

Spike Recovery from Large Random Tensors

Tensor Journal Club

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Technology Innovation Institute

April 19th 2023



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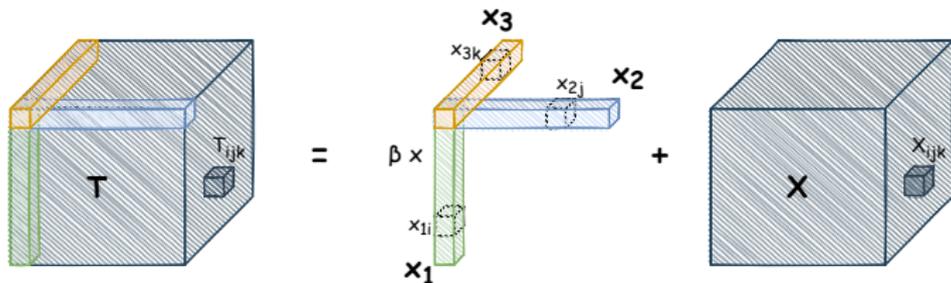
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We consider the following model: $(x_1 \otimes x_2 \otimes x_3)_{ijk} = x_{1i}x_{2j}x_{3k}$

$$\mathbf{T} = \underbrace{\beta \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_d}_{\text{signal}} + \frac{1}{\sqrt{n}} \underbrace{\mathbf{X}}_{\text{noise}} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$$

where $\beta \geq 0$, $\|\mathbf{x}_i\| = 1$, $X_{i_1 \dots i_d} \sim \mathcal{N}(0, 1)$ i.i.d. and $n = \sum_{i=1}^d n_i$.

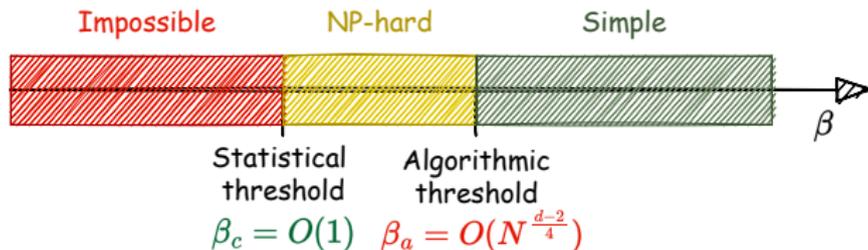
- ▶ Is it possible to recover the signal in theory? for which **critical** value of β ?
- ▶ What **alignment** $\langle \mathbf{x}_i, \mathbf{u}_i \rangle$ between the signal and an estimator $\mathbf{u}_i(\mathbf{T})$?
- ▶ Is there an algorithm that can recover the signal in **polynomial time**?

Related Works: Symmetric Case

Introduced initially by (Montanari & Richard, 2014)

$$\mathbf{Y} = \beta \mathbf{x}^{\otimes d} + \frac{1}{\sqrt{N}} \mathbf{W} \in \mathbb{R}^{N \times \dots \times N}$$

where $\|\mathbf{x}\| = 1$ and \mathbf{W} has random Gaussian entries and is **symmetric**. This is a natural extension of the classical spiked matrix model $\mathbf{Y} = \beta \mathbf{x} \mathbf{x}^\top + \frac{1}{\sqrt{N}} \mathbf{W}$.



Other works in the literature include: (Montanari et al., 2015), (Hopkins et al., 2020), (Kim et al., 2017), (Ben Arous et al., 2019), (Jagannath et al., 2020), (Perry et al., 2020), (Ros et al., 2020), (Goulart et al., 2022).

Of which Goulart et al. "*A random matrix perspective on random tensors*", JMLR 2022.

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Random Matrix Approach (Goulart et al., 2022)

The optimization problem of maximum likelihood estimator (MLE) for $d = 3$:

$$\min_{\lambda > 0, \|\mathbf{u}\|=1} \left\| \mathbf{Y} - \lambda \mathbf{u}^{\otimes 3} \right\|_F^2 \Leftrightarrow \max_{\|\mathbf{u}\|=1} \langle \mathbf{Y}, \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u} \rangle$$

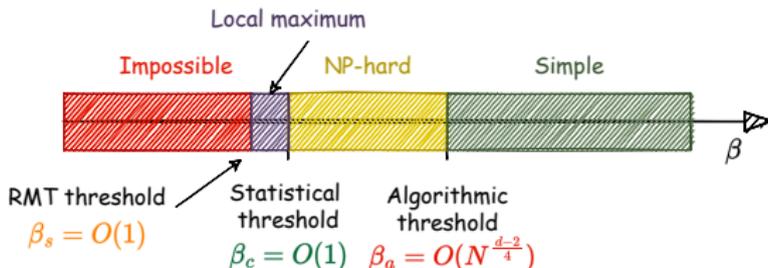
The critical points satisfy (Lim, 2005):

$$\mathbf{Y}(\mathbf{u}, \mathbf{u}) = \lambda \mathbf{u} \Leftrightarrow \mathbf{Y}(\mathbf{u})\mathbf{u} = \lambda \mathbf{u}, \quad \|\mathbf{u}\| = 1$$

where $(\mathbf{Y}(\mathbf{u}, \mathbf{u}))_i = \sum_{j,k} u_j u_k Y_{ijk}$ et $(\mathbf{Y}(\mathbf{u}))_{ij} = \sum_k u_k Y_{ijk}$. The MLE $\hat{\mathbf{x}}$ corresponds to the dominant eigenvector of $\mathbf{Y}(\hat{\mathbf{x}})$: $\mathbf{Y}(\hat{\mathbf{x}})\hat{\mathbf{x}} = \|\mathbf{Y}\|\hat{\mathbf{x}}$.

Hence, the approach from (Goulart et al., 2021) consists in studying:

$$\mathbf{Y}(\mathbf{u}) = \beta \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{x} \mathbf{x}^\top + \frac{1}{\sqrt{N}} \mathbf{W}(\mathbf{u}) \in \mathbb{R}^{N \times N}$$



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The optimization problem of MLE for $d = 3$:

$$\min_{\lambda > 0, \|\mathbf{u}_i\|=1} \|\mathbf{T} - \lambda \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \mathbf{u}_3\|_F^2 \Leftrightarrow \max_{\prod_{i=1}^3 \|\mathbf{u}_i\|=1} \langle \mathbf{T}, \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \mathbf{u}_3 \rangle$$

The critical points satisfy (Lim, 2005):

$$\mathbf{T}(\cdot, \mathbf{u}_2, \mathbf{u}_3) = \lambda \mathbf{u}_1, \mathbf{T}(\mathbf{u}_1, \cdot, \mathbf{u}_3) = \lambda \mathbf{u}_2, \mathbf{T}(\mathbf{u}_1, \mathbf{u}_2, \cdot) = \lambda \mathbf{u}_3$$

where $\|\mathbf{u}_i\| = 1$ for all $i \in [3]$ and $(\mathbf{T}(\cdot, \mathbf{u}_2, \mathbf{u}_3))_i = \sum_{j,k} u_{2j} u_{3k} T_{ijk}$.

