Tensor eigenvalue/vector distributions via field theoretical methods

Naoki Sasakura

Yukawa Institute for Theoretical Physics, Kyoto University

Based on
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NS, ArXiv:2210.15129 [hep-th]

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§ Introduction

Eigenvalue distributions of matrix models play important roles in understanding atoms, 2-dim quantum gravity, QCD, etc.

H ~ random matrix : Semicircle law E.Wigner 1958

Solving matrix models via $\rho(e)$

E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, 1978

Gross-Witten-Wadia transition, topological change of $\rho(e)$

D. J. Gross and E. Witten, S. R. Wadia, 1980

What roles eigenvalue/vector distributions can take in tensor models ?

Tensor eigenvalue/vector distributions are previously studied in

- Expectation numbers of real tensor eigenvalues

  P. Breiding, SIAM Journal on Applied Algebra and Geometry 1, 254-271 (2017).
  P. Breiding, Transactions of the American Mathematical Society 372, 7857-7887 (2019).

- Estimation of the largest eigenvalue

  O. Evnin, Lett. Math. Phys. 111, 66 (2021) doi:10.1007/s11005-021-01407-z [arXiv:2003.11220 [math-ph]].

- Extension of Wigner semicircle law

  R. Gurau, [arXiv:2004.02660 [math-ph]].

The problem of tensor eigenvalue distribution is essentially the same as counting the critical points of the Hamiltonian (complexity) of the spherical *p*-spin model for spin glasses.

$$H = C_{abc} w_a w_b w_c, \quad w_a w_a = 1 \qquad C_{abc} : \text{random (Gaussian)}$$

$(p = 3)$

This has comprehensively been solved vie matrix model techniques in

Auffinger, A., Arous, G.B. and Černý, J. (2013), "Random Matrices and Complexity of Spin Glasses." Comm. Pure Appl. Math., 66: 165-201. https://doi.org/10.1002/cpa.21422

Accordingly, the end results of this talk are not new. However, the method we use is different, i.e., field theoretical, and may give different insights from previous studies in the future.

In this talk, we consider real symmetric order-three tensor C_{abc}

Tensor eigenvalues/vectors of C_{abc} :

$$C_{abc}v_b v_c = \zeta\, v_a \qquad \zeta : \text{Eigenvalue} \quad v_a : \text{Eigenvector}$$

L.Qi 2005, L.H.Lim 2005, D.Cartwright and B.Sturmfels 2013

There exist some differences from the matrix case:

- A system of N non-linear equations
- Not unique: can be rescaled by $\zeta \to c\,\zeta,\ v_a \to c\, v_a$
- Even if C_{abc} is real, ζ, v_a are not necessarily real.

Accordingly there are some different notions of eigenvalues/vectors.

Ex. Z-eigenvalue (Qi) : $\zeta\ (\geq 0)$ with real v_a $(v_a v_a = 1)$

In this talk we consider symmetric real order-three tensors with Gaussian distributions, and compute the distributions of eigenvectors and eigenvalues:

C_{abc} : symmetric real tensor, Gaussian distribution

Eigenvector distribution → Eigenvalue distribution

$$C_{abc}v_b v_c = v_a$$

v_a : real

$$\zeta = 1$$

$$\zeta = \frac{1}{|v|}$$

$$C_{abc}w_b w_c = \zeta w_a \quad w_a = \frac{v_a}{|v|}$$

(Z-eigenvalues of $\zeta > 0$)

What is new in this talk is that we use field theoretical methods instead of matrix models.

§ Field theoretical expression

Eigenvector distribution for a given C :

$$\rho(v, C) = \sum_{i=1}^{\#\mathrm{sol}(C)} \prod_{a=1}^{N} \delta(v_a - v_a^i)$$

$$v_a^i - C_{abc} v_b^i v_c^i = 0 \qquad v_a^i \in \mathbb{R}$$

$$i = 1,2,\cdots,\#\mathrm{sol}(C)$$

$$= |\,\mathrm{Det} M\,| \prod_{a=1}^{N} \delta(C_{abc} v_b v_c - v_a)$$

$$M_{ab} = \frac{\partial}{\partial v_a}(v_b - C_{bcd} v_c v_d) = \delta_{ab} - 2C_{abc} v_c \qquad \text{: Hessian}$$

Eigenvector distribution for C_{abc} with Gaussian distribution :

$$\rho(v) = \langle \rho(v, C) \rangle_C = A^{-1} \int_{\mathbb{R}^{\#C}} dC\, e^{-\alpha C^2} \rho(v, C)$$

$$\alpha > 0 \quad C^2 = C_{abc} C_{abc}$$

We employ the following three different ways to compute $\rho(v)$ (or similar quantity) with different treatments of $|\mathrm{Det}M|$.

(1) Just ignore taking the absolute value

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$$\rho_{\mathrm{sign}}(v, C) = \mathrm{Det}M \prod_{a=1}^{N} \delta(C_{abc}v_b v_c - v_a)$$
$$= \sum_{i=1}^{\#\mathrm{sol}(C)} (-1)^{k(M_i)} \prod_{a=1}^{N} \delta(v_a - v_a^i)$$

$k(M_i)$: the number of negative eigenvalues of M at $v = v^i$

Field theoretical expressions:

$$\mathrm{Det}M = \int d\bar{\psi} d\psi \, e^{\bar{\psi} M \psi}$$

$$\prod_{a=1}^{N} \delta(C_{abc}v_b v_c - v_a) = (2\pi)^{-N} \int_{\mathbb{R}^N} d\lambda \, e^{i\,\lambda_a (v_a - C_{abc} v_b v_c)}$$

(2) Analytic continuation via replica trick

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$$\rho_R(v,C) = \left\{\mathrm{Det}(M^2+\epsilon\, I)\right\}^R \prod_{a=1}^{N} \delta(C_{abc}v_b v_c - v_a)$$

$$\left\{\mathrm{Det}(M^2+\epsilon I)\right\}^R \to |\,\mathrm{Det}\, M\,| \text{ by } R \to 1/2,\ \epsilon \to +0$$

ϵ needed for unique determination

The determinant part can be expressed as

Linear in C

$$\left\{\mathrm{Det}(M^2+\epsilon I)\right\}^R = (-1)^{NR}\int d\bar\psi d\psi d\bar\varphi d\varphi\, e^{-\bar\varphi^i\varphi_i+\epsilon\bar\psi^i\psi_i-\bar\psi^i M\psi_i-\bar\varphi^i M\varphi_i}$$

Fermions : $\bar\psi^i_a,\ \psi_{ia},\ \bar\varphi^i_a,\ \varphi_{ia}\ \ (i=1,2,\cdots,R,\ a=1,2,\cdots,N)$

Two kinds of fermions are introduced for linearity in M.

