Renormalization group maps for Ising models and tensor networks

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Outline

Joint work with Slava Rychkov

- Wilson-Kadanoff RG (real-space RG)
- Tensor networks
- Simple RG map for tensor network
- CDL problem - instability of high T fixed point
- Better RG map - disentangler
- Low temperatures - first order transition
- Outlook - second order transition

arXiv:2107.11464, arxiv:2210.06669

Thanks for the invitation. Ask questions!

Wilson-Kadanoff RG (real space RG)

Ising type models: spins take on only values ± 1.
Nearest neighbor interaction

$$H(\sigma) = -\beta \sum_{<i,j>} \sigma_i \sigma_j$$

More general interaction

$$H(\sigma) = \sum_{Y} d(Y)\,\sigma(Y), \qquad \sigma(Y) = \prod_{i \in Y} \sigma_i$$

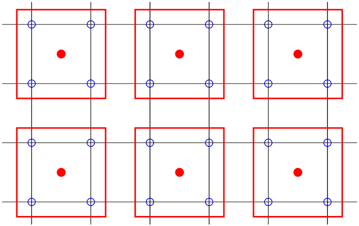
where the sum is over all finite subsets including the empty set.

Note that β has been absorbed into the Hamiltonian.

Blocking

Lattice divided into blocks; each block assigned a block spin variable.

Block spins also take on only the values ± 1.



Wilson Kadanoff RG

Original spins: σ Block spins: $\overline{\sigma}$

RG Kernel: $T(\overline{\sigma}, \sigma)$, e.g., majority rule

Satisfies

$$\sum_{\overline{\sigma}} T(\overline{\sigma}, \sigma) = 1, \quad \forall \sigma$$

for all original spin configurations σ.

Renormalized Hamiltonian $\bar{H}(\overline{\sigma})$ is formally defined by

$$e^{-\bar{H}(\overline{\sigma})} = \sum_{\sigma} T(\overline{\sigma}, \sigma)\, e^{-H(\sigma)}$$

Note: β has been absorbed into the Hamiltonians.

Key point: only makes sense in finite volume.

RG maps preserves Z

$$\sum_{\overline{\sigma}} T(\overline{\sigma},\sigma) = 1 \quad \forall \sigma, \qquad\qquad e^{-\bar{H}(\overline{\sigma})} = \sum_{\sigma} T(\overline{\sigma},\sigma)\, e^{-H(\sigma)}$$

$$\sum_{\overline{\sigma}} e^{-\bar{H}(\overline{\sigma})} = \sum_{\sigma} e^{-H(\sigma)}$$

So free energy of the original model can be recovered from the renormalized Hamiltonian.

Study the critical behavior of the system by studying iterations of the renormalization group map:

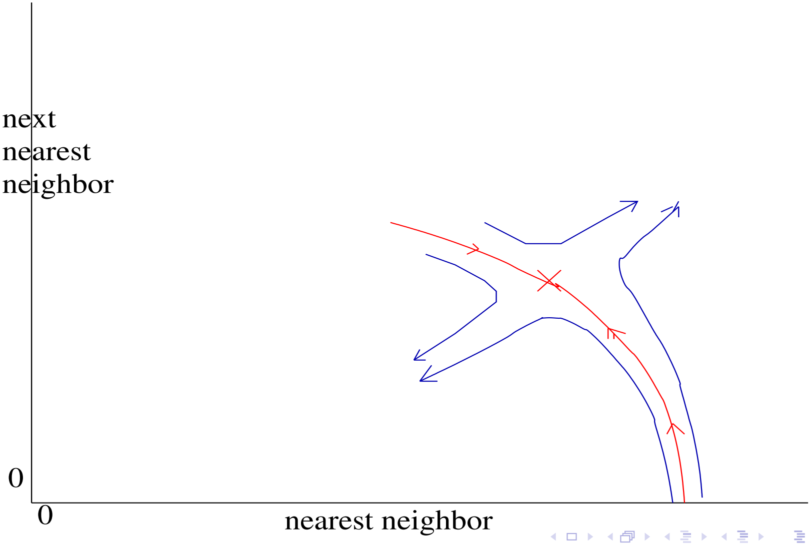
$$\mathcal{R}(H) = \bar{H}$$

What is known about this map?

