Random tensor network states & Holography

Based on joint work with Newton Cheng, Geoff Penington, Michael Walter and Freek Witteveen, available at arXiv:2206.10482

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Outline

Mathematical formalism of quantum physics in a nutshell

- System with 1 subsystem: Complex Hilbert space $\mathrm{H} \equiv \mathbf{C}^d$.
  System with N subsystems: Complex Hilbert space $\mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_N \equiv \mathbf{C}^{d_1} \otimes \cdots \otimes \mathbf{C}^{d_N}$.
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  Mixed state of H: convex combination of pure states $\rho = \sum_{k=1}^{r} \lambda_k |\varphi_k\rangle\langle\varphi_k|$.
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- *Entanglement/Separability*: A state ρ on $\mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_N$ is *separable* if it is a convex combination of product states, i.e. $\rho = \sum_{k=1}^{r} \lambda_k \rho_k^1 \otimes \cdots \otimes \rho_k^N$. Otherwise it is *entangled*.

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  Particular case: If $\rho = |\varphi\rangle\langle\varphi|$ for some $|\varphi\rangle \in \mathrm{H}_1 \otimes \mathrm{H}_2$, then ρ is separable iff $|\varphi\rangle$ is product, i.e. $|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$ for some $|\varphi_1\rangle \in \mathrm{H}_1$ and $|\varphi_2\rangle \in \mathrm{H}_2$.
  Equivalently, ρ is separable iff its *reduced state* ρ_1 on H_1 is pure.
  ↳ $\rho_1 := \mathrm{Tr}_2(\rho) := \mathrm{I}_1 \otimes \mathrm{Tr}_2(\rho)$

  $\longrightarrow$ How to quantify the amount of entanglement in ρ?

  - Simplest idea: $\mathrm{rank}(\rho_1)$, which is between 1 and $\min(d_1, d_2)$.
    ↳ i.e. tensor rank of $|\varphi\rangle$
  - Smoother version: $S(\rho_1)$, which is between 0 and $\log \min(d_1, d_2)$.
    ↳ $:= -\mathrm{Tr}(\rho_1 \log \rho_1)$ (von Neumann entropy of ρ_1)

Underlying graph: set of vertices V, set of edges E.

We write $V = V_0 \sqcup \partial V$, with V_0 the set of *bulk vertices* and ∂V the set of *boundary vertices*.

$\hookrightarrow \forall\, v \in V_0,\ d(v) > 1$ $\qquad \hookrightarrow \forall\, v \in \partial V,\ d(v) = 1$

We associate to each $e \in E$ a space $\mathrm{H}_e \equiv (\mathbf{C}^D)^{\otimes 2}$ and to each $v \in V$ a space $\mathrm{H}_v \equiv (\mathbf{C}^D)^{\otimes d(v)}$.

$\longrightarrow$ We can identify $\mathrm{H}_E := \bigotimes_{e\in E} \mathrm{H}_e$ with $\mathrm{H}_V := \bigotimes_{v\in V} \mathrm{H}_v$, since $\mathrm{H}_E \equiv \mathrm{H}_V \equiv (\mathbf{C}^D)^{\otimes 2|E|}$.

For each $e \in E$, pick $|\psi_e\rangle \in \mathrm{H}_e$, and set $|\psi_E\rangle := \bigotimes_{e\in E} |\psi_e\rangle \in \mathrm{H}_E$.

For each $v \in V_0$, pick $|\varphi_v\rangle \in \mathrm{H}_v$, and set $|\varphi_{V_0}\rangle := \bigotimes_{v\in V_0} |\varphi_v\rangle \in \mathrm{H}_{V_0}$.

Construct $|\varphi_{\partial V}\rangle := \langle \varphi_{V_0} | \psi_E \rangle \in \mathrm{H}_{\partial V} \equiv (\mathbf{C}^D)^{\otimes |\partial V|}$.

$\longrightarrow$ $|\varphi_{\partial V}\rangle$ is a *multipartite pure state constructed from an underlying graph* (up to normalization).

