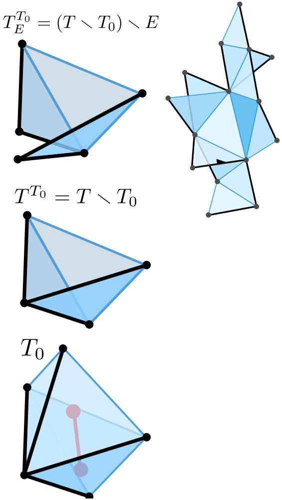

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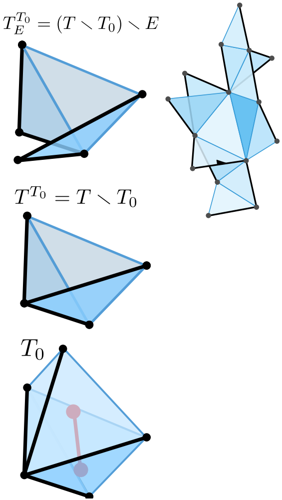
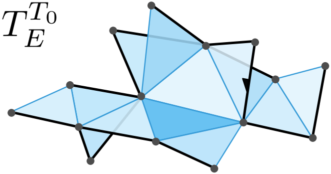
- Claim: both can be conveniently represented as 2d triangulations!

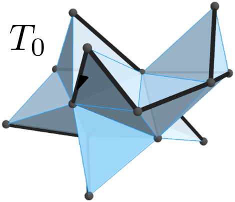


Encoding in plane trees

$T_E^{T_0}$



T_0



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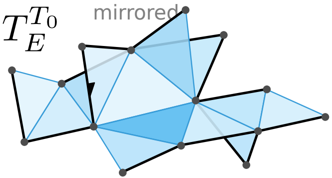

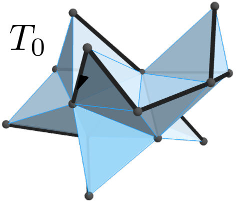

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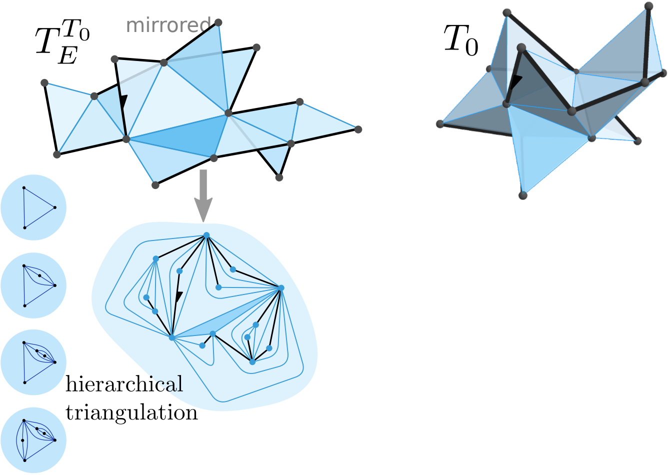

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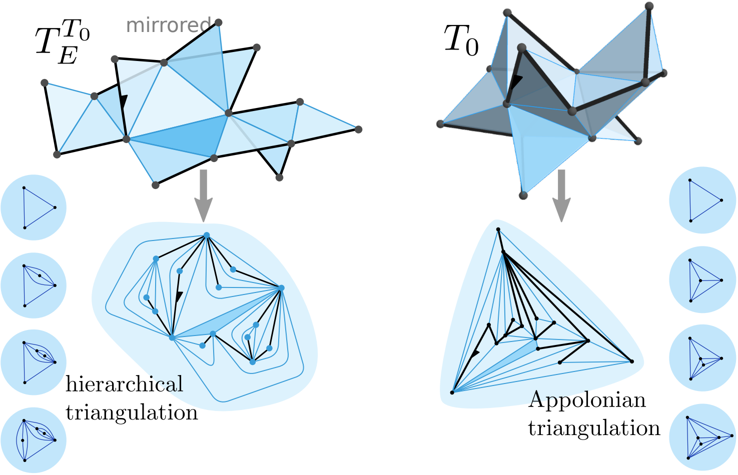