(3) Introducing both bosons and fermions

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$$\rho_\epsilon(v, C) = \frac{\mathrm{Det}(M^2 + \epsilon\, I)}{\sqrt{\mathrm{Det}(M^2 + \epsilon\, I)}} \prod_{a=1}^{N} \delta(C_{abc} v_b v_c - v_a)$$

$$\lim_{\epsilon \to +0} \frac{\mathrm{Det}(M^2 + \epsilon\, I)}{\sqrt{\mathrm{Det}(M^2 + \epsilon\, I)}} = |\,\mathrm{Det} M\,|$$

Rewrite numerator by fermions and denominator by bosons.

$$\frac{\mathrm{Det}(M^2 + \epsilon\, I)}{\sqrt{\mathrm{Det}(M^2 + \epsilon\, I)}} = (-\pi)^{-N} \int d\phi d\sigma d\bar{\psi} d\psi d\bar{\varphi} d\varphi\, e^{-S}$$

$$S = \sigma^2 + 2i\sigma M\phi + \epsilon\phi^2 + \bar{\varphi}\varphi + \bar{\psi} M \varphi + \bar{\varphi} M \psi + \epsilon \bar{\psi}\psi$$

σ_a, ϕ_a $(a = 1,2,\cdots,N)$: bosons $\quad \bar{\psi}_a, \psi_a, \bar{\varphi}_a, \varphi_a$: fermions

In either case of (1),(2),(3), what we want to compute has generally the form:

$$\rho_{\cdot}(v) = \int dC\, d\lambda\, d\bar{\psi} d\psi\, d\phi \cdots \;\; e^{S}$$

$$M_{ab} = \delta_{ab} - 2C_{abc}v_c$$

$$S = -\alpha C^2 + i\lambda_a(v_a - C_{abc}v_b v_c) - (\bar{\psi},\psi,\phi,\cdots)^2 - (\bar{\psi},\psi,\phi,\cdots)M(\bar{\psi},\psi,\phi,\cdots)$$

↑ Linear in C

$$= (C,\lambda)\begin{pmatrix} -\alpha & * \\ * & 0 \end{pmatrix}\begin{pmatrix} C \\ \lambda \end{pmatrix} + (C,\lambda)\begin{pmatrix} * \\ * \end{pmatrix} + \cdots$$

Since S is Gaussian (+linear) in C and λ, these can be integrated out.

Then we obtain an effective theory of bosons and fermions with quartic interactions.

§ Computations of effective theories : case (1)

Signed distribution : $|\mathrm{Det}M| \to \mathrm{Det}M$

$$\rho_{\text{signed}}(v) = 3^{(N-1)/2}\pi^{-N/2}\alpha^{N/2}v^{-2N}e^{-v^2/\alpha}\int d\bar{\psi}d\psi\, e^{S}$$

S is a four-fermi theory

$$S = -\bar{\psi}_{\|}\psi_{\|} + \bar{\psi}_{\perp}\psi_{\perp} - \frac{v^2}{6\alpha}(\bar{\psi}_{\perp}\psi_{\perp})^2 \qquad (\bar{\psi}\psi) = \bar{\psi}_a\psi_a, \text{ etc.}$$

The parallel and transverse components of $\bar{\psi}_a, \psi_a$ against v_a are decoupled. Parallel components $\bar{\psi}_{\|}, \psi_{\|}$ are free, and can trivially be integrated out. But it generates an overall sign, which matters.

The transverse part can also be computed, and we obtain an exact expression of $\rho(v)$.

$$\int d\bar{\psi}_\perp d\psi_\perp \, (\bar{\psi}_\perp \psi_\perp)^{2n} \, e^{\bar{\psi}_\perp \psi_\perp} = \left[\frac{d^{2n}}{dk^{2n}} \int d\bar{\psi}_\perp d\psi_\perp \, e^{k\bar{\psi}_\perp \psi_\perp} \right]_{k=1} = (1-N)_{2n}$$

Pochhammer symb.