- ▶ In contrast to the symmetric case, the choice of the associated *contraction* matrix is not straightforward. For instance:

$$\mathbf{T}(\mathbf{u}_3) \equiv \mathbf{T}(\cdot, \cdot, \mathbf{u}_3) = \beta \langle \mathbf{x}_3, \mathbf{u}_3 \rangle \mathbf{x}_1 \mathbf{x}_2^\top + \frac{1}{\sqrt{n}} \mathbf{X}(\cdot, \cdot, \mathbf{u}_3) \in \mathbb{R}^{n_1 \times n_2}$$

Objectives:

- ▶ Evaluate the asymptotic limits of $\hat{\lambda}$ and $\langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle$ associated (a priori) to the MLE when $n_i \rightarrow \infty$.
- ▶ Define a *symmetric* random matrix that is equivalent to \mathbf{T} .

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Associated Random Matrix to \mathbf{T}

Stein's Lemma. Let $X \sim \mathcal{N}(0, 1)$, then $\mathbb{E}[Xf(X)] = \mathbb{E}[f'(X)]$.

Recall $\lambda = \mathbf{T}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \frac{1}{\sqrt{n}} \sum_{ijk} u_{1i} u_{2j} u_{3k} X_{ijk} + \beta \prod_{i=1}^3 \langle \mathbf{x}_i, \mathbf{u}_i \rangle$.

$$\mathbb{E}[\lambda] = \frac{1}{\sqrt{n}} \sum_{ijk} \mathbb{E} \left[u_{2j} u_{3k} \frac{\partial u_{1i}}{\partial X_{ijk}} \right] + \mathbb{E} \left[u_{1i} u_{3k} \frac{\partial u_{2j}}{\partial X_{ijk}} \right] + \mathbb{E} \left[u_{1i} u_{2j} \frac{\partial u_{3k}}{\partial X_{ijk}} \right] + \dots$$

$$\begin{bmatrix} \frac{\partial \mathbf{u}_1}{\partial X_{ijk}} \\ \frac{\partial \mathbf{u}_2}{\partial X_{ijk}} \\ \frac{\partial \mathbf{u}_3}{\partial X_{ijk}} \end{bmatrix} \simeq -\frac{1}{\sqrt{n}} \left(\underbrace{\begin{bmatrix} \mathbf{0}_{n_1 \times n_1} & \mathbf{T}(\mathbf{u}_3) & \mathbf{T}(\mathbf{u}_2) \\ \mathbf{T}(\mathbf{u}_3)^\top & \mathbf{0}_{n_2 \times n_2} & \mathbf{T}(\mathbf{u}_1) \\ \mathbf{T}(\mathbf{u}_2)^\top & \mathbf{T}(\mathbf{u}_1)^\top & \mathbf{0}_{n_3 \times n_3} \end{bmatrix}}_{\Phi_3(\mathbf{T}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)} - \lambda \mathbf{I}_n \right)^{-1} \begin{bmatrix} u_{2j} u_{3k} e_i^{n_1} \\ u_{1i} u_{3k} e_j^{n_2} \\ u_{1i} u_{2j} e_k^{n_3} \end{bmatrix}$$

The resolvent matrix: $\mathbf{R}(z) = (\Phi_3(\mathbf{T}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) - z\mathbf{I}_n)^{-1}$.

When $n_i \rightarrow \infty$, the non-vanishing terms involve the **trace** of $\mathbf{R}(z)$,

$$\lambda + \frac{1}{n} \text{tr} \mathbf{R}(\lambda) = \beta \prod_{i=1}^3 \langle \mathbf{x}_i, \mathbf{u}_i \rangle$$

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Associated Random Matrix to \mathbf{T}

For an order- d tensor the associated random matrix is $\Phi_d(\mathbf{T}, \mathbf{u}_1, \dots, \mathbf{u}_d)$

$$\Phi_d : (\mathbf{X}, \mathbf{a}_1, \dots, \mathbf{a}_d) \mapsto \begin{bmatrix} \mathbf{0}_{n_1 \times n_1} & \mathbf{X}^{12} & \mathbf{X}^{13} & \dots & \mathbf{X}^{1d} \\ (\mathbf{X}^{12})^\top & \mathbf{0}_{n_2 \times n_2} & \mathbf{X}^{23} & \dots & \mathbf{X}^{2d} \\ (\mathbf{X}^{13})^\top & (\mathbf{X}^{23})^\top & \mathbf{0}_{n_3 \times n_3} & \dots & \mathbf{X}^{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\mathbf{X}^{1d})^\top & (\mathbf{X}^{2d})^\top & (\mathbf{X}^{3d})^\top & \dots & \mathbf{0}_{n_d \times n_d} \end{bmatrix}$$

with $\mathbf{X}^{ij} \equiv \mathbf{X}(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \cdot, \mathbf{a}_{i+1}, \dots, \mathbf{a}_{j-1}, \cdot, \mathbf{a}_{j+1}, \dots, \mathbf{a}_d) \in \mathbb{R}^{n_i \times n_j}$.

Remark. $(d-1)\lambda$ is an eigenvalue of $\Phi_d(\mathbf{T}, \mathbf{u}_1, \dots, \mathbf{u}_d)$ with

$$\Phi_d(\mathbf{T}, \mathbf{u}_1, \dots, \mathbf{u}_d) \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_d \end{bmatrix} = (d-1)\lambda \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_d \end{bmatrix}$$

since $\mathbf{T}(\mathbf{u}_1, \dots, \mathbf{u}_{j-1}, \cdot, \mathbf{u}_{j+1}, \dots, \mathbf{u}_d) = \lambda \mathbf{u}_j$.

$$\text{rank}(\Phi_d(\mathbf{T}, \mathbf{u}_1, \dots, \mathbf{u}_d)) = \sum_{i=1}^d \min \left(n_i, \sum_{j \neq i} n_j \right)$$

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Spectral Measure of $\Phi_d(\mathbf{T}, \mathbf{u}_1, \dots, \mathbf{u}_d)$

Stieltjes Transform. The Stieltjes transform of a probability measure ν is

$$g_\nu(z) = \int \frac{d\nu(\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathcal{S}(\nu).$$