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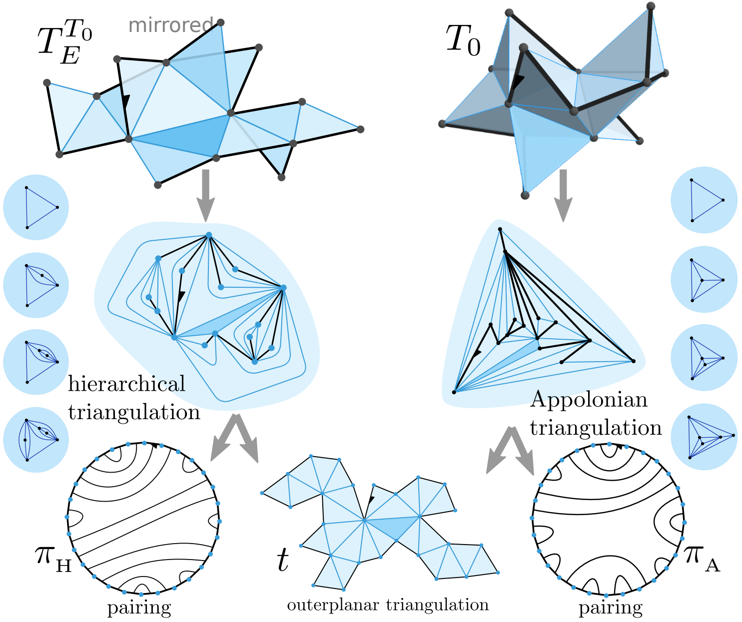

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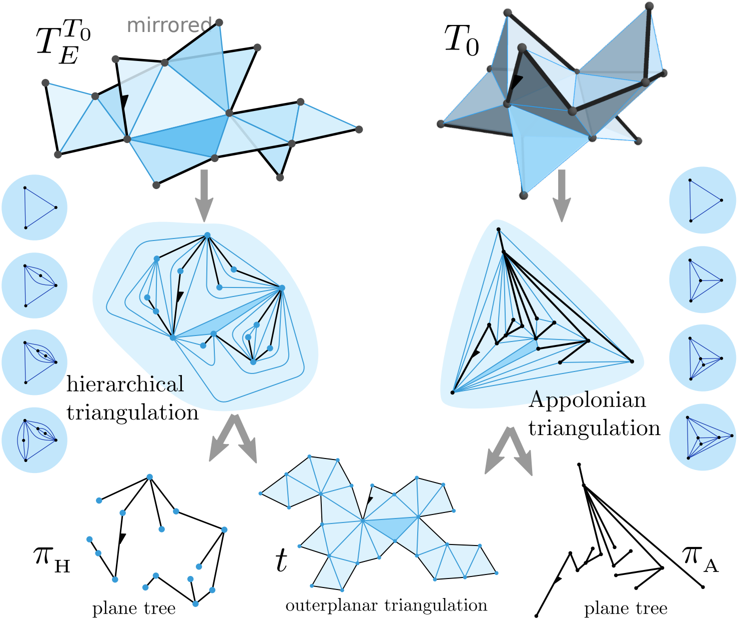

Main bijective result

- A triple $(t, \pi_{\mathrm{H}}, \pi_{\mathrm{A}})$ is called a triple tree if

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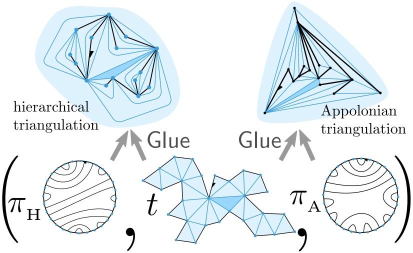

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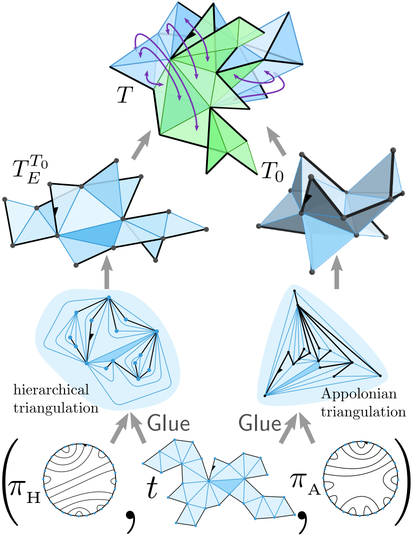