Physicist: lots!

Has a fixed point with stable manifold of co-dim 2

Eigenvalues of linearization $> 1 \implies$ critical exponents



Mathematician: very little

Existence of map at high temp or large magnetic field
Griffiths and Pearce; Israel; Kashapov; Yin

Non-existence of map at low temp for various kernels
Griffiths and Pearce; Israel; van Enter, Fernández and Sokal

Non-existence of map near critical temp for some kernels

Essentially no results even for first iteration of the map near critical surface !

Numerical studies

Wilson (Rev. Mod. Phys. 1975) - 217 terms in H !
"A number of details are omitted."

Lots of Monte Carlo studies using Wilson-Kadanoff RG

Swendsen: compute the linearization of the RG map from correlation functions. Avoids computing $\mathcal{R}$ itself.

Brandt,Ron,Swendsen Saw significant dependence of $\bar{H}$ on truncation method.

"Even though the individual multispin interactions usually have smaller coupling constants than two-spin interactions, the fact that they are very numerous can lead to multispin interactions dominating the effects of two-spin interactions."

Open problems for real space RG maps

1. Study the first order transition at low temperatures - probably not doable with this type of RG. Contour methods - Gawedzki, Kotecký, Kupiainen

2. Develop a systematic numerical approach to compute the RG map.

3. Prove there is a Banach space of Hamiltonians and a rigorously defined RG map on it which has a non-trival fixed point with a stable manifold of co-dimension two.

Tensor network

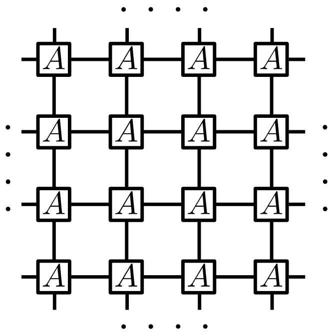
Let H be a real Hilbert space (finite or infinite dimensional)
A tensor (of order 4) is a map

$$A : H \times H \times H \times H \to \mathbb{R}$$

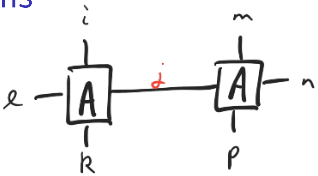
which is linear in each argument. Let e_i be o.n. basis for H.

$$A_{ijkl} = A(e_i, e_j, e_k, e_l)$$

Tensor network is formed by contracting copies of A:



Contractions



$$\sum_{j} A_{ijkl} \, A_{mnpj}$$

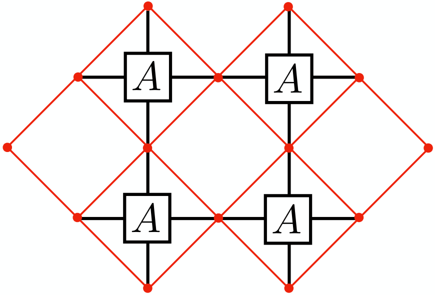
Contractions - cont

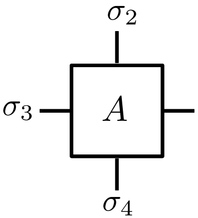
With periodic boundary conditions contraction of network gives a number - like a partition function.

Rescaling: multiply A by a scalar and contraction of network changes just by power of that scalar.

Analagous to adding a constant term to the Hamiltonian.

Ising model as tensor network - cond matter

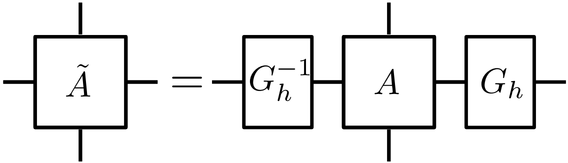




$$\sigma_1 = e^{\beta(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)}$$

Gauge transformations

Let G_h be invertible 2-leg tensor (matrix). Define $\tilde{A}$ by



Contraction of $\tilde{A}$ network is same as contraction of A network.