$|\psi_E\rangle \in (\mathbf{C}^D)^{\otimes 2|E|}$ $\quad |\varphi_{V_0}\rangle \in (\mathbf{C}^D)^{\otimes (2|E| - |\partial V|)}$ $\quad |\varphi_{\partial V}\rangle \in (\mathbf{C}^D)^{\otimes |\partial V|}$

Tensor network states: construction

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General problem: How is the geometry of the bulk reflected in the entanglement-related properties of the resulting boundary state?

Minimal cut / Maximal flow

Given $A \subset \partial V$, its *min-cut* $\delta(A)$ and the number of ways of achieving it $N(A)$ are

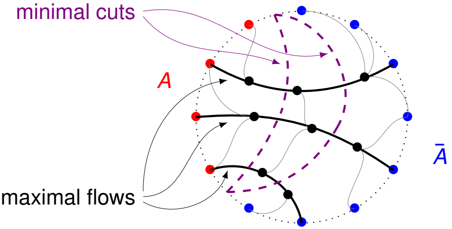
$$\delta(A) := \min\left\{\delta(AX{:}\bar{A}\bar{X}),\ X \subset V_0\right\} \text{ and } N(A) := \left|\left\{X \subset V_0,\ \delta(AX{:}\bar{A}\bar{X}) = \delta(A)\right\}\right|,$$

with $\delta(Y{:}Y')$ the number of edges having one end in Y and one end in Y', for $Y, Y' \subset V$ disjoint.

Fact: There exist exactly $\delta(A)$ edge-disjoint paths going from A to $\bar{A}$, called the *max-flows*.

Assumption: Min-cuts are *non-crossing*. This means that, for any $A \subset \partial V$, if $N(A) > 1$, then two distinct ways of achieving $\delta(A)$ have no edge in common.

$\longrightarrow$ Min-cuts can be ordered and $V = \bigsqcup_{i=0}^{N(A)} V_i$, with V_i the vertices 'between' min-cuts i and $i+1$.



$\delta(A) = 3,\ N(A) = 2$

Example: Graphs that define a Riemannian geometry in the continuum limit.
$\longrightarrow$ 'Pipe' from A to $\bar{A}$, where distinct 'bottlenecks' are disjoint.

Tensor network states: motivation

• *AdS/CFT correspondence:* Duality between quantum gravity theory in anti-de-Sitter space of dimension $d+1$ and quantum conformal field theory of dimension d.
$\longrightarrow$ *Holographic principle* that conjectures quantitative relations between geometric properties of the bulk and entanglement properties of the boundary.

• *Tensor networks:* Discrete toy-models for AdS/CFT correspondence that (1) are mathematically rigorous and tractable, (2) reproduce several of the conjectured formulas.

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Example: Holographic states are expected to satisfy an *area law of entanglement* (e.g. the Ryu-Takayanagi formula), which tensor network states (TNS) do by construction.

Indeed: Let $|\varphi_{\partial V}\rangle \in (\mathbf{C}^D)^{\otimes|\partial V|}$ be a TNS. Given a subset of boundary vertices $A \subset \partial V$, let ρ_A be the reduced state of $|\varphi_{\partial V}\rangle$ on $(\mathbf{C}^D)^{\otimes|A|}$, i.e. $\rho_A := \mathrm{Tr}_{\bar{A}}\left(|\varphi_{\partial V}\rangle\langle\varphi_{\partial V}|\right)$.

By construction, $\mathrm{rank}(\rho_A) \leqslant D^{\delta(A)} \ll D^{|A|}$, i.e. $S(\rho_A) \leqslant \delta(A)\log D \ll |A|\log D$.
↳ tensor rank of $|\varphi_{\partial V}\rangle$ across the bipartition $A:\bar{A}$

$\longrightarrow$ The entropy of ρ_A scales proportionally to the 'area' $\delta(A)$ and not to the 'volume' $|A|$.

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Remark: TNS are useful in many other fields, e.g. in *quantum condensed matter physics*, where physically relevant states (such as ground states of gapped local Hamiltonians) are also expected to satisfy an area law of entanglement (Hastings, Landau/Vazirani/Vidick).