For $\mathbf{S} \in \text{Sym}_n$ with λ_i its eigenvalues and denote its resolvent $\mathbf{R}_\mathbf{S}(z) = (\mathbf{S} - z\mathbf{I}_n)^{-1}$, the ESM of \mathbf{S} and its associated Stieltjes transform are:

$$\nu_\mathbf{S} = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}, \quad g_{\nu_\mathbf{S}}(z) = \frac{1}{n} \sum_{i=1}^n \frac{1}{\lambda_i - z} = \frac{1}{n} \text{tr} \mathbf{R}_\mathbf{S}(z), \quad z \in \mathbb{C} \setminus \mathcal{S}(\nu_\mathbf{S})$$

Definition 1. Let ν be the probability measure with Stieltjes transform $g(z) = \sum_{i=1}^d g_i(z)$ verifying $\Im[g(z)] > 0$ for $\Im[z] > 0$, where $g_i(z)$ satisfies $g_i^2(z) - (g(z) + z)g_i(z) - c_i = 0$, for $z \notin \mathcal{S}(\nu)$.

Assumption 1. As $n_i \rightarrow \infty$ with $\frac{n_i}{\sum_j n_j} \rightarrow c_i \in (0, 1)$, there exists a sequence of critical points $(\hat{\lambda}, \hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_d)$ s.t. $\hat{\lambda} \xrightarrow{\text{a.s.}} \lambda$, $|\langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle| \xrightarrow{\text{a.s.}} \rho_i$ with $\lambda \notin \mathcal{S}(\nu)$ and $\rho_i > 0$.

Theorem 1 (SGC'21). Under Assumption 1, the ESM of $\Phi_d(\mathbf{T}, \hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_d)$ converges weakly to ν defined in Definition 1 (i.e. $\frac{1}{n} \text{tr} \mathbf{R}(z) \xrightarrow{\text{a.s.}} g(z)$).

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Spectral Measure of $\Phi_d(\mathbf{T}, \mathbf{u}_1, \dots, \mathbf{u}_d)$

Repeat

$$\triangleright g_i \leftarrow c_i / (g_i - g - z)$$

$$\triangleright g \leftarrow \sum_i g_i$$

Until convergence of g .

$$c_1 = \frac{c}{4}, \quad c_2 = \frac{c}{2}, \quad c_3 = 1 - \frac{3c}{4}$$

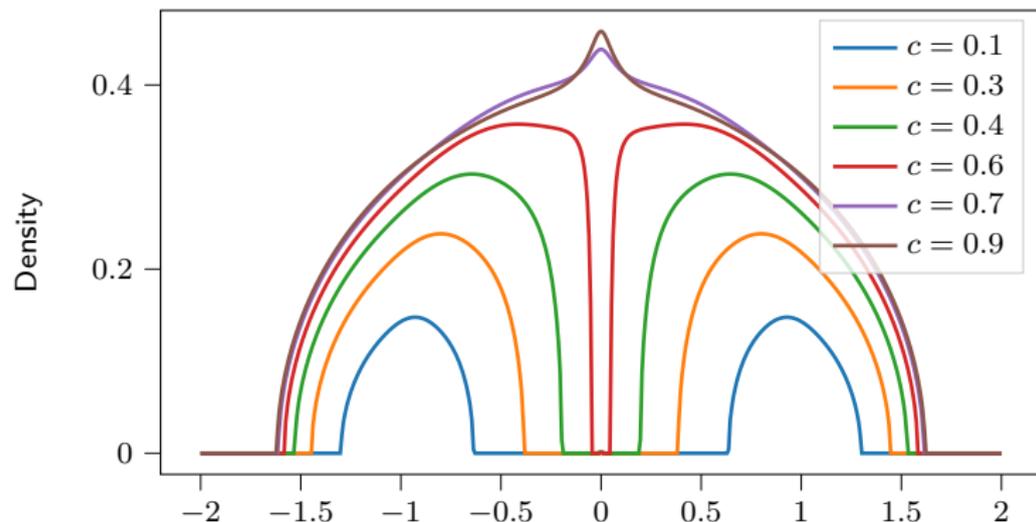


Figure: Density of the limiting spectral measure $\nu(dx) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \Im[g(x + i\epsilon)]$.

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Spectral Measure of $\Phi_d(\mathbf{T}, \mathbf{u}_1, \dots, \mathbf{u}_d)$

Corollary 1. When $c_i = \frac{1}{d}$ for all $i \in [d]$, the ESM of $\Phi_d(\mathbf{T}, \hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_d)$ converges to a **semi-circle law** ν of support $\left[-2\sqrt{\frac{d-1}{d}}, 2\sqrt{\frac{d-1}{d}}\right]$, where

$$\nu(dx) = \frac{d}{2(d-1)\pi} \sqrt{\left(\frac{4(d-1)}{d} - x^2\right)^+}, \quad g(z) = \frac{-zd + d\sqrt{z^2 - \frac{4(d-1)}{d}}}{2(d-1)}$$

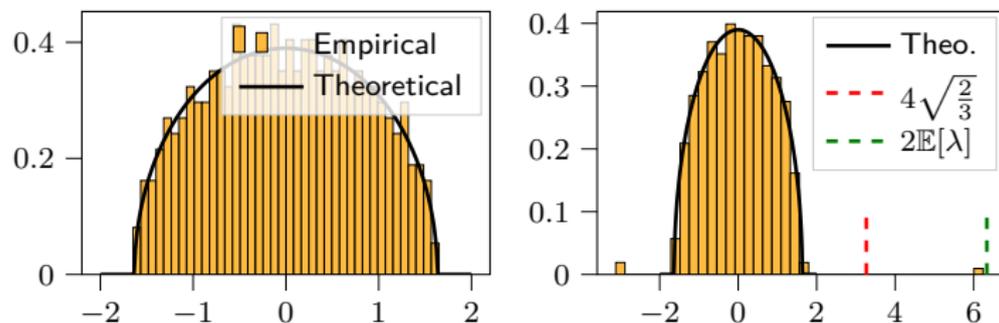


Figure: Spectrum of $\Phi_3(\mathbf{T}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ at initialization (left) and convergence (right) of tensor power iteration applied on \mathbf{T} . $n_1 = n_2 = n_3 = 150$ and $\beta = 3$.