Infinite temperature tensor

Infinite temperature ($\beta = 0$) Ising tensor is gauge equivalent to tensor with only one non-zero component : $A_{0000} = 1$.

$$A(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \exp(\beta[\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1]) = 1$$

Original basis $|+\rangle, |-\rangle$

New basis $|0\rangle, |1\rangle$

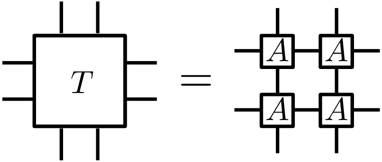
$$|0\rangle = [|+\rangle + |-\rangle]/\sqrt{2}$$

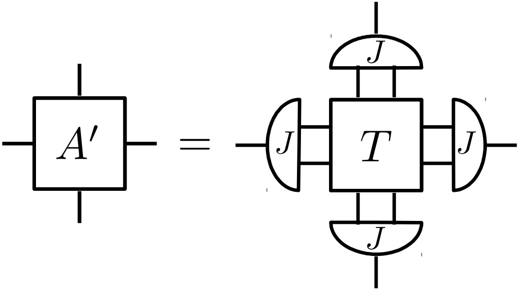
$$|1\rangle = [|+\rangle - |-\rangle]/\sqrt{2}$$

$$A^{G}_{0000} = 1, \quad A^{G}_{ijkl} = 0 \quad ijkl \neq 0000$$

Simplest RG for tensor network

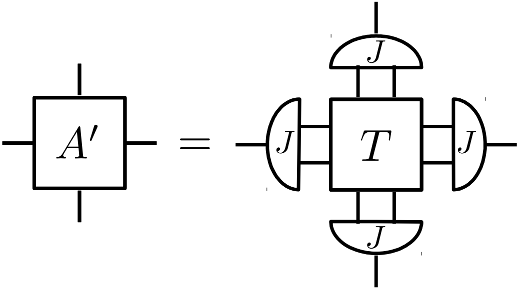
Levin,Nave (2007)





J is isometry of $H \otimes H$ onto H'. $\quad dim(H') = dim(H)^2$

Simplest RG for tensor network



J is isometry of $H \otimes H$ onto H'. Many such isometries.
This freedom is equivalent to a gauge transformation.

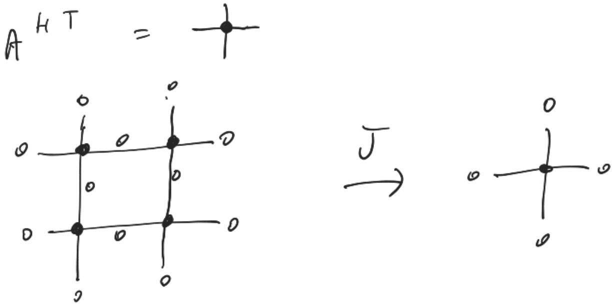
Wilson-Kadanoff vs tensor network RG

- *Growth of number of variables*
  WK RG: Spins only have two values but Hamiltonian becomes non-local with many multi-body terms
  TN RG: Tensor stays local, but leg dimension grows
- *Computability of RG map*
  WK RG: no explict way to compute it - ∞ volume limit
  TN RG: Explicitly computable, but disentanglers complicate it

High temperature fixed point

Let A^{HT} be tensor with one nonzero component $A^{HT}_{0000} = 1$.

Assume $J(e_0 \otimes e_0) = e_0$. Then A^{HT} is a fixed point of RG.



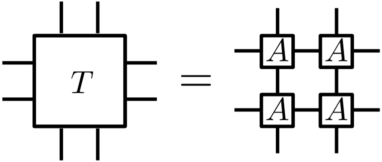
Is it stable?