Random tensor network states

$\rightarrow |\psi_e\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |u_i\rangle \otimes |u_i\rangle$

- Edge tensors $|\psi_e\rangle \in (\mathbf{C}^D)^{\otimes 2}$ are fixed. E.g. maximally entangled states.
- Vertex tensors $|\varphi_v\rangle \in (\mathbf{C}^D)^{\otimes d(v)}$ are picked at random. E.g. independent Gaussian tensors.

  $|\varphi_v\rangle$ has independent complex Gaussian entries with mean 0 and variance 1 $\leftarrow$

$\longrightarrow$ Resulting random boundary tensor $|\varphi_{\partial V}\rangle \in (\mathbf{C}^D)^{\otimes |\partial V|}$.

Note: We can show that: $\forall\, \varepsilon > 0,\ \mathbf{P}\left(\left| \| \varphi_{\partial V} \| - 1 \right| > \varepsilon \right) \leqslant e^{-c|V_0|(\sqrt{D}\varepsilon)^{1/|V_0|}}$.

$\hookrightarrow$ by a concentration inequality for polynomials in Gaussian variables

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Question: Given a subset of boundary vertices $A \subset \partial V$, what is the distribution of the random reduced state $\rho_A := \mathrm{Tr}_{\bar{A}}\left(|\varphi_{\partial V}\rangle\langle\varphi_{\partial V}|\right)$?
In particular, for large D, what is typically its spectrum and hence its entropy?

Known: In the case where $N(A) = 1$, for large D, ρ_A is expected to have close to maximal entropy, i.e. $\mathbf{E}\left(S(\rho_A)\right) \underset{D\to\infty}{=} \delta(A)\log D - o(1)$ (Hayden/Nezami/Qi/Thomas/Walter/Yang, Hastings).

$\longrightarrow$ What about the case where $N(A) > 1$? Is the asymptotic spectrum of ρ_A richer?

Motivation: Not all holographic states are expected to have a flat spectrum.

Outline

A few definitions from free probability

Definition (S-transform)

Given a probability distribution μ with density $d\mu$ on $\mathbf{R}$, its *S-transform* is the formal power series

$$S_\mu(z) := \frac{1+z}{z} M_\mu^{-1}(z), \text{ where } M_\mu(z) := \sum_{p=1}^{\infty} M_\mu^{(p)} z^p \text{ for } M_\mu^{(p)} := \int_{\mathbf{R}} x^p d\mu(x).$$

Fact: One-to-one correspondence between μ with compactly supported density and S_μ.

Definition (Free product)

Given probability distributions μ, ν with compactly supported densities on $\mathbf{R}$, their *free product* $\mu \boxtimes \nu$ is the unique probability distribution on $\mathbf{R}$ satisfying

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• Marčenko-Pastur distribution μ_{MP}: characterized by $S_{\mu_{MP}}(z) = \frac{1}{1+z}$.

It is supported on $]0,4]$ and has density $d\mu_{MP}(x) := \frac{\sqrt{4/x-1}}{2\pi} \mathbf{1}_{]0,4]}(x) dx$.

• Free product of N Marčenko-Pastur distributions $\mu_{MP}^{\boxtimes N}$: characterized by $S_{\mu_{MP}^{\boxtimes N}}(z) = \left(\frac{1}{1+z}\right)^N$.

It is supported on $]0, (N+1)^{N+1}/N^N]$ (Banica/Belinschi/Capitaine/Collins, Collins/Nechita/Życzkowski).

Given a Hermitian matrix M on $\mathbf{C}^d$, denote by $\mu_M := \frac{1}{d}\sum_{\lambda\in\mathrm{spec}(M)} \delta_\lambda$ its spectral distribution.

• Let $W_d = GG^*$ with G a $d\times d$ matrix whose entries are independent complex Gaussians with mean 0 and variance $1/d$ (i.e. W_d is a normalized Wishart matrix of size and parameter d).
• Let $W_{d,N} = HH^*$ with $H = G_1\times\cdots\times G_N$ and $G_1,\ldots,G_N$ independent $d\times d$ matrices whose entries are independent complex Gaussians with mean 0 and variance $1/d$.