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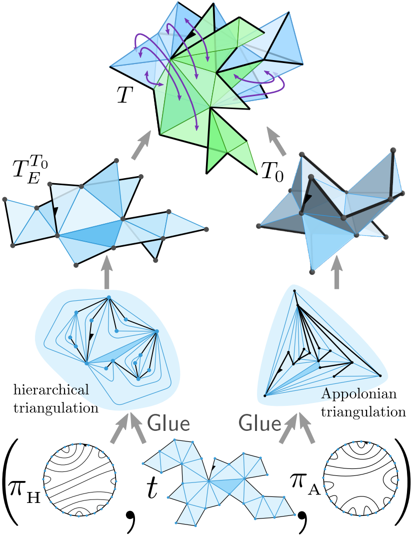

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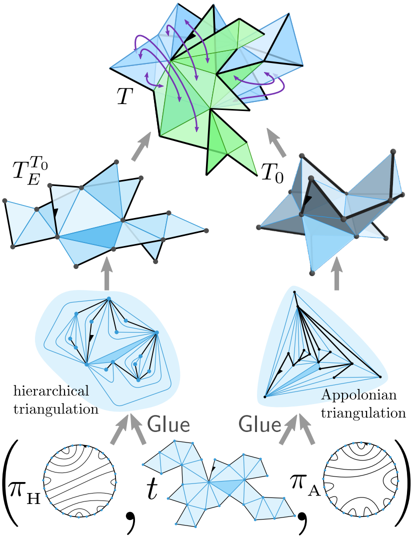

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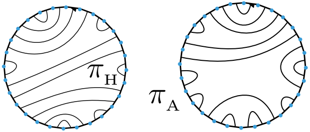

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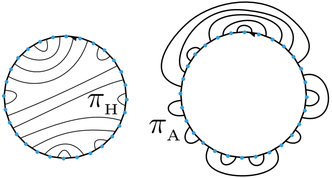

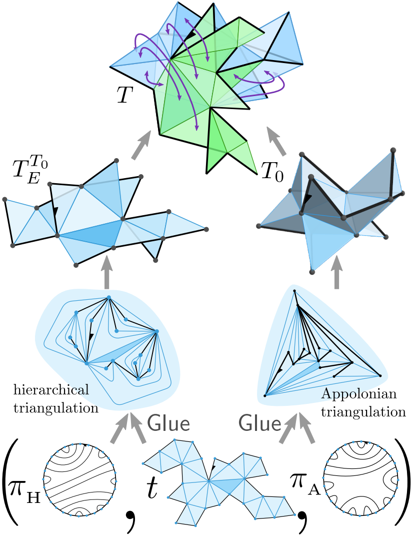

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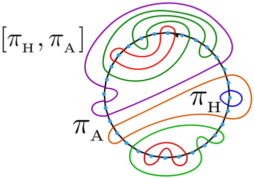

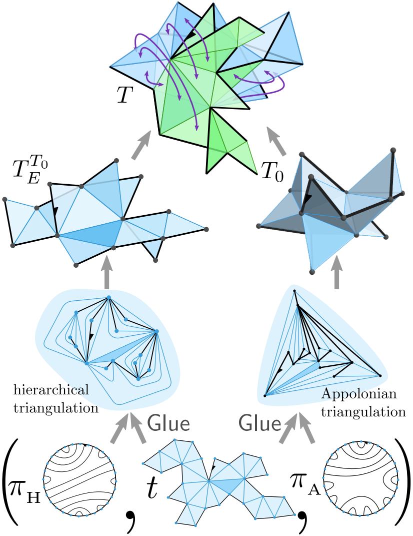

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- Consider the partition function ($z = e^{-k_3}$, $x = e^{k_0}$ in notation DT)

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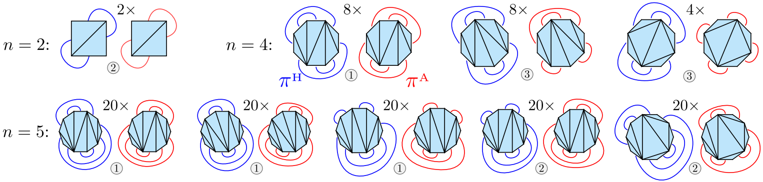