$$(a)_n = a(a+1)\cdots(a+n-1)$$

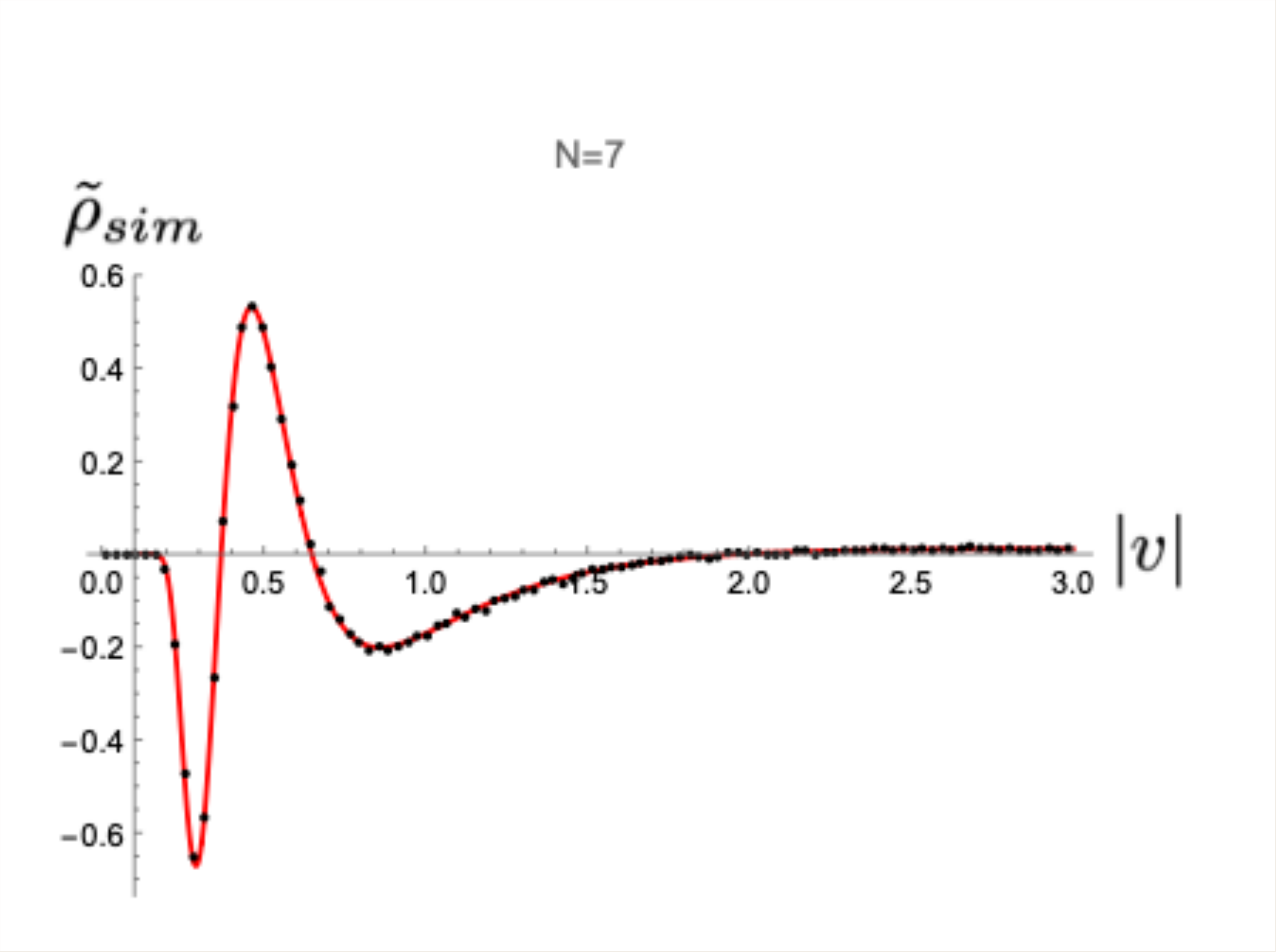
$$\rho(v) = -\,3^{1/2} 2^{-1+N/2} \alpha \, \pi^{-N/2} e^{-\alpha/v^2} \, |\, v \,|^{-N-2} \, U\left(1 - \frac{N}{2}, \frac{3}{2}, \frac{3\alpha}{2v^2}\right)$$

Confluent hypergeometric function of the second kind

Can also be expressed by Hermite polynomials.

Comparison with Monte Carlo simulation

Randomly generate C_{abc} with normal distribution and solve $C_{abc}v_b v_c = v_a$ by Mathematica.



§ Computations of effective theories : case (2)

Analytic continuation via replicas of fermions

$$|\mathrm{Det}M| \to \left\{\mathrm{Det}(M^2 + \epsilon I)\right\}^R \qquad (\epsilon \to +0,\ R \to 1/2)$$

Parallel components are free and can be integrated out.

$$\rho(v, R, \epsilon) = 3^{(N-1)/2}\pi^{-N/2}\alpha^{N/2}v^{-2N}e^{-\alpha/v^2}(-1)^{(N-1)R}\int_{\perp} d\bar{\psi}d\psi d\bar{\varphi}d\varphi\, e^S$$

Only transverse components

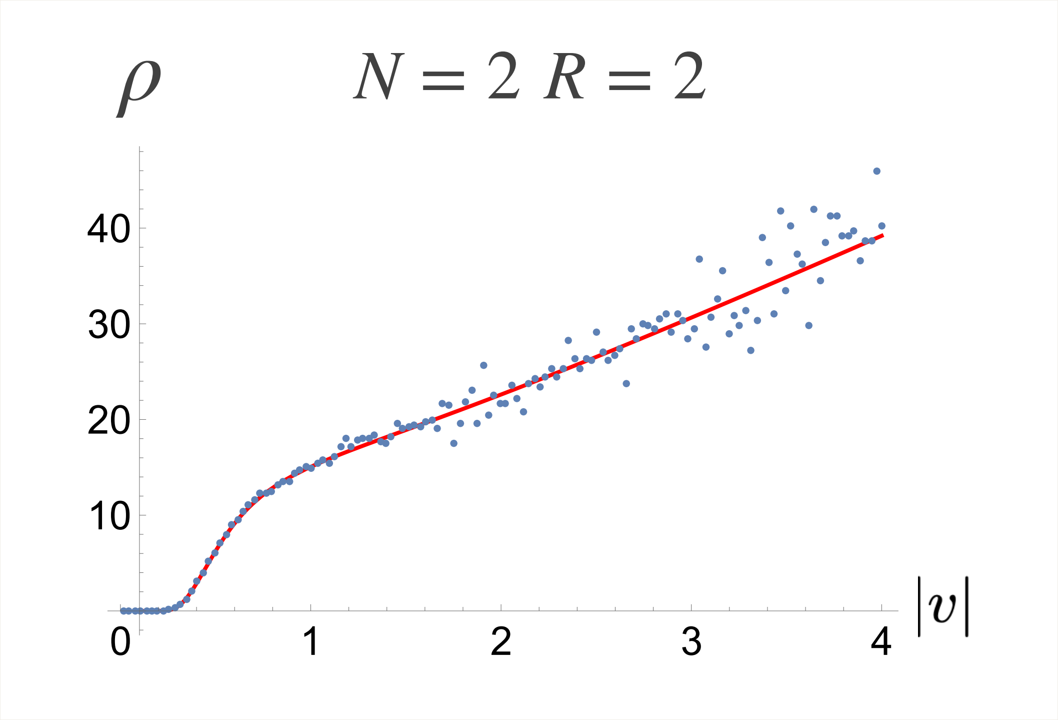
$$S = \epsilon\,\bar{\psi}^i\psi_i - \bar{\psi}^i\varphi_i - \bar{\varphi}^i\psi_i - \bar{\varphi}^i\varphi_i - \frac{v^2}{6\alpha}\Big((\bar{\psi}^i\bar{\psi}^j)(\varphi_i\varphi_j) + (\bar{\psi}^i\varphi_j)(\bar{\psi}^j\varphi_i)$$
$$+ (\bar{\varphi}^i\bar{\varphi}^j)(\psi_i\psi_j) + (\bar{\varphi}^i\psi_j)(\bar{\varphi}^j\psi_i) + 2(\bar{\psi}^i\bar{\varphi}^j)(\varphi_i\psi_j) + 2(\bar{\psi}^i\psi_j)(\bar{\varphi}^j\varphi_i)\Big)$$