$$\mathbf{u}_1 \leftarrow \frac{\mathbf{T}(\cdot, \mathbf{u}_2, \mathbf{u}_3)}{\|\mathbf{T}(\cdot, \mathbf{u}_2, \mathbf{u}_3)\|}, \quad \mathbf{u}_2 \leftarrow \frac{\mathbf{T}(\mathbf{u}_1, \cdot, \mathbf{u}_3)}{\|\mathbf{T}(\mathbf{u}_1, \cdot, \mathbf{u}_3)\|}, \quad \mathbf{u}_3 \leftarrow \frac{\mathbf{T}(\mathbf{u}_1, \mathbf{u}_2, \cdot)}{\|\mathbf{T}(\mathbf{u}_1, \mathbf{u}_2, \cdot)\|}$$

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$$\mathbf{T}(\mathbf{x}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3) = \hat{\lambda} \langle \mathbf{x}_1, \hat{\mathbf{u}}_1 \rangle \underbrace{\Rightarrow}_{\text{Stein}} \left[\hat{\lambda} + g_2(\hat{\lambda}) + g_3(\hat{\lambda}) \right] \langle \mathbf{x}_1, \hat{\mathbf{u}}_1 \rangle = \beta \prod_{i=2}^3 \langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle$$

Assumption 1. As $n_i \rightarrow \infty$ with $\sum_j \frac{n_i}{n_j} \rightarrow c_i \in (0, 1)$, there exists a sequence of critical points $(\hat{\lambda}, \hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_d)$ s.t. $\hat{\lambda} \xrightarrow{\text{a.s.}} \lambda$, $|\langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle| \xrightarrow{\text{a.s.}} \rho_i$ with $\lambda \notin \mathcal{S}(\nu)$ and $\rho_i > 0$.

Theorem 2 (SGC'21). For all $d \geq 3$, under Assumption 1, there exists $\beta_s > 0$ such that for all $\beta > \beta_s$

$$\hat{\lambda} \xrightarrow{\text{a.s.}} \lambda, \quad |\langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle| \xrightarrow{\text{a.s.}} q_i(\lambda)$$

where λ satisfies $f(\lambda, \beta) = 0$ with

$$f(z, \beta) = z + g(z) - \beta \prod_{i=1}^d q_i(z), \quad q_i(z) = \sqrt{1 - \frac{g_i^2(z)}{c_i}}$$

for $\beta \in [0, \beta_s]$, λ is bounded and $|\langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle| \xrightarrow{\text{a.s.}} 0$.

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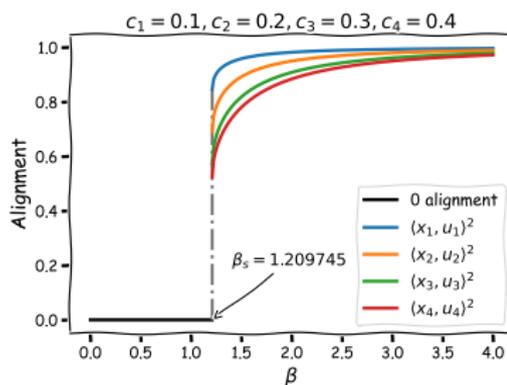
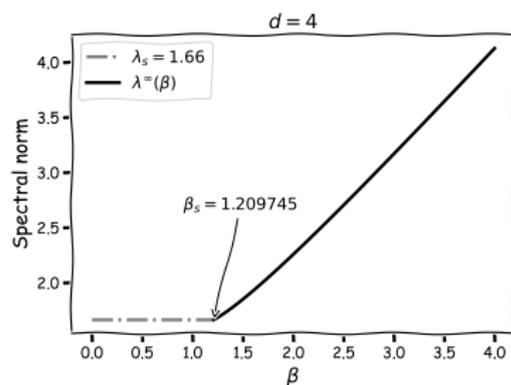
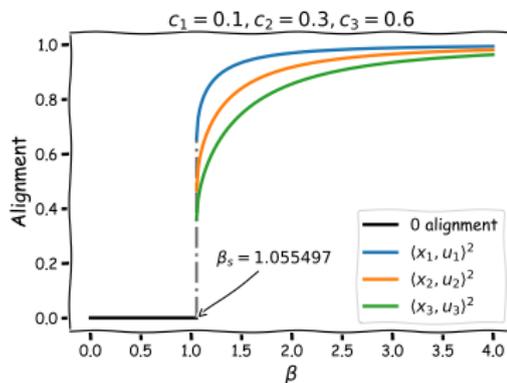
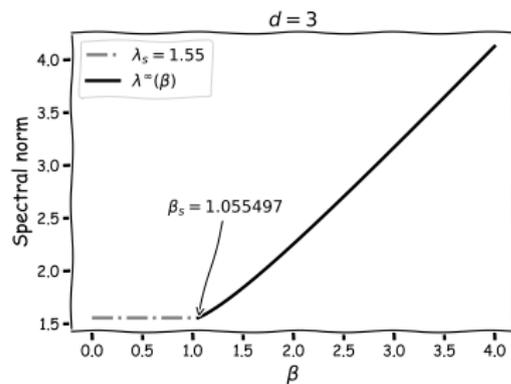
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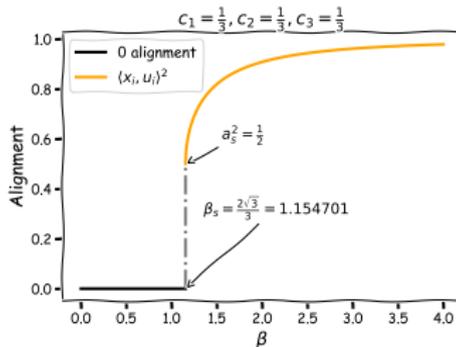
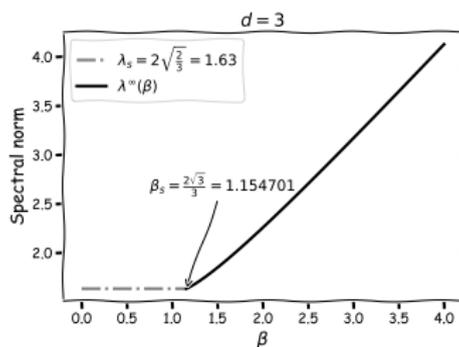
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Corollary 2 (SGC'21). If $d = 3$ with $c_i = \frac{1}{3}$, then for all $\beta > \frac{2\sqrt{3}}{3}$

$$\left\{ \begin{array}{l} \hat{\lambda} \xrightarrow{\text{a.s.}} \sqrt{\frac{\beta^2}{2} + 2 + \frac{\sqrt{3}\sqrt{(3\beta^2-4)^3}}{18\beta}} \\ |\langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle| \xrightarrow{\text{a.s.}} \frac{\sqrt{9\beta^2-12 + \frac{\sqrt{3}\sqrt{(3\beta^2-4)^3}}{\beta}} + \sqrt{9\beta^2+36 + \frac{\sqrt{3}\sqrt{(3\beta^2-4)^3}}{\beta}}}{6\sqrt{2}\beta} \end{array} \right.$$