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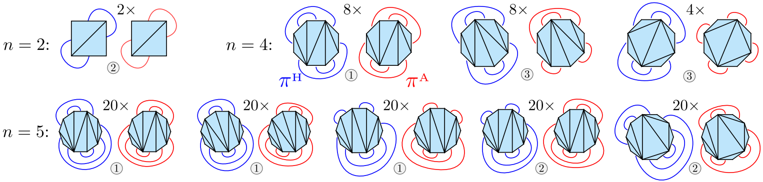

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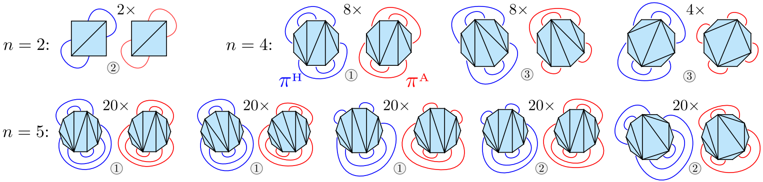

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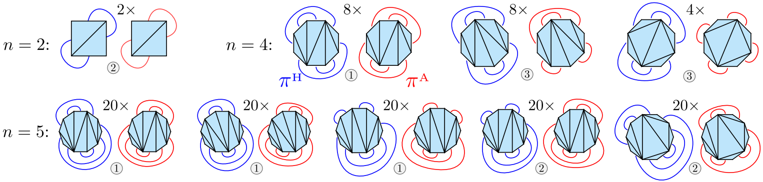

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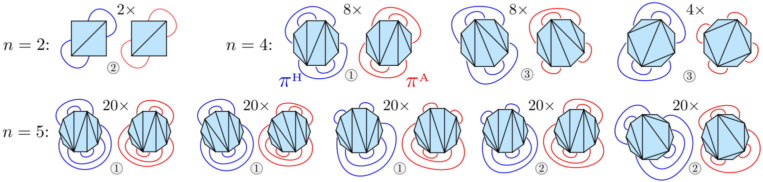

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- How about phase diagram? Need to compare Monte Carlo simulations with DT...

Dynamical triangulations & Monte Carlo simulation

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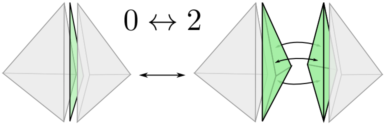

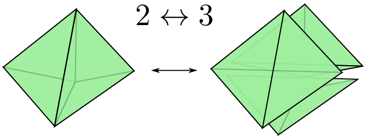

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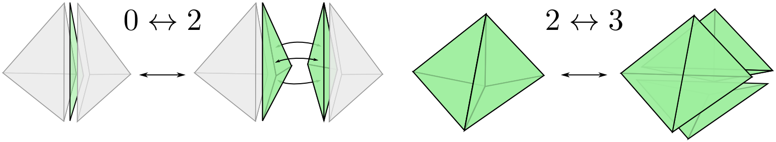
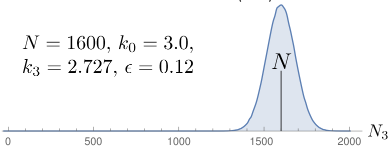


- Note $N_3 = \#$tetrahedra not fixed! Instead $\mathbb{P}(T) \propto e^{-k_3 N_3 - \epsilon (N_3 - n)^2/n} e^{k_0 N_0}$ and tune k_3 and ϵ such that $\langle N_3 \rangle \approx n$.



- Perform measurements only when $N_3 = n$.

Summary of known results [Agistein, Migdal, Ambjorn, Varsted, Catterall, Thorleifsson, . . . , ’90s]

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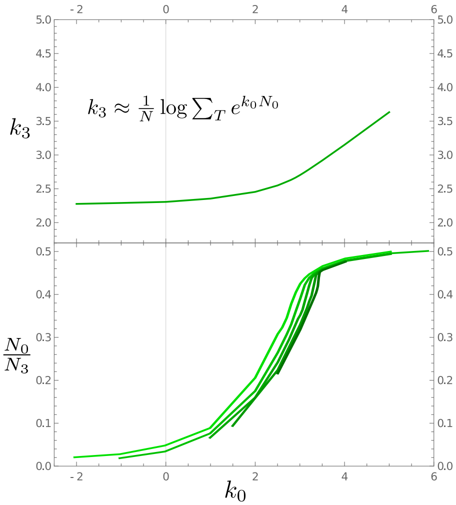