$(\bar{\psi}\psi) = \bar{\psi}_a\psi_a$, etc.

$\bar{\psi}^i_a,\ \psi^i_a,\ \bar{\varphi}^i_a,\ \varphi^i_a\ (a = 1,2,\cdots,N-1,\ i = 1,2,\cdots,R)$: Fermions

For integer N, R, the fermionic integral is a finite polynomial function of $z = v^2/6\alpha$ of order $(N-1)R$. We used a Mathematica package to do the explicit computations.

$$\mathscr{L}_{N=2,R=1} = 1 + 4z$$
$$\mathscr{L}_{N=3,R=1} = 1 + 4z + 28z^2$$
$$\mathscr{L}_{N=2,R=2} = 1 + 24z + 48z^2$$
$$\mathscr{L}_{N=3,R=2} = 1 + 40z + 552z^2 + 1248z^3 + 5136z^4$$



Agrees with Monte Carlo

It is not easy to obtain the expression for larger values of N, R, because of the complication of the fermionic integral.

Therefore it seems difficult to obtain the general expression for any R and analytically continue it to $R = 1/2$.

We therefore compute the partition function for large-N by using Schwinger-Dyson equation. And we put $R = 1/2$.

In more details :

Assume

$$\langle \bar{\psi}^i_a \psi_{bj} \rangle = Q_{11} I_{ab} \delta^i_j$$

$$\langle \bar{\psi}^i_a \varphi_{bj} \rangle = Q_{12} I_{ab} \delta^i_j$$

$Q_{\alpha\beta}$ to be determined

$$\langle \bar{\varphi}^i_a \psi_{bj} \rangle = Q_{21} I_{ab} \delta^i_j$$

$$\langle \bar{\varphi}^i_a \varphi_{bj} \rangle = Q_{22} I_{ab} \delta^i_j$$

$$\text{others} = 0$$

In the leading order of $N-1$

$$\langle (\bar{\psi}^i \varphi_j)(\bar{\psi}^j \varphi_i) \rangle \sim \langle \bar{\psi}^i \varphi_j \rangle \langle \bar{\psi}^j \varphi_i \rangle = (N-1)^2 R Q_{12}^2 \quad \text{etc.}$$

$$S_{\text{eff}} = (N-1)R\Bigg(\epsilon Q_{11} - Q_{12} - Q_{21} - Q_{22} - \frac{v^2(N-1)}{6\alpha}(Q_{12}^2 + Q_{21}^2 + 2Q_{11}Q_{22}) - \underline{\log(-\det Q)} \Bigg)$$

Coming from integrating out fermions

$Q_{\alpha\beta}$ are determined by

$$\frac{\partial S_{\text{eff}}}{\partial Q_{\alpha\beta}} = 0$$

There are four independent solutions. Uniquely chosen from the free theory limit at $v = 0$:

$$Q_{11} = \frac{1}{1+\epsilon},\ Q_{12} = Q_{21} = \frac{-1}{1+\epsilon},\ Q_{22} = \frac{\epsilon}{1+\epsilon},\ \text{at } v = 0$$

The expressions of the solutions are complicated, so here it is suppressed.

In the $\epsilon \to +0$ limit, the solution has two regions:

$x = v^2(N-1)/3\alpha$

- $0 < x \leq 1/4$

$$Q_{11} = \frac{-\sqrt{1-4x}+1}{2x\sqrt{1-4x}},\ Q_{12} = Q_{21} = \frac{1-\sqrt{1-4x}-4x}{2x\sqrt{1-4x}},\ Q_{22} = 0$$

- $1/4 < x$

$$Q_{11} = \frac{\sqrt{-1+4x}}{2x\sqrt{\epsilon}} - \frac{1}{2x} + \cdots,\ Q_{12} = Q_{21} = -\frac{1}{2x} + \frac{\sqrt{\epsilon}}{2x\sqrt{-1+4x}} + \cdots,$$

$$Q_{22} = -\frac{\sqrt{\epsilon}\sqrt{-1+4x}}{2x} + \frac{\epsilon}{2x} + \cdots$$

The solution for $x > 1/4$ diverges in $\epsilon \to +0$, but $S_{\text{eff}}^{\epsilon}$ converges.

- $0 < x \leq 1/4$

$$S_{\text{eff}}^{\epsilon=+0}(x) = \frac{(N-1)R}{2}\left(2 + \log 16 + \frac{1-\sqrt{1-4x}}{x} - 4\log(1-\sqrt{1-4x}) + 4\log x\right)$$

- $1/4 < x$

$$S_{\text{eff}}^{\epsilon=+0}(x) = \frac{(N-1)R}{2}\left(\frac{1}{x} + 2 + 2\log x\right)$$

Only the latter case ($1/4 < x$) matters for large-N, because $x = v^2(N-1)/3\alpha \to \infty$.