Norms

Use Hilbert-Schmidt (Frobenius) norm:

$$||A||^2 = \sum_{ijkl} A_{ijkl}^2$$

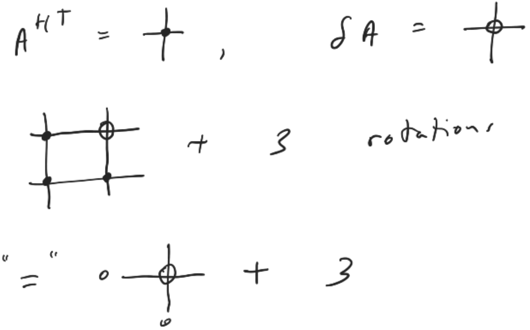
If A, B are tensors of any order and C is formed by contracting some indices of A with some indices of B, then by Cauchy-Schwaz inequality $||C|| \leq ||A|| ||B||$.



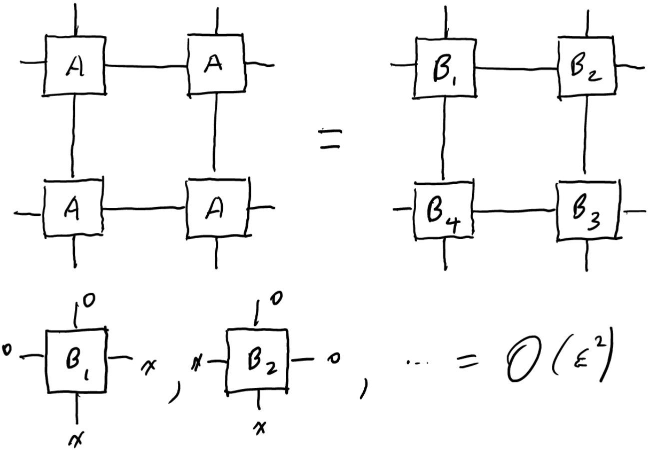
$$||T|| \leq ||A||^4$$

CDL Problem (Corner double line)

Now perturb A^{HT}: $A = A^{HT} + \delta A$, $||\delta A|| = O(\epsilon)$.
Compute A' to first order:

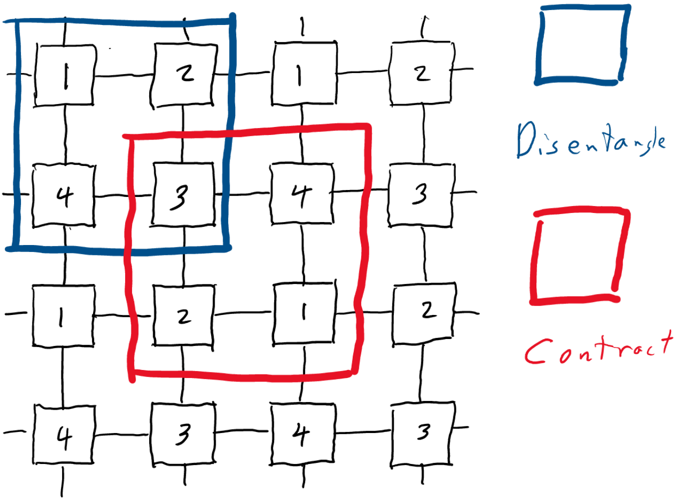


Disentanglers



0: leg fixed to 0 index, x: leg can only be nonzero index
subtensor

Disentanglers - cont



Stability of high temp fixed point

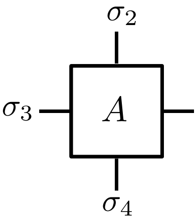
Theorem:

There is a tensor RG map such that if $A = A^{HT} + \delta A$ with $||\delta A||$ small, then the image has the form $A' = A^{HT} + \delta A'$ with

$$||\delta A'|| \leq C||\delta A||^{3/2}$$

(The tensor A is normalized so that $A_{0000} = 1$ and the RG map includes a normalization step so that $A'_{0000} = 1$.)