For hyper-cubic tensors of order d , we have

$$\beta_s = \sqrt{\frac{d-1}{d}} \left(\frac{d-2}{d-1} \right)^{1-\frac{d}{2}}, \quad \lim_{\beta \rightarrow \beta_s} \rho_i(\beta) = \sqrt{\frac{d-2}{d-1}}$$

Spiked Matrix Model

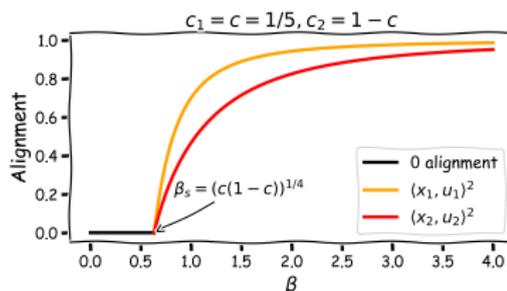
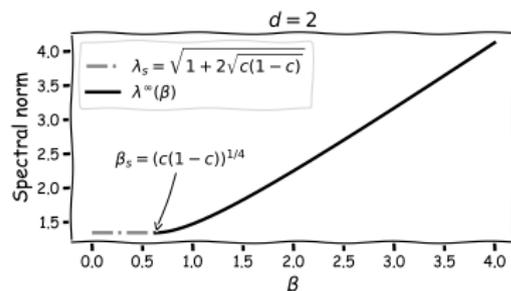
$$\text{For } d = 3, n_3 = 1 \Rightarrow \mathbf{T} = \beta \mathbf{x}_1 \mathbf{x}_2^\top + \frac{1}{\sqrt{n_1 + n_2}} \mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$$

Corollary 3 (SGC'21). If $d = 3$ with $c_1 = c$ et $c_2 = 1 - c$ for $c \in [0, 1]$, the spiked tensor model becomes a **spiked matrix model** (i.e. $c_3 = 0$).

Let $\kappa(\beta, c) = \beta \sqrt{\frac{\beta^2(\beta^2+1)-c(c-1)}{(\beta^4+c(c-1))(\beta^2+1-c)}}$, for $\beta > \beta_s = \sqrt[4]{c(1-c)}$

$$\hat{\lambda} \xrightarrow{\text{a.s.}} \sqrt{\beta^2 + 1 + \frac{c(1-c)}{\beta^2}}, \quad |\langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle| \xrightarrow{\text{a.s.}} \frac{1}{\kappa(\beta, c_i)}, \quad i \in \{1, 2\}$$

while for $\beta \in [0, \beta_s]$, $\hat{\lambda} \xrightarrow{\text{a.s.}} \sqrt{1 + 2\sqrt{c(1-c)}}$ et $|\langle \mathbf{x}_i, \hat{\mathbf{u}}_i \rangle| \xrightarrow{\text{a.s.}} 0$.



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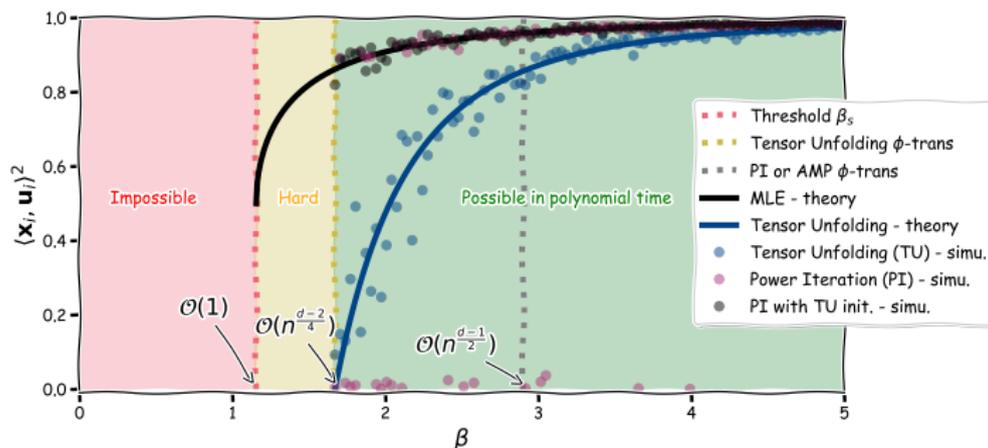
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$$\min_{\lambda > 0, \|\mathbf{u}_i\|=1} \|\mathbf{T} - \lambda \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_d\|_F^2 \Rightarrow \text{NP-hard (Hillar et al., 2013)}$$

- ▶ Tensor unfolding: $\mathcal{M}_i(\mathbf{T}) = \beta \mathbf{x}_i \mathbf{y}_i^\top + \frac{1}{\sqrt{n}} \mathcal{M}_i(\mathbf{X}) \in \mathbb{R}^{n_i \times \prod_{j \neq i} n_j}$.
- ▶ Using Corollary 3, we find $\beta_a = (\prod_i n_i)^{1/4} / \sqrt{\sum_i n_i}$.
- ▶ Coincides with $O\left(N^{\frac{d-2}{4}}\right)$ of (Ben Arous et al, 2021) for $n_i = N$.
- ▶ Same threshold for tensor power iteration initialized with tensor unfolding (Auddy et al., 2021).



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**Hotteling-type Tensor
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Orthogonalized Tensor
Deflation

We consider the following rank-2 order-3 spiked tensor model

$$\mathbf{T}_1 = \sum_{i=1}^2 \beta_i \mathbf{x}_{1i} \otimes \mathbf{x}_{2i} \otimes \mathbf{x}_{3i} + \frac{1}{\sqrt{n}} \mathbf{X} \in \mathbb{R}^{p \times p \times p}$$

where $\beta_i \geq 0$, $\|\mathbf{x}_{mi}\| = 1$, $X_{ijk} \sim \mathcal{N}(0, 1)$ i.i.d. and $n = 3p$. Assume

$$\alpha \equiv \langle \mathbf{x}_{11}, \mathbf{x}_{12} \rangle = \langle \mathbf{x}_{21}, \mathbf{x}_{22} \rangle = \langle \mathbf{x}_{31}, \mathbf{x}_{32} \rangle \in [0, 1]$$