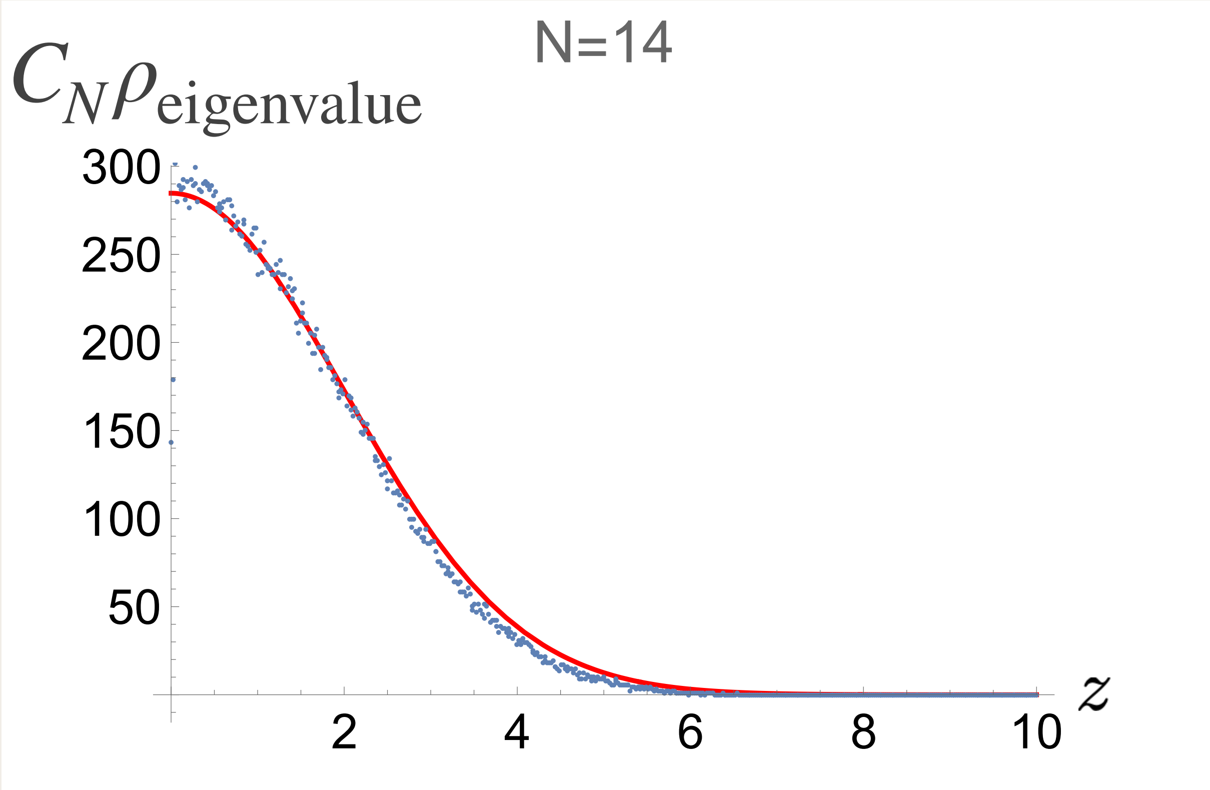
Putting $R = 1/2$ to the expression, we obtain

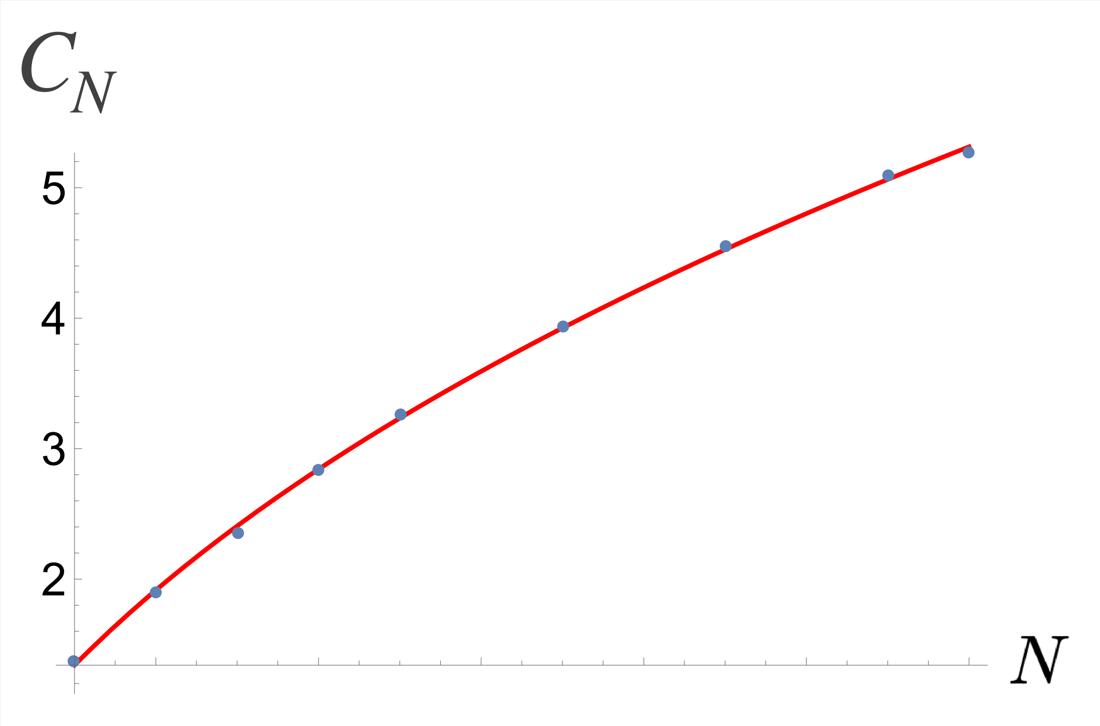
$$\rho(v) = (N-1)^{(N-1)/2} e^{-(N-1)/2} \pi^{-N/2} \alpha^{1/2} |v|^{-N-1} e^{-\frac{\alpha}{4|v|^2}}$$

Eigenvalue ($\zeta = 1/|v|$) distribution is given by Gaussian

$$\rho_{\text{eigenvalue}}(\zeta) = 2(N-1)^{(N-1)/2} e^{-(N-1)/2} \alpha^{1/2} \Gamma(N/2)^{-1} e^{-\frac{\alpha}{4}\zeta^2}$$

Agrees with Monte Carlo up to overall factors depending on N.