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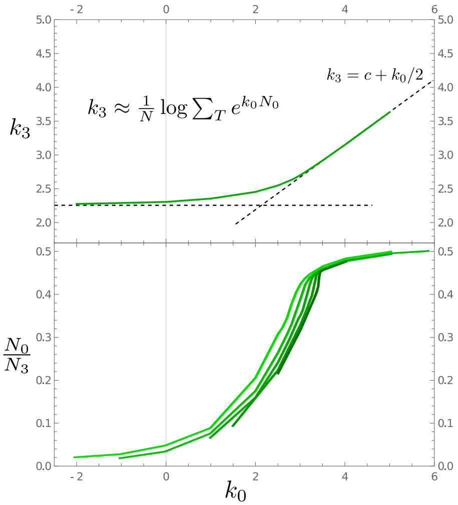

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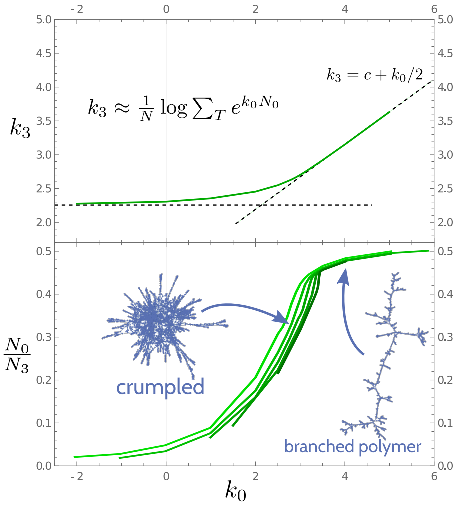

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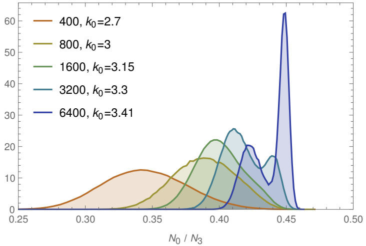
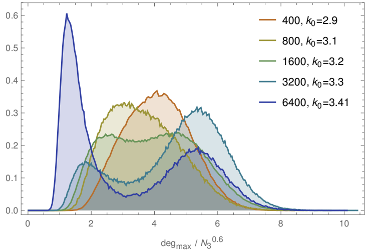
- Order parameters: N_0/N_3 and max vertex degree.
- Phase transition is 1st order: double peaks in histograms become more pronounced as $N \to \infty$.





Towards simulating triple trees

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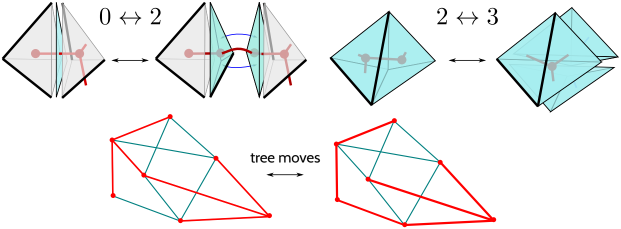

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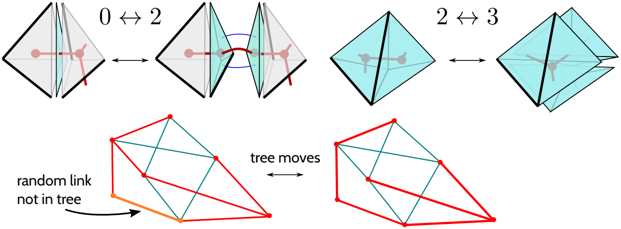

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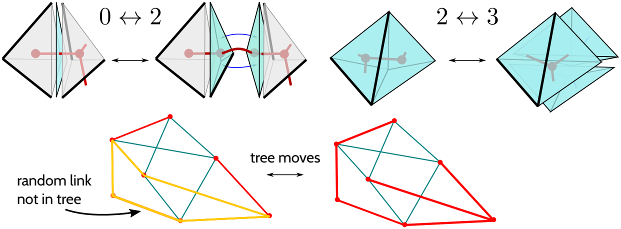

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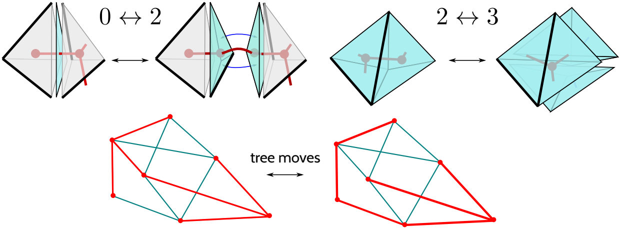