Low temperature tensor



$$= e^{\beta(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)+h(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}$$

For large β and h of order one, two significant components

$$A_{++++} = e^{4\beta+4h}, \quad A_{----} = e^{4\beta-4h}$$

Let A^+, A^- be tensors with one non-zero component

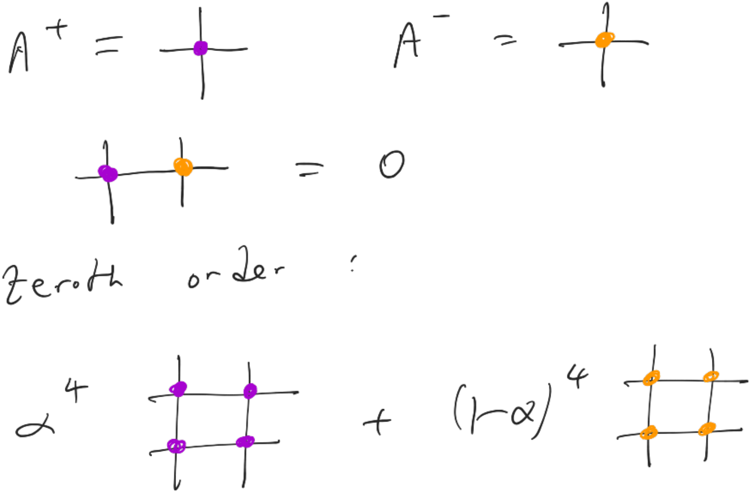
$$A^+_{++++} = 1, \quad A^-_{----} = 1$$

Then after a rescaling

$$A = \alpha A^+ + (1-\alpha)A^- + B, \quad B = O(e^{-4\beta})$$

Zeroth order RG

$$A = \alpha A^+ + (1-\alpha)A^- + O(e^{-4\beta}), \quad \alpha = e^{4h}/(e^{4h}+e^{-4h})$$



$$A' = \alpha^4 A^+ + (1-\alpha)^4 A^- + B'$$

Rescale

$$A' = \alpha' A^+ + (1-\alpha') A^- + B'$$

$$\alpha' = r(\alpha) = \frac{\alpha^4}{\alpha^4 + (1-\alpha)^4} + O(||B||^4)$$

There is a CDL problem here too. With suitable disentangler we can show there is a $\rho < 1$ such that

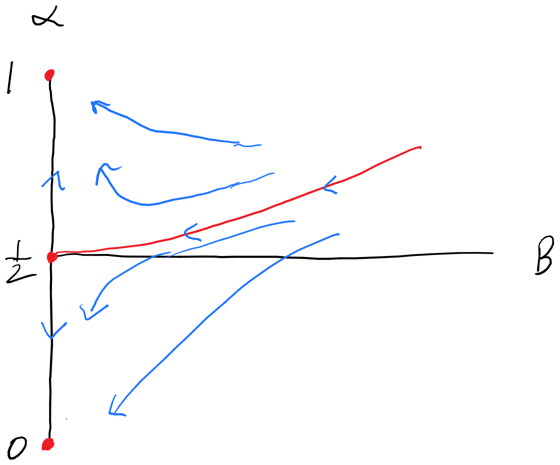
$$||B'|| \leq \rho ||B||$$

Theorem (RG flow)

Theorem: There is an $\epsilon > 0$ such that for $\alpha \in [0, 1]$ and B with $||B|| < \epsilon$ the RG map $(\alpha, B) \to (\alpha', B')$ satisfies
$\alpha' = r(\alpha) + O(B^2), \quad B' = \epsilon^{1/2}\, O(B)$
The map has two stable fixed points at $(\alpha, B) = (0, 0), (1, 0)$ and one unstable fixed point at $(1/2, 0)$.
There is a function $\alpha_c(B)$ such that $\{\alpha_c(B), B) : ||B|| < \epsilon\}$ is the local stable manifold. The stable manifold has co-dimension 1. (Corresponds to phase coexistence curve.)

Ising model: Theorem applies for large β and arbitrary h.

Picture RG flow



RG flow cont

Corollary For $\alpha \in [0,1]$ and $||B|| < \epsilon$, the free energy is analytic off the stable manifold and continuous across it. The magnetization is discontinuous across the stable manifold.