Next order will be needed to compute C_N.

§ Computations of effective theories : case (3)

$$\lim_{\epsilon \to +0} \frac{\mathrm{Det}(M^2 + \epsilon I)}{\sqrt{\mathrm{Det}(M^2 + \epsilon I)}} = |\mathrm{Det} M|$$

Numerator by Fermions, denominator by bosons

No analytic continuation needed. Just evaluate the expression.

After integrating over C and λ, we have a boson-fermion four-interaction system. Parallel components are free and can be integrated out :

$$\rho(v) = \lim_{\epsilon \to +0} 3^{(N-1)/2} \pi^{1-3N/2} \alpha^{N/2} v^{-2N} e^{-\alpha/v^2} (-1)^{N-1} \int_{\perp} d\phi d\sigma d\bar{\psi} d\psi d\bar{\varphi} d\varphi \, e^S$$

↑ Transverse

$$S = K_B + K_F + V_F + V_B + V_{BF}$$

σ_a, ϕ_a: bosons, $\bar{\psi}_a, \psi_a, \bar{\varphi}_a, \varphi_a$: fermions $(a = 1,2,\cdots,N)$

$$K_B = -\sigma^2 - 2i\sigma\phi - \epsilon\phi^2$$

$$K_F = -\bar{\varphi}\varphi - \bar{\psi}\varphi - \bar{\varphi}\psi - \epsilon\bar{\psi}\psi$$

$(\bar{\psi}\psi) = \bar{\psi}_a\psi_a$, etc.

$$V_F = -\frac{v^2}{6\alpha}\left((\bar{\psi}\varphi)^2 + (\bar{\varphi}\psi)^2 + 2(\bar{\psi}\bar{\varphi})(\varphi\psi) + 2(\bar{\psi}\psi)(\bar{\varphi}\varphi)\right)$$

$$V_B = -\frac{2v^2}{3\alpha}(\sigma^2\phi^2 + (\sigma\phi)^2)$$

$$V_{BF} = \frac{2iv^2}{3\alpha}((\bar{\psi}\sigma)(\varphi\phi) + (\bar{\varphi}\sigma)(\psi\phi) + (\bar{\psi}\phi)(\varphi\sigma) + (\bar{\varphi}\phi)(\psi\sigma))$$

We can perform similar Schwinger-Dyson analysis for large-N as in case (2). The result turns out to be the same as in (2).

In principle, we can improve the result by taking into account higher orders of Schwinger-Dyson analysis.

What turns out to be more interesting is that there exist exact expressions of the eigenvalue distributions for any N in terms of polynomial, exponential and error functions. This has been checked for $N \leq 8$.

After fermionic integration, the bosonic integrand turns out to be a total derivative (+ a term) and the bosonic integral can be exactly performed.

More explicitly :

The fermionic integral can be proven to have the form

$$\int_{\perp} d\bar{\psi} d\psi d\bar{\varphi} d\varphi \, e^{K_F+V_F+V_{BF}} = a_0 + a_1(\sigma\phi) + a_2(\sigma\phi)^2$$
$$+a_3\sigma^2\phi^2 + a_4(\sigma^2\phi^2(\sigma\phi) - (\sigma\phi)^3) + a_5(\sigma^2\phi^2 - (\sigma\phi)^2)^2$$

where $a_0, a_1, \cdots, a_5$ are some finite polynomial functions of v^2/α. The explicit form can be determined by using a Mathematica package.

Then, after some changes of variables, the bosonic integral turns into the form

$$G_N = \int_0^{1/8z} dx \, e^{-x} x^{(N-3)/2} (1-4zx)^{-(N+2)/2} (4a_0 + 2(-2ia_1 + a_2 + (N-1)a_3)x$$
$$+(8iza_1 - (3+4z)(a_2+a_3) - i(N-2)a_4)x^2 + 8z(a_2+a_3)x^3)$$

$$z = v^2/6\alpha$$

To derive this, we assumed

$$64z^2a_0 - 8iza_1 + (4z-1)a_2 + (-1-4z+8(N-1)z)a_3 - i(N-2)a_4 + N(N-2)a_5 = 0$$

This is actually satisfied for $N \le 8$.

Otherwise, the integrand would contain a factor of $1/(1-8zx)$, which ruins the integrability.

Putting the explicit values of $a_0, \cdots, a_5$, one finds

For odd N

$$G_N = \int_0^{1/8z} dx \frac{d}{dx} \frac{e^{-x}\sum_{n=0}^{(N+1)/2} b_n x^n}{(1-4zx)^{N/2}} = 2^{N/2} e^{-1/8z} \sum_{n=0}^{(N+1)/2} b_n (8z)^{-n} - b_0$$

For even N

$$G_N = \int_0^{1/8z} dx \left(c_0 x^{-1/2} e^{-x} + \frac{d}{dx} \frac{x^{1/2} e^{-x} \sum_{n=0}^{N/2} b_n x^n}{(1-4zx)^{N/2}} \right)$$

$$= c_0 \gamma[1/2, 1/8z] + 2^{N/2} e^{-1/8z} \sum_{n=0}^{N/2} \frac{b_n}{(8z)^{n+1/2}}$$

b_i, c_0 : determined from a_i

The end results are

$$G_{N=5} = \pi^4 \left(1 - 12z + 12z^2 + \sqrt{2}e^{-\frac{1}{8z}}(1 + 12z + 12z^2)\right),$$

$$G_{N=7} = \pi^6 \left(1 - 30z + 180z^2 - 120z^3 + \frac{\sqrt{2}e^{-\frac{1}{8z}}}{8z}(1 + 8z + 120z^2 - 480z^3 + 2640z^4)\right).$$