Fact: $\mu_{W_d} \underset{d\to\infty}{\longrightarrow} \mu_{MP}$ and $\mu_{W_{d,N}} \underset{d\to\infty}{\longrightarrow} \mu_{MP}^{\boxtimes N}$.

Convergence in moments: $\forall\, p\in\mathbf{N}$, $\begin{cases} \frac{1}{d}\mathbf{E}\,\mathrm{Tr}(W_d^p) \underset{d\to\infty}{\longrightarrow} \mathrm{M}^{(p)}_{\mu_{MP}} = \mathrm{Cat}_p := \frac{1}{p+1}\binom{2p}{p} \\ \frac{1}{d}\mathbf{E}\,\mathrm{Tr}(W_{d,N}^p) \underset{d\to\infty}{\longrightarrow} \mathrm{M}^{(p)}_{\mu_{MP}^{\boxtimes N}} = \mathrm{FCat}_{p,N} := \frac{1}{Np+1}\binom{Np+p}{p} \end{cases}$.

(Catalan numbers; Fuss-Catalan numbers)

This actually implies convergence in probability of the spectral distribution ('weak' convergence).

Connections with random matrices and combinatorics of permutations

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(Cat_p → Catalan numbers; $\mathrm{FCat}_{p,N}$ → Fuss-Catalan numbers)

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Remark: $(\pi_1,\pi_2)\in\mathcal{S}(p)\times\mathcal{S}(p)\mapsto|\pi_1^{-1}\pi_2|\in\{0,\ldots,p-1\}$ is a distance.
($|\pi_1^{-1}\pi_2|$ → minimal number of transpositions needed to write $\pi_1^{-1}\pi_2$)

Hence by the triangle inequality, for any $\pi_1,\pi_2,\pi_3\in\mathcal{S}(p)$, $|\pi_1^{-1}\pi_2|+|\pi_2^{-1}\pi_3|\geqslant|\pi_1^{-1}\pi_3|$, with equality iff $\pi_1\to\pi_2\to\pi_3$ is a geodesic.

Cat_p counts the number of $\pi\in\mathcal{S}(p)$ s.t. $\mathrm{id}\to\pi\to\gamma$ is a geodesic.
(γ → ordered full cycle $(1,\ldots,p)$)

$\mathrm{FCat}_{p,N}$ counts the number of $(\pi_1,\ldots,\pi_N)\in\mathcal{S}(p)^N$ s.t. $\mathrm{id}\to\pi_1\to\cdots\to\pi_N\to\gamma$ is a geodesic.

Outline

Lemma (Limiting moments of random TNS)

Let $|\varphi_{\partial V}\rangle$ be a random TNS. For any $A \subset \partial V$, the random reduced state ρ_A is s.t.

$$\forall\, p \in \mathbf{N},\ \mathbf{E}\left(\mathrm{Tr}\left(\rho_A^p\right)\right) \underset{D\to\infty}{\sim} \mathrm{FCat}_{p,N(A)-1}\, D^{-\delta(A)(p-1)}.$$

Remark: In addition, $\dfrac{\mathbf{Var}\left(\mathrm{Tr}\left(\rho_A^p\right)\right)}{\left[\mathbf{E}\left(\mathrm{Tr}\left(\rho_A^p\right)\right)\right]^2} \underset{D\to\infty}{=} O\left(\dfrac{1}{D}\right)$. So $\mathrm{Tr}\left(\rho_A^p\right)$ concentrates around its average.