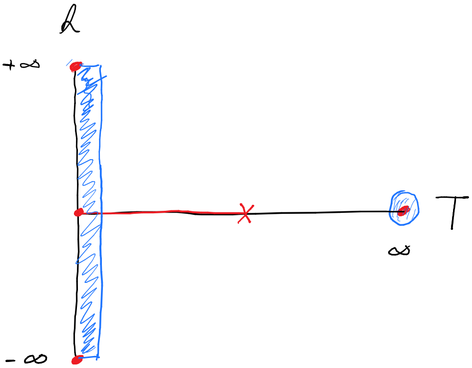
Remarks:

- Corollary has been proven before by other methods. Here you see that the discontinuity in the magnetization arises from the discontinuity in the RG trajectory as the initial tensor crosses the stable manifold.
- In physics literature the unstable fixed point is called a discontinuity fixed point of the RG map (Nienhuis-Nauenberg 1975).

Summary

Back to Ising model:

$$H = \beta \sum_{<ij>} \sigma_i \sigma_j + h \sum_i \sigma_i, \quad \beta = 1/T$$



Outlook

Presented some first steps in the rigorous study of tensor RG maps without truncation.

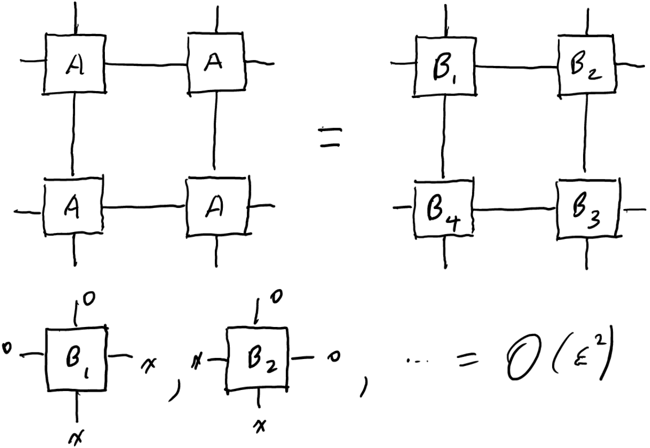
Goal : prove there is a RG map for tensor networks with a non-trivial fixed point which describes the second order transition in Ising type models.

There are many tensor RG maps that have been studied numerically. Which one is best for above?
How good is the approximate fixed point?

Appendices

Sketch of the proof - 1

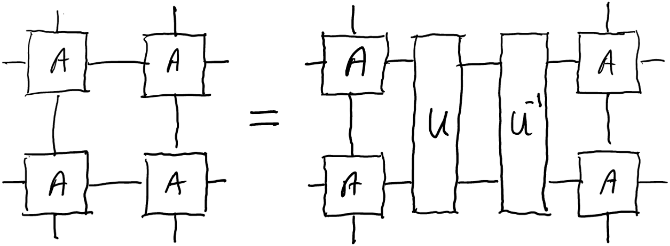
Recall the disentangler :



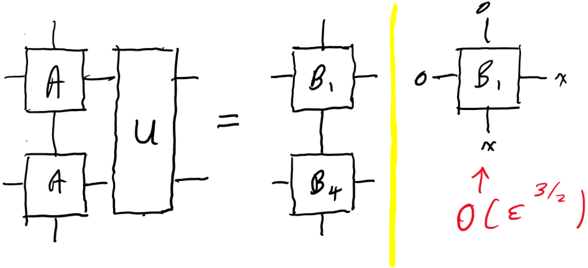
This reduces the proof to proving the existence of the disentangler.

NB: We will cheat a bit in the following - more on this later

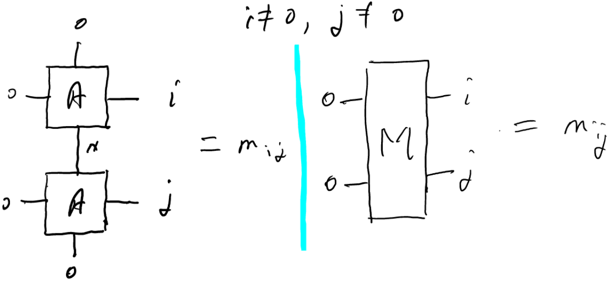
Sketch of the proof - 2



Sketch of the proof -3



Sketch of the proof -4



$$M = \sum_{i \neq 0, j \neq 0} m_{ij} \, \left|\begin{smallmatrix}0\\0\end{smallmatrix}\right\rangle \left\langle\begin{smallmatrix}i\\j\end{smallmatrix}\right|$$

