Counting problems for random tensors

Let's consider two types of random tensors

① We say that $A \in \mathbb{R}^{n_1 \times n_2 \times \dots \times n_d}$

$A = (a_{i_1 \dots i_d})$ is a Gaussian tensor,

if $a_{1\dots 1}, \dots, a_{n_1, \dots, n_d} \overset{iid}{\sim} N(0,1)$,

i.e. the entries of A are iid standard.

In the case $n := n_1 = \dots = n_d$ we have the action of the symmetric group S_d on $\mathbb{R}^{n \times \dots \times n}$.

$$\pi . (a_{i_1, \dots, i_d}) := (a_{i_{\pi(1)}, \dots, i_{\pi(d)}}) \quad , \quad \pi \in S_d$$

(generalization of transposition). We call A symmetric, if $\pi . A = A$ for all $\pi \in S_d$.

② We say that a symmetric tensor $A \in \mathbb{R}^{n \times \dots \times n}$

is a Gaussian symmetric tensor, if

$$A = \frac{1}{d!} \sum_{\pi \in S_d} \pi . B \; ,$$

where B is Gaussian.

Example $d=2$: Gaussian symmetric matrices = GOE

Write $\langle A, B\rangle := \sum_{i_1,\dots,i_d} a_{i_1,\dots,i_d}\, b_{i_1,\dots,i_d}$

$$\|A\| = \sqrt{\langle A, A\rangle}$$

Then, the prob. density for Gaussian tensors is $\propto e^{-1/2\|A\|^2}$.

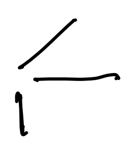
Let me discuss the following two counting problems:

- expected rank
- expected number of zeros of a systems of polynomials.

Expected rank

Def • Let $A \in \mathbb{R}^{n_1\times\dots\times n_d}$. We say that A has rank-one, if

$$A = a_1 \otimes \dots \otimes a_d = \left((a_1)_{i_1} \cdots (a_d)_{i_d} \right).$$

$a_i \in \mathbb{R}^{n_i}$, so $A =$  $(d=3)$.

• We define

$$\operatorname{rank}(A) := \min\{r \mid A = A_1 + \dots + A_r,\ \operatorname{rank}(A_i) = 1\}$$

- We say that r is a typical rank, if

$$\operatorname*{Prob}_{A \text{ Gaussian}} \{\operatorname{rank}(A) = r\} > 0.$$

Ten Berge (1991): typical ranks in $\mathbb{R}^{n \times n \times 2}$ are n and $n+1$

<u>Theorem</u> (Bergqvist, 2013)

$$\operatorname*{Prob}_{A \text{ Gaussian} \in \mathbb{R}^{2\times2\times2}} \{\operatorname{rank}(A) = 2\} = \pi/4.$$

<u>Kinematic Formula</u>

space of lines in $\mathbb{R}^{n+1}$ through 0.

Let $\mathbb{R}P^N := (\mathbb{R}^{N+1} \setminus \{0\}) / \sim$ be real projective space, where $x \sim y$ iff $x \parallel y$.

We have a 2:1 map $\pi: S^N \to \mathbb{R}P^N$, $x \mapsto [x]$

For $A \subset \mathbb{R}P^N$ we define

$$\operatorname{vol}(A) := \frac{1}{2} \operatorname{vol}(\pi^{-1}(A))$$

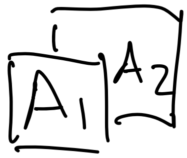
Let now $L \subset \mathbb{R}P^N$ be a linear space of codimension m and $M \subset \mathbb{R}P^N$ be a submanifold of dimension m. Then (Howard 1993):

$$\mathop{\mathbb{E}}_{U \sim \mathrm{Unif}(\underbrace{O(N+1)}_{\text{orth. group}})} \#(M \cap U \cdot L) = \frac{\mathrm{vol}(M)}{\mathrm{vol}(\mathbb{R}P^m)}.$$

<u>Back to Bergqvist</u>:

Ten Berge showed that for $A \in \mathbb{R}^{n \times n \times 2}$ with prob. 1:

$$\mathrm{rank}(A) = n \iff \det(A_1 + tA_2) = 0 \text{ has } n \text{ real zeros}$$

$A = [A_1 | A_2] =$ 

Now, important observation: $L := \{sA_1 + tA_2 \mid s,t \in \mathbb{R}\} \subseteq \mathbb{R}^{n \times n}$

$\leadsto$ descends to a random line in $\mathbb{R}P^{n^2-1}$.
and its $O(n^2)$-invariant. And there is a <u>unique</u> $O(n^2)$-invariant prob. distribution on lines in $\mathbb{R}P^{n^2-1}$

$$\leadsto \mathbb{E}\, \#\{t \mid \det(A_1 + tA_2) = 0\} = \frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{R}P^{n^2-2})}$$

$X = \{\det = 0\}$

For $2 \times 2 \times 2$: with prob 1 we either have 2 real intersection points or zero.

$\rightsquigarrow 2 \cdot \mathbb{P}\text{rob} \{ \det(A_1 + tA_2) = 0 \text{ has 2 real zeros} \}$

$= \mathbb{E}\, \# \{ t \mid \det(A_1 + tA_2) = 0 \}$

$$= \frac{\text{vol}(\{\det = 0\})}{\text{vol}(\mathbb{RP}^2)} = \pi/2$$

$$\rightsquigarrow \mathbb{P}\text{rob} \{ \text{rank}(A) = 2 \} = \frac{\pi/2}{2} = \pi/4.$$

Expected number of zeros

Observation: there is a one-to-one correspondence between symmetric tensors and homogeneous polynomials in n variables of degree d.

$$A \in \mathbb{R}^{\overbrace{n \times \dots \times n}^{d}} \longleftrightarrow f_A(X) = \langle A, X \otimes \dots \otimes X \rangle$$

$$= \sum a_{i_1 \dots i_d} x_{i_1} \cdots x_{i_d}$$

Then, for $A_1, \dots, A_{n-1}$ iid Gaussian symmetric tensors in $\mathbb{R}^{n \times \dots \times n}$

$$\mathbb{E}\, \# \{ X \in \mathbb{RP}^{n-1} \mid f_{A_1}(x) = \dots = f_{A_{n-1}}(x) = 0 \}$$

$$= \frac{\text{vol}(V_{d,n})}{\text{vol}(\mathbb{RP}^{n-1})},$$

where $V_{d,n} = \{ [X \otimes \dots \otimes X] \mid X \in \mathbb{R}^n \setminus \{0\} \}$

One can compute that $\operatorname{vol}(V_{d,n}) = \sqrt{d}^{\,n-1} \operatorname{vol}(\mathbb{RP}^{n-1})$

$\Rightarrow \mathbb{E}\, \# \{ x \in \mathbb{RP}^{n-1} \mid f_{A_1}(x) = \dots = f_{A_{n-1}}(x) = 0 \}$
$= \sqrt{d}^{\,n-1}$

first observed by Kostlan-Edelman-Shub.