Tensor Deflation. Compute $\hat{\lambda}_2 \hat{\mathbf{u}}_{12} \otimes \hat{\mathbf{u}}_{22} \otimes \hat{\mathbf{u}}_{32}$ as best rank-one approximation of \mathbf{T}_2 with

$$\mathbf{T}_2 = \mathbf{T}_1 - \hat{\lambda}_1 \hat{\mathbf{u}}_{11} \otimes \hat{\mathbf{u}}_{21} \otimes \hat{\mathbf{u}}_{31}$$

where $\hat{\lambda}_i \hat{\mathbf{u}}_{1i} \otimes \hat{\mathbf{u}}_{2i} \otimes \hat{\mathbf{u}}_{3i}$ is a critical point of

$$\arg \min_{\lambda_i > 0, \|\mathbf{u}_{mi}\|=1} \|\mathbf{T}_i - \lambda_i \mathbf{u}_{1i} \otimes \mathbf{u}_{2i} \otimes \mathbf{u}_{3i}\|_F^2$$

Such a critical point satisfy

$$\mathbf{T}_i(\cdot, \hat{\mathbf{u}}_{2i}, \hat{\mathbf{u}}_{3i}) = \hat{\lambda}_i \hat{\mathbf{u}}_{1i} \quad \mathbf{T}_i(\hat{\mathbf{u}}_{1i}, \cdot, \hat{\mathbf{u}}_{3i}) = \hat{\lambda}_i \hat{\mathbf{u}}_{2i} \quad \mathbf{T}_i(\hat{\mathbf{u}}_{1i}, \hat{\mathbf{u}}_{2i}, \cdot) = \hat{\lambda}_i \hat{\mathbf{u}}_{3i}$$

Illustration of Signal Recovery with Deflation

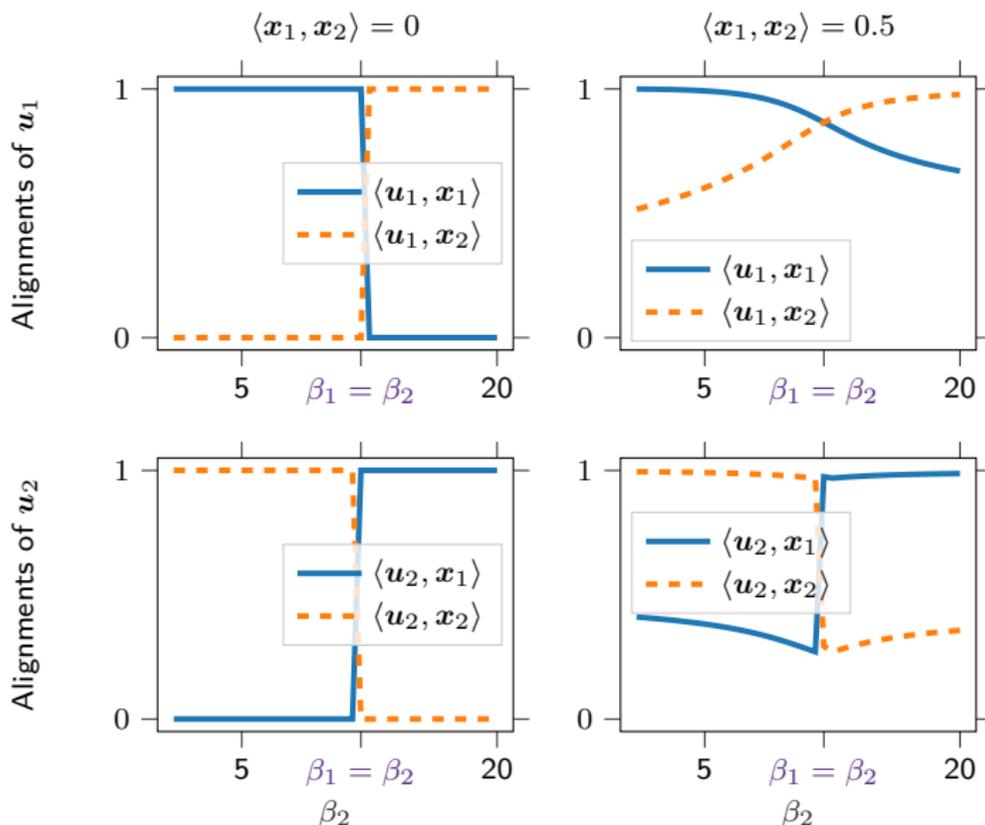


Figure: Deflation on $\mathbf{T}_1 = \sum_{i=1}^2 \beta_i \mathbf{x}_i^{\otimes 3}$ with $\mathbf{x}_i = \mathbf{e}_i \in \mathbb{R}^P$.

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For both deflation steps:

$$\mathbf{T}_i \rightarrow \Phi_3(\mathbf{T}_i, \hat{\mathbf{u}}_{1i}, \hat{\mathbf{u}}_{2i}, \hat{\mathbf{u}}_{3i}) \rightarrow \text{Stieltjes transform } g(z)$$

since \mathbf{T}_i 's are low-rank perturbations of $\frac{1}{\sqrt{n}}\mathbf{X}$.

Assumption 2. Assume that as $n \rightarrow \infty$, there exists a sequence of critical points $(\hat{\lambda}_i, \hat{\mathbf{u}}_{1i}, \hat{\mathbf{u}}_{2i}, \hat{\mathbf{u}}_{3i})$ such that

$$\hat{\lambda}_i \xrightarrow{\text{a.s.}} \lambda_i \quad |\langle \hat{\mathbf{u}}_{mi}, \mathbf{x}_{mj} \rangle| \xrightarrow{\text{a.s.}} \rho_{ij} \quad |\langle \hat{\mathbf{u}}_{m1}, \hat{\mathbf{u}}_{m2} \rangle| \xrightarrow{\text{a.s.}} \eta$$

with $\lambda_i > 2\sqrt{\frac{2}{3}}$ and $\rho_{ij}, \eta > 0$.

Theorem 3 (SGC'21). Under Assumption 2, the ESM of $\Phi_3(\mathbf{T}_i, \hat{\mathbf{u}}_{1i}, \hat{\mathbf{u}}_{2i}, \hat{\mathbf{u}}_{3i})$ converges to the semi-circle law ν of compact support $\left[-2\sqrt{\frac{2}{3}}, 2\sqrt{\frac{2}{3}}\right]$, with Stieltjes transform

$$g(z) = \frac{-3z + 3\sqrt{z^2 - \frac{8}{3}}}{4}, \quad z > 2\sqrt{\frac{2}{3}}$$

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First Deflation Step

Theorem 4 (SGD'22). Under Assumption 2, λ_1 , ρ_{11} and ρ_{12} satisfy

$$\begin{cases} f_g(\lambda_1) = \sum_{i=1}^2 \beta_i \rho_{1i}^3 \\ h_g(\lambda_1) \rho_{1j} = \sum_{i=1}^2 \beta_i \alpha_{ij} \rho_{1i}^2 \end{cases} \quad \text{for } j \in [2]$$

where $\alpha_{ij} = \alpha$ if $i \neq j$ and 1 otherwise, and denote $f_g(z) = z + g(z)$ and $h_g(z) = -\frac{1}{g(z)}$.