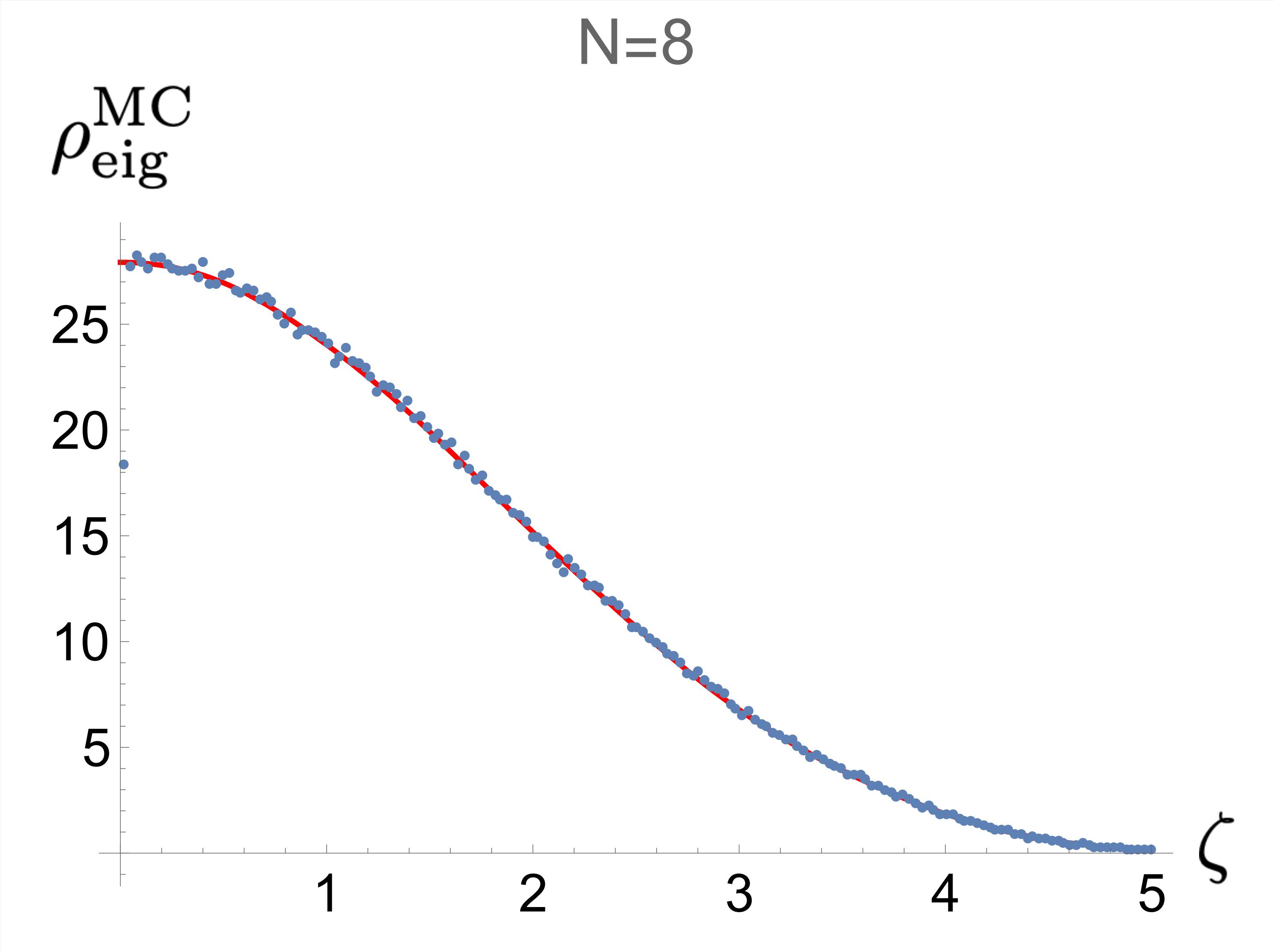
$$G_{N=4} = \pi^{\frac{5}{2}} \left(6\sqrt{2}e^{-\frac{1}{8z}}\sqrt{z}(1 + 2z) + (1 - 6z)\,\gamma\left[\frac{1}{2}, \frac{1}{8z}\right]\right),$$

$$G_{N=6} = \pi^{\frac{9}{2}} \left(\frac{2\sqrt{2}e^{-\frac{1}{8z}}(1 + 15z + 180z^3)}{3\sqrt{z}} + (1 - 20z + 60z^2)\,\gamma\left[\frac{1}{2}, \frac{1}{8z}\right]\right),$$

$$G_{N=8} = \pi^{\frac{13}{2}} \left(\frac{\sqrt{2}e^{-\frac{1}{8z}}(1 + 210z^2 - 2100z^3 + 12600z^4 + 25200z^5)}{15z^{\frac{3}{2}}} + (1 - 42z + 420z^2 - 840z^3)\,\gamma\left[\frac{1}{2}, \frac{1}{8z}\right]\right).$$

$$z = v^2/6\alpha$$

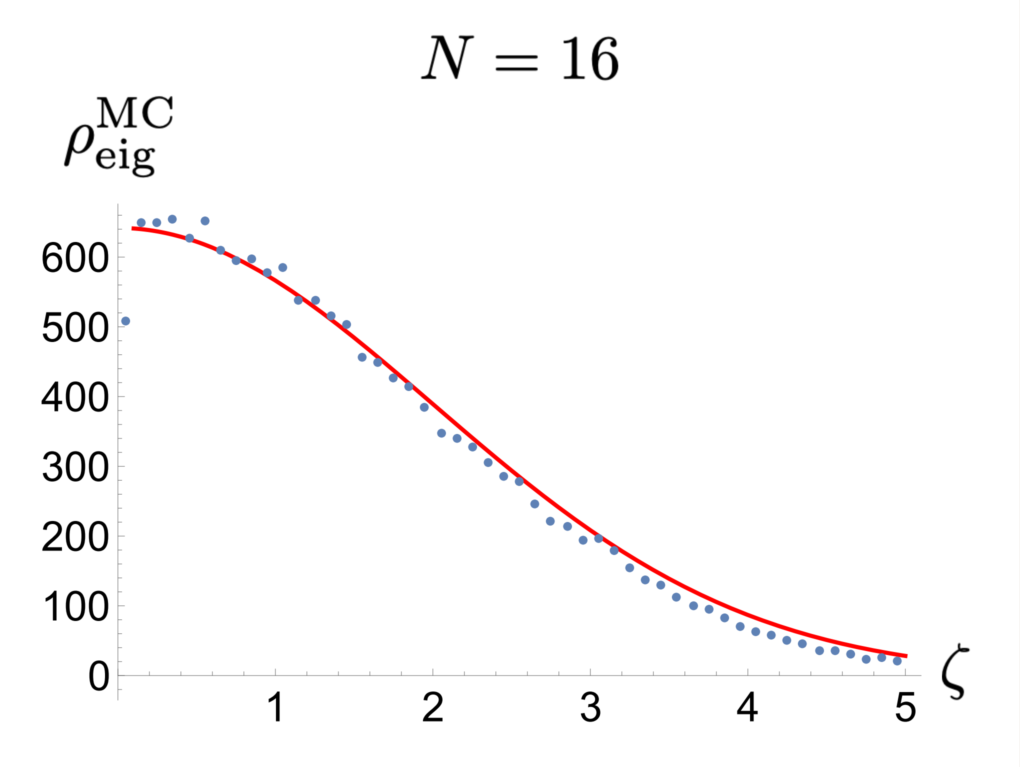
The exact formula agrees with the Monte Carlo result.