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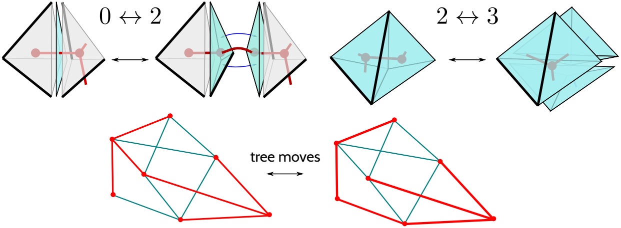

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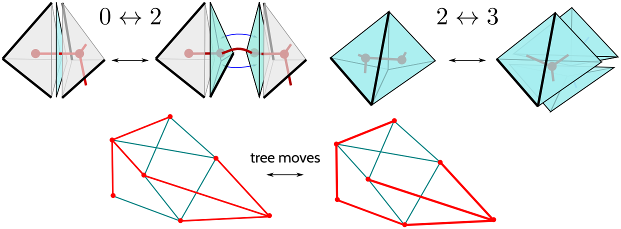
- $\texttt{LC}_n$: Reject if T^{T_0} does not collapse on E. Worst case $O(n)$ time per move $\approx$ pretty bad.

Towards simulating triple trees

- Three new ensembles $\texttt{TripleTrees}_n \subset \texttt{LC}_n \subset \texttt{TwoTrees}_n$:

$$\begin{aligned} \texttt{TwoTrees}_n &= \{(T, T_0, E) : T \in \texttt{DT}_n \text{ and no restrictions on } T_0, E\} \\ \texttt{LC}_n &= \{(T, T_0, E) \in \texttt{TwoTrees}_n : \text{local construction}\} \\ \texttt{TripleTrees}_n &= \{(T, T_0, E) \in \texttt{LC}_n : T_E^{T_0} \text{ is tree of triangles}\} \end{aligned}$$

- Adapt Markov chain to sample from these with $\mathbb{P}(T, T_0, E) \propto x^{N_0} = e^{k_0 N_0}$, rejecting whenever restrictions on (T, T_0, E) are violated.



- $\texttt{TwoTrees}_n$: Any move in $O(\log n)$ time (using link/cut tree data structure).
- $\texttt{LC}_n$: Reject if T^{T_0} does not collapse on E. Worst case $O(n)$ time per move $\approx$ pretty bad.
- $\texttt{TripleTrees}_n$: Back to $O(\log n)$ per move (trees are easier!) but high rejection.

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- Caution: we do not know for sure that the Markov chain is ergodic/irreducible in $\texttt{TripleTrees}_n$.

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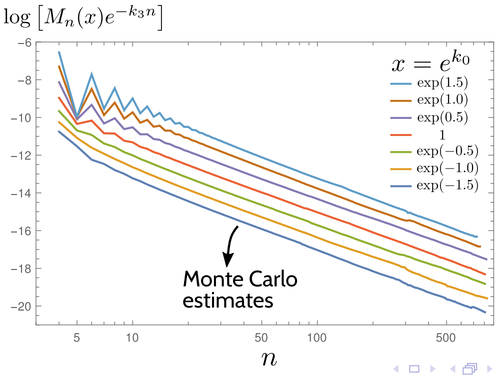

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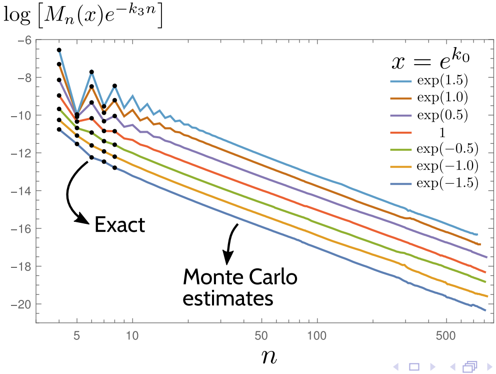
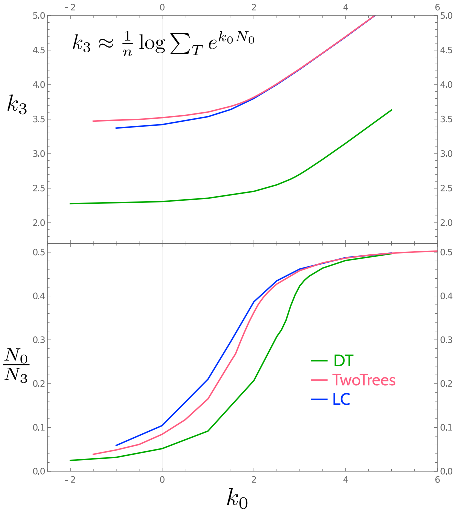
- ...and they agree with the exact enumeration for small n,