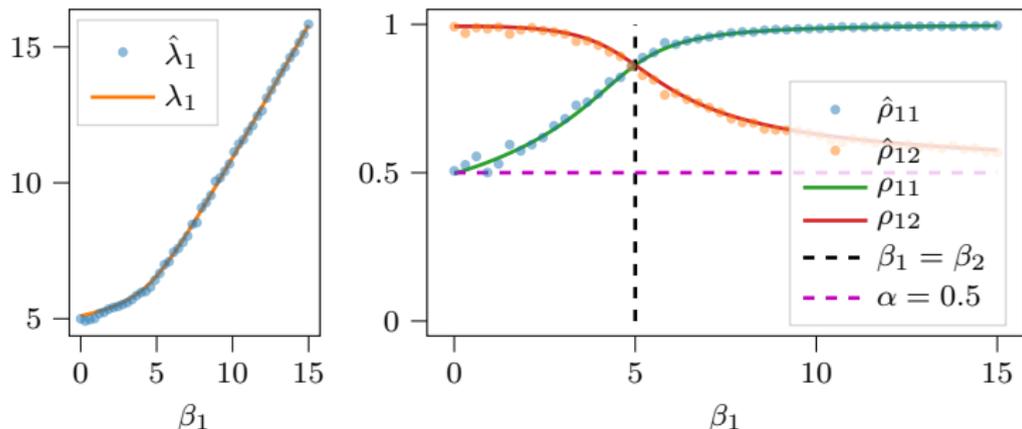


Figure: Simulated versus asymptotic singular value and alignments corresponding to the first deflation step. We considered $\beta_2 = 5$, $\alpha = 0.5$, $p = 100$ and varying $\beta_1 \in [0, 15]$.

Second Deflation Step

Theorem 5 (SGD'22). Under Assumption 2, λ_1 , ρ_{11} and ρ_{12} satisfy

$$\begin{cases} f_g(\lambda_2) + \lambda_1 \eta^3 = \sum_{i=1}^2 \beta_i \rho_{2i}^3 \\ h_g(\lambda_2) \rho_{2j} + \lambda_1 \eta^2 \rho_{1j} = \sum_{i=1}^2 \beta_i \alpha_{ij} \rho_{2i}^2 \\ h_g(\lambda_2) \eta + q_g(\lambda_1) \eta^2 = \sum_{i=1}^2 \beta_i \rho_{1i} \rho_{2i}^2 \end{cases} \quad \text{for } j \in [2]$$

where $\alpha_{ij} = \alpha$ if $i \neq j$ and 1 otherwise, and denote $f_g(z) = z + g(z)$, $h_g(z) = -\frac{1}{g(z)}$ and $q_g(z) = z + \frac{g(z)}{3}$.

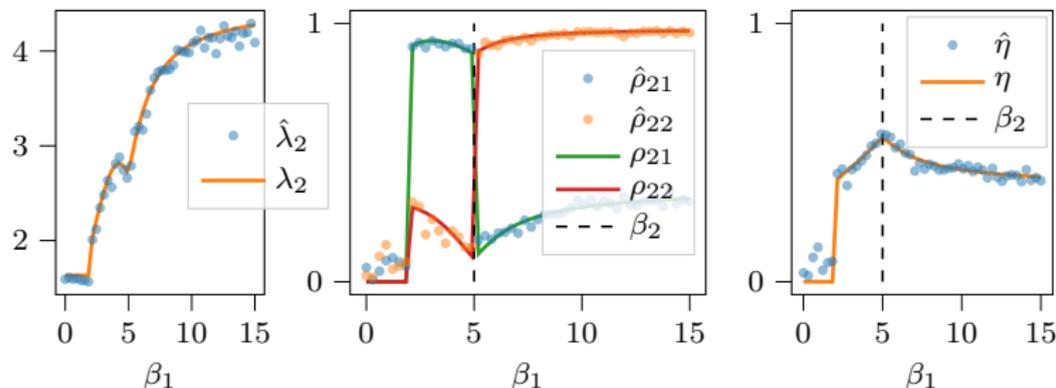


Figure: Simulated versus asymptotic singular value and alignments corresponding to the first deflation step. We considered $\beta_2 = 5$, $\alpha = 0.5$, $p = 100$ and varying $\beta_1 \in [0, 15]$.

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$$\mathbf{T}_1 = \sum_{i=1}^2 \beta_i \mathbf{x}_{1i} \otimes \mathbf{x}_{2i} \otimes \mathbf{x}_{3i} + \frac{1}{\sqrt{n}} \mathbf{X} \in \mathbb{R}^{p \times p \times p}$$

where $\beta_i \geq 0$, $\|\mathbf{x}_{mi}\| = 1$, $X_{ijk} \sim \mathcal{N}(0, 1)$ i.i.d. and $n = 3p$. Assume

$$\alpha \equiv \langle \mathbf{x}_{11}, \mathbf{x}_{12} \rangle = \langle \mathbf{x}_{21}, \mathbf{x}_{22} \rangle = \langle \mathbf{x}_{31}, \mathbf{x}_{32} \rangle \in [0, 1]$$

Orthogonalized Deflation. Compute $\hat{\lambda}_2 \hat{\mathbf{u}}_{12} \otimes \hat{\mathbf{u}}_{22} \otimes \hat{\mathbf{u}}_{32}$ as best rank-one approximation of \mathbf{T}_2 with

$$\mathbf{T}_2 \equiv \mathbf{T}_1 \times_1 (\mathbf{I}_p - \gamma \hat{\mathbf{u}}_{11} \hat{\mathbf{u}}_{11}^\top) = \mathbf{T}_1 - \gamma \hat{\mathbf{u}}_{11} \otimes \mathbf{T}_1(\hat{\mathbf{u}}_{11})$$

where $\gamma \in [0, 1]$ and $\hat{\lambda}_i \hat{\mathbf{u}}_{1i} \otimes \hat{\mathbf{u}}_{2i} \otimes \hat{\mathbf{u}}_{3i}$ is a critical point of

$$\arg \min_{\lambda_i > 0, \|\mathbf{u}_{mi}\|=1} \|\mathbf{T}_i - \lambda_i \mathbf{u}_{1i} \otimes \mathbf{u}_{2i} \otimes \mathbf{u}_{3i}\|_F^2$$