Limiting spectral distribution of random tensor network states

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Theorem (Limiting spectral distribution of random TNS)

Let $|\varphi_{\partial V}\rangle$ be a random TNS. For any $A \subset \partial V$, let ρ_A be the random reduced state and $\hat{\rho}_A$ be the restriction of ρ_A to its support. Set $\mu_A^{(D)} := \dfrac{1}{D^{\delta(A)}} \displaystyle\sum_{\lambda \in \mathrm{spec}(\hat{\rho}_A)} \delta_{D^{\delta(A)}\lambda}$. Then,

$$\mu_A^{(D)} \underset{D\to\infty}{\longrightarrow} \mu_{MP}^{\boxtimes(N(A)-1)} \text{ weakly.}$$

Remark: This means that, for any $f : \mathbf{R} \to \mathbf{R}$ continuous,

$$\forall\, \varepsilon > 0,\ \lim_{D\to+\infty} \mathbf{P}\left(\left|\int_{\mathbf{R}} f(x)\, d\mu_A^{(D)}(x) - \int_{\mathbf{R}} f(x)\, d\mu_{MP}^{\boxtimes(N(A)-1)}(x)\right| \leqslant \varepsilon\right) = 1.$$

Interpretation and consequences

Particular cases:

- If $N(A)=1$: $\mu_A^{(D)} \underset{D\to\infty}{\longrightarrow} \delta_1$, i.e. $\mathrm{spec}\left(\hat{\rho}_A\right) \underset{D\to\infty}{\simeq} \mathrm{spec}\left(\frac{I}{D^{\delta(A)}}\right)$, with I the identity of size $D^{\delta(A)}$.
  Note: In this case we can actually show strong convergence, i.e.
  $$\forall\, \varepsilon>0,\ \lim_{D\to\infty} \mathbf{P}\left(\mathrm{supp}\left(\mu_A^{(D)}\right)\subset[1-\varepsilon,1+\varepsilon]\right)=1.$$
- If $N(A)=2$: $\mu_A^{(D)} \underset{D\to\infty}{\longrightarrow} \mu_{MP}$, i.e. $\mathrm{spec}\left(\hat{\rho}_A\right) \underset{D\to\infty}{\simeq} \mathrm{spec}\left(\frac{W}{D^{\delta(A)}}\right)$, with W a normalized Wishart matrix of size and parameter $D^{\delta(A)}$.

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$\longrightarrow$ If $N(A)>1$, the asymptotic spectrum of $\hat{\rho}_A$ is not flat.

Corollary (Limiting entropy of random TNS)

Let $|\varphi_{\partial V}\rangle$ be a random TNS. For any $A \subset \partial V$, the random reduced state ρ_A is s.t.

$$\mathbf{E}\left(S(\rho_A)\right) \underset{D\to\infty}{=} \delta(A)\log D - \sum_{k=2}^{N(A)} \frac{1}{k} + o(1).$$

$\longrightarrow$ Area law of entanglement, with finite correction when $N(A)>1$.

Ingredients in the proof

Given $\pi \in \mathcal{S}(p)$, denote by U^π the associated unitary on $(\mathbf{C}^d)^{\otimes p}$.

- $\mathrm{Tr}\left(\rho_A^p\right) = \mathrm{Tr}\left(U^\gamma_{A^p} \rho_A^{\otimes p}\right) = \mathrm{Tr}\left(U^\gamma_{A^p} \otimes U^{\mathrm{id}}_{\bar{A}^p} |\varphi\rangle\langle\varphi|^{\otimes p}_{\partial V}\right) = \mathrm{Tr}\left(U^\gamma_{A^p} \otimes U^{\mathrm{id}}_{\bar{A}^p} \otimes |\varphi\rangle\langle\varphi|^{\otimes p}_{V_0} |\psi\rangle\langle\psi|^{\otimes p}_{E}\right)$
  - ↳ 'replica trick'
  - ↳ $\rho_A = \mathrm{Tr}_{\bar{A}}(|\varphi\rangle\langle\varphi|_{\partial V})$
  - ↳ $|\varphi_{\partial V}\rangle = \langle \varphi_{V_0} | \psi_E\rangle$

- For $|\varphi\rangle \in \mathbf{C}^d$ a Gaussian vector, $\mathbf{E}\left(|\varphi\rangle\langle\varphi|^{\otimes p}\right) = \sum_{\pi\in\mathcal{S}(p)} U^\pi$.