$$
\begin{aligned}
M(z,x) = {} & 2x^2z^2 + (8x + 12x^3)z^4 + (60x + 40x^2)z^5 + (336x + 996x^2 + 420x^3 + 618x^4)z \\
& + (5460x + 10416x^2 + 6496x^3 + 1652x^4)z^7 \\
& + (63344x + 135776x^2 + 150544x^3 + 75360x^4 + 46360x^5)z^8 + \cdots .
\end{aligned}
$$

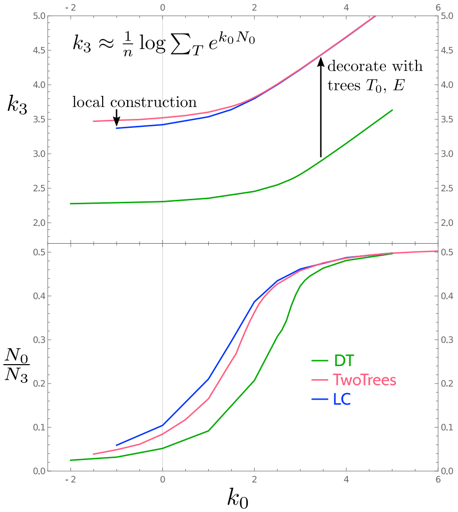


Data: first impressions



Data: first impressions

- Inclusion of T_0, E adds much more entropy than the restriction $\mathtt{LC}_n \subset \mathtt{TwoTrees}_n$ takes away.



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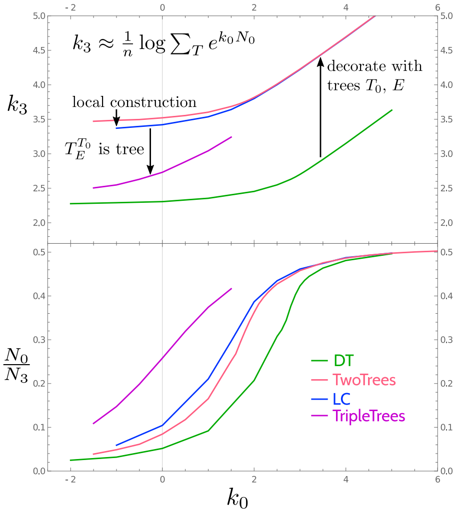

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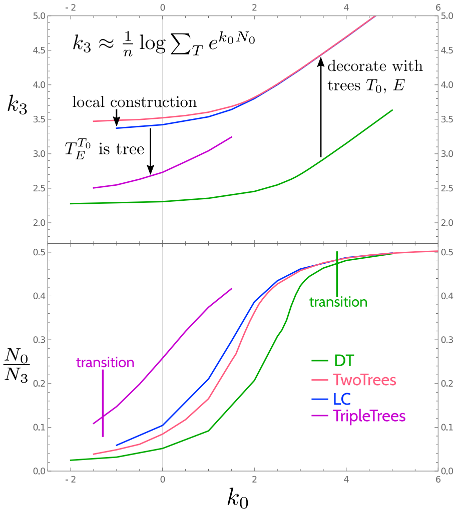