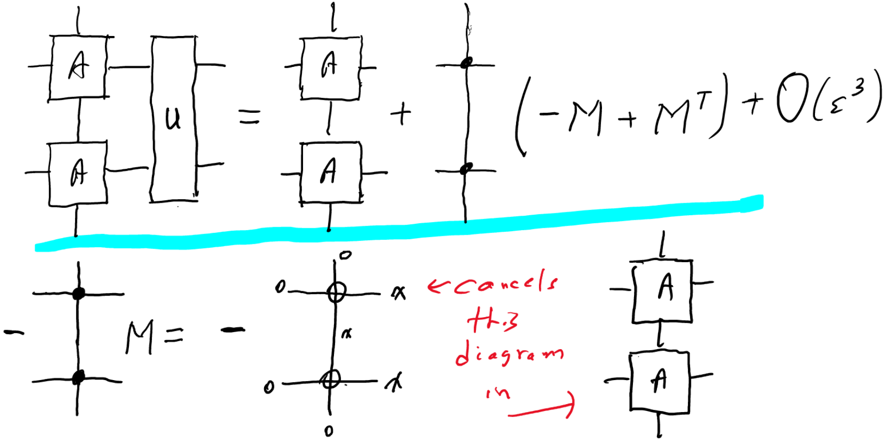
$$U = \exp(-M + M^T)$$

Note $||M|| = O(\epsilon^2)$.

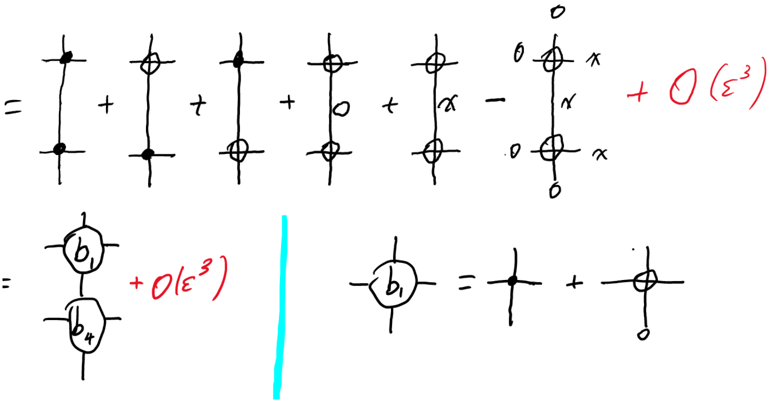
$$U = I - M + M^T + O(\epsilon^4)$$

Sketch of the proof -5

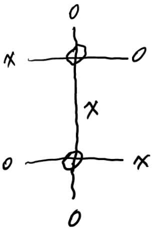
Compute to order $O(\epsilon^2)$



Sketch of the proof - 6



Sketch of the proof - the cheat



not cancelled

not $O(\varepsilon^3)$

Lattice gas variables

RG calculations usually done using the spin variables $\sigma_i = \pm 1$.

lattice gas variables: $n_i = (1 - \sigma_i)/2$ which take on the values $0, 1$.

In lattice gas variables

$$\bar{H}(\bar{n}) = \sum_Y c(Y)\, \bar{n}(Y), \qquad \bar{n}(Y) = \prod_{i \in Y} \bar{n}_i$$

Y summed over all finite subsets of block spins
Take H to be n.n. critical Ising
You can compute the $c(Y)$ very accurately.
Compute them for about $10,000$ $Y's$.
Order by decreasing $|c(Y)|$ and plot.
arXiv:0905.2601

Decay of lattice gas coefs

Bottom curve: $|c(Y_n)|$ vs. n. Top curve: $\sum_{i=n}^{N} |c(Y_n)|$ vs. n.

Two lines : $c2^{-n/850}$