Hence: $\mathbf{E}\left(\mathrm{Tr}\left(\rho_A^p\right)\right) = \sum_{\substack{\pi_x\in\mathcal{S}(p),\, x\in V \\ \pi_x=\gamma,\, x\in A \\ \pi_x=\mathrm{id},\, x\in\bar{A}}} \mathrm{Tr}\left(\bigotimes_{x\in V} U^{\pi_x}_{x^p} |\psi\rangle\langle\psi|^{\otimes p}_E\right) = \sum_{\substack{\pi_x\in\mathcal{S}(p),\, x\in V \\ \pi_x=\gamma,\, x\in A \\ \pi_x=\mathrm{id},\, x\in\bar{A}}} D^{-w((\pi_x)_{x\in V})}$,

with $w((\pi_x)_{x\in V}) = \sum_{(x,y)\in E} |\pi_x^{-1}\pi_y|$, since $\forall\, (x,y)\in E,\ \mathrm{Tr}\left(U^{\pi_x}_{x^p} \otimes U^{\pi_y}_{y^p} |\psi\rangle\langle\psi|^{\otimes p}_{xy}\right) = D^{-|\pi_x^{-1}\pi_y|}$.

Now, for any $(\pi_x)_{x\in V}$, $w((\pi_x)_{x\in V}) \geqslant \delta(A)(p-1)$, with equality iff there is a geodesic path $\gamma = \pi_0 \to \pi_1 \to \cdots \to \pi_{N(A)-1} \to \pi_{N(A)} = \mathrm{id}$ s.t. for all $0 \leqslant i \leqslant N(A)$ and all $x \in V_i$, $\pi_x = \pi_i$.

Therefore: $\mathbf{E}\left(\mathrm{Tr}\left(\rho_A^p\right)\right) \underset{D\to\infty}{\sim} \mathrm{FCat}_{p,N(A)-1} D^{-\delta(A)(p-1)}$.

Generalizations

- What about the case where the edge tensors have different local dimensions D_e?
  If all of them are of the same order D, i.e. $D_e = \alpha_e D$, then the results can be quite straightforwardly generalized.

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- What about the case where the edge tensors $|\psi_e\rangle \in (\mathbf{C}^D)^{\otimes 2}$ are not maximally entangled?
  If they have bounded entanglement spectrum, i.e. (almost) all their Schmidt coefficients are of order $1/\sqrt{D}$, then the results can be quite straightforwardly generalized.
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  To understand the regime of unbounded entanglement spectrum, different tools are needed.
- What about the case where the minimal cuts are not necessarily edge-disjoint?
  The minimizing configurations of permutations can still be identified, but counting them may become cumbersome.

Future directions

- What about estimating other quantities than entropies of random boundary states?

  Example: For $A, B \subset \partial V$ with $A \cap B = \emptyset$ and $A \cup B \neq \partial V$, it is known that the average *mutual information* and *entanglement negativity* of the random bipartite boundary state ρ_{AB} are non-vanishing as D grows iff $\delta(AB) < \delta(A) + \delta(B)$.

  But what are the conditions for ρ_{AB} to be typically *entangled or separable*, satisfying or not a given *entanglement criterion*, etc?
  In the simplest case of a 'network' having one bulk vertex, i.e. a (normalized) Gaussian tensor $|\varphi_{ABC}\rangle \in \mathbf{C}^{d_A} \otimes \mathbf{C}^{d_B} \otimes \mathbf{C}^{d_C}$ as boundary state, *threshold phenomena* have been identified for properties such as separability, PPT, realignment, reduction, extendibility, etc (Aubrun/Szarek/Ye, Aubrun, Aubrun/Nechita, Jivulescu/Lupa/Nechita, Lancien).
  $\longrightarrow$ Can these results be generalized to more complicated networks?

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- What about implications in terms of *quantum error-correcting codes*?

  Setting: Add one non-contracted leg to each bulk vertex tensor, and view the resulting tensor as a map from the bulk (*'logical'*) space to the boundary (*'physical'*) space.
  It is known that, if $N(A) = 1$, the *entanglement wedge* of A is protected against errors in $\bar{A}$ (Harlow/Pastawki/Preskill/Yoshida, Hayden/Nezami/Qi/Thomas/Walter/Yang).
  $\longrightarrow$ What happens in the case of non-unique min-cuts?

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