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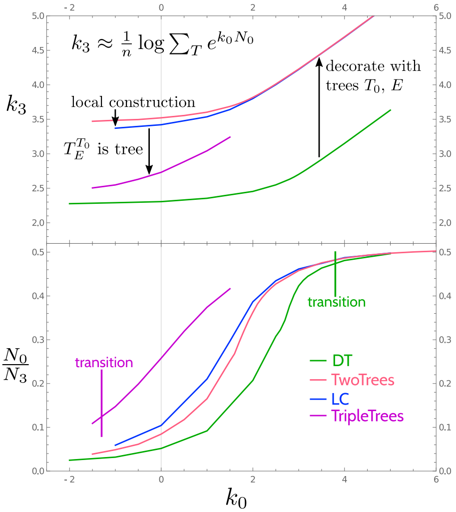
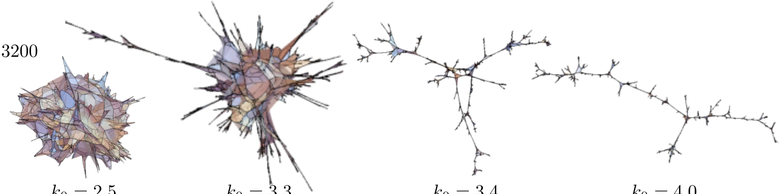
- Not unsurprisingly: spanning trees favour the branched polymer phase.
- Qualitative changes? Let's have a look!



DT

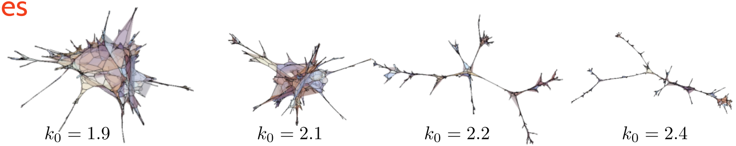
$n = 3200$



$k_0 = 2.5$ $k_0 = 3.3$ $k_0 = 3.4$ $k_0 = 4.0$

TwoTrees

$n = 1600$



$k_0 = 1.9$ $k_0 = 2.1$ $k_0 = 2.2$ $k_0 = 2.4$

DT

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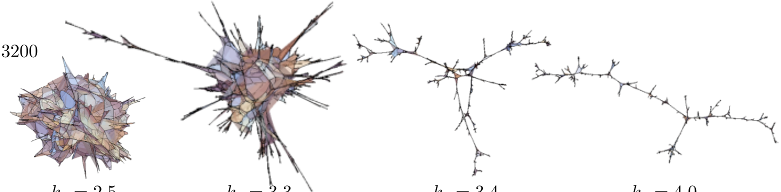

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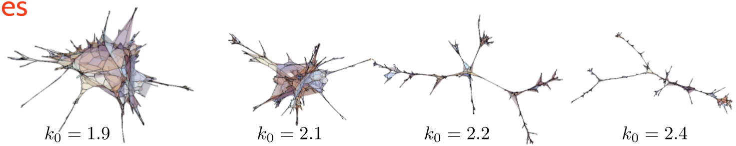
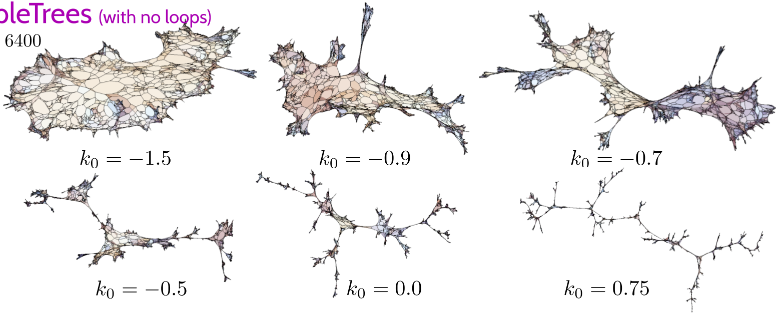


$k_0 = 1.9$ $k_0 = 2.1$ $k_0 = 2.2$ $k_0 = 2.4$

TripleTrees (with no loops)

$n = 6400$



$k_0 = -1.5$ $k_0 = -0.9$ $k_0 = -0.7$

$k_0 = -0.5$ $k_0 = 0.0$ $k_0 = 0.75$

Conclusions

- Incorporating local construction data into triangulations allows to avoid two important roadblocks (certified topology and exponential bound).
- Encoding in trees may facilitate analytic investigation and increase chances of criticality: trees are simple and don't mind being critical!
- Enumeration of triple trees is still out of reach, but the formulation in planar map language should enlarge attack surface (and enthuse more mathematicians).
- Glimpse of changes in phase diagram compared to DT, but the numerics is challenging.
- Naturally the phase diagram of locally constructible triangulations is larger (3d) with triple trees in one corner. Any new phase transitions?