The Schwinger-Dyson analysis in the leading order carried out in case (2) did not correctly produce the overall factor of the distribution, since this is in the next order.

From the exact results of $N \leq 8$ one can guess the N-dependence of the overall factor. We obtain

$$\rho_{\text{eig}}(\zeta) \sim 2^{-N/2+2}\alpha^{1/2}\pi^{-1/2}\frac{\Gamma[N+1]}{\Gamma\left[\frac{N}{2}+1\right]\Gamma\left[\frac{N}{2}\right]}e^{-\frac{\alpha}{4}\zeta^2}$$



The formula agrees well with the Monte Carlo result without extra tuning of the overall factor.

§ Summary and future prospects

Eigenvalue distributions of tensors could be an interesting dynamical quantity which characterizes the dynamical aspects of tensor models.

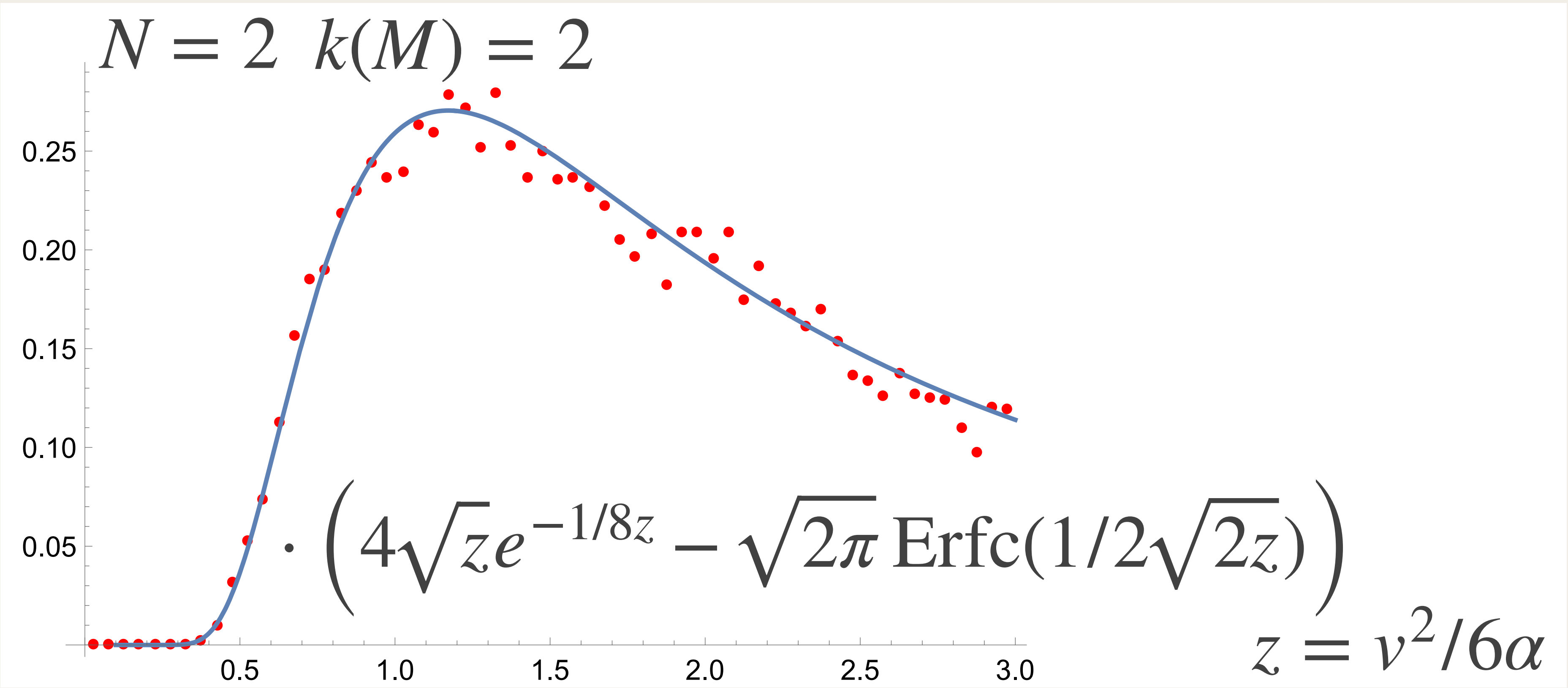
We compute real eigenvalue/vector distribution for order-three real symmetric tensors with Gaussian distribution.

The large-N limit of the eigenvalue distribution is given by Gaussian, which contrasts with Wigner's semicircle law in the matrix model.

The difference from the previous studies is that we used field theoretical methods, while the previous studies used the matrix model techniques.

One would extend the computation in various directions.

- Count only the eigenvectors at which Hessian matrix M has signature k (= the number of negative eigenvalues).



- Compute correlations among eigenvectors. In matrix models, eigenvalues are repulsive. How about tensor models ?

- Obtain exact formulas of eigenvalue / vector distributions for any N, R