Such a critical point satisfy

$$\mathbf{T}_i(\cdot, \hat{\mathbf{u}}_{2i}, \hat{\mathbf{u}}_{3i}) = \hat{\lambda}_i \hat{\mathbf{u}}_{1i} \quad \mathbf{T}_i(\hat{\mathbf{u}}_{1i}, \cdot, \hat{\mathbf{u}}_{3i}) = \hat{\lambda}_i \hat{\mathbf{u}}_{2i} \quad \mathbf{T}_i(\hat{\mathbf{u}}_{1i}, \hat{\mathbf{u}}_{2i}, \cdot) = \hat{\lambda}_i \hat{\mathbf{u}}_{3i}$$

Associated Random Matrix (Second Deflation Step)

Let $\hat{\kappa} = \langle \hat{\mathbf{u}}_{11}, \hat{\mathbf{u}}_{12} \rangle$

$$\mathbf{T}_2 \rightarrow M_\gamma \equiv \frac{1}{\sqrt{n}} \begin{bmatrix} 0 & \mathbf{X}(\hat{\mathbf{u}}_{32}) & \mathbf{X}(\hat{\mathbf{u}}_{22}) \\ \mathbf{X}(\hat{\mathbf{u}}_{32})^\top & 0 & \mathbf{X}(\hat{\mathbf{u}}_{12}) - \gamma \hat{\kappa} \mathbf{X}(\hat{\mathbf{u}}_{11}) \\ \mathbf{X}(\hat{\mathbf{u}}_{22})^\top & \mathbf{X}(\hat{\mathbf{u}}_{12})^\top - \gamma \hat{\kappa} \mathbf{X}(\hat{\mathbf{u}}_{11})^\top & 0 \end{bmatrix}$$

Remark. If $\gamma = 1$ then $\hat{\kappa} = 0$.

$$\begin{aligned} \lambda_2 \langle \hat{\mathbf{u}}_{11}, \hat{\mathbf{u}}_{12} \rangle &= \mathbf{T}_2(\hat{\mathbf{u}}_{11}, \hat{\mathbf{u}}_{22}, \hat{\mathbf{u}}_{32}) \\ &= \mathbf{T}_1(\hat{\mathbf{u}}_{11}, \hat{\mathbf{u}}_{22}, \hat{\mathbf{u}}_{32}) - \underbrace{\langle \hat{\mathbf{u}}_{11}, \hat{\mathbf{u}}_{11} \rangle}_{=1} \mathbf{T}_1(\hat{\mathbf{u}}_{11}, \hat{\mathbf{u}}_{22}, \hat{\mathbf{u}}_{32}) = 0 \end{aligned}$$

which yields a **semi-circle law** as in Hotteling's deflation.

Assumption 3. Assume that as $n \rightarrow \infty$, there exists a sequence of critical points $(\hat{\lambda}_2, \hat{\mathbf{u}}_{12}, \hat{\mathbf{u}}_{22}, \hat{\mathbf{u}}_{32})$ such that for $m \neq 1$

$$\begin{aligned} \hat{\lambda}_2 &\xrightarrow{\text{a.s.}} \lambda_2 & |\langle \hat{\mathbf{u}}_{12}, \mathbf{x}_{1i} \rangle| &\xrightarrow{\text{a.s.}} \theta_{2i} & |\langle \hat{\mathbf{u}}_{m2}, \mathbf{x}_{mi} \rangle| &\xrightarrow{\text{a.s.}} \rho_{2i} \\ |\langle \hat{\mathbf{u}}_{11}, \hat{\mathbf{u}}_{12} \rangle| &\xrightarrow{\text{a.s.}} \kappa & |\langle \hat{\mathbf{u}}_{m1}, \hat{\mathbf{u}}_{m2} \rangle| &\xrightarrow{\text{a.s.}} \eta \end{aligned}$$

with $\lambda_2 > \lambda_+$ and $\theta_{2i}, \rho_{2i}, \eta > 0$.

Associated Random Matrix (Second Deflation Step)

Theorem 6 (SMD'23). Under Assumption 3, the ESM of M_γ converges weakly to a deterministic measure μ having Stieltjes transform $s(z) = a(z) + 2b(z)$ verifying $\Im[s(z)] > 0$ for $\Im[z] > 0$, where $a(z)$ and $b(z)$ satisfy, for $z \notin \text{Supp}(\mu)$

$$\begin{cases} [2b(z) + z]a(z) + \frac{1}{3} = 0 \\ (a(z) + z - \tau b(z))b(z) + \frac{1}{3} = 0 \end{cases}$$

with $\tau = \gamma\kappa^2 - 1 + \kappa(\gamma - 1)$.

Repeat

- ▶ $a \leftarrow -1/(3(2b + z))$
- ▶ $b \leftarrow -1/(3(a + z - \tau b))$

Until convergence of a and b .

$$\mu(dx) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \Im[s(x + i\epsilon)]$$

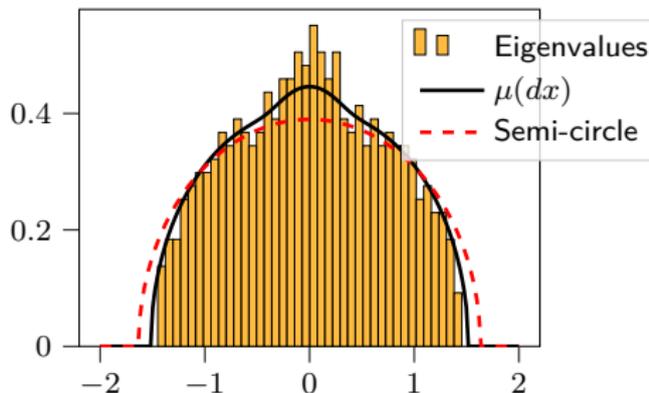


Figure: Histogram of the eigenvalues of M_γ and limiting measure μ . We considered $p = 200$, $\beta_1 = 20$, $\beta_2 = 15$, $\alpha = 0.8$, $\gamma = 0.85$.

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Theorem 6 (SMD'23). Under Assumption 3, λ_2 , θ_{2i} , ρ_{2i} , κ and η satisfy

$$\left\{ \begin{array}{l} f_s(\lambda_2) - \frac{\gamma\kappa\eta^2}{3}g(\lambda_1) - 2\gamma\kappa^2b(\lambda_2) = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{2i}^2 - \gamma\kappa \sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}^2 \\ [f_s(\lambda_2) - a(\lambda_2)]\theta_{2j} - \gamma\rho_{1j} \left[\frac{\eta^2}{3}g(\lambda_1) + 2\kappa b(\lambda_2) \right] = \sum_{i=1}^2 \beta_i\alpha_{ij}\rho_{2i}^2 - \gamma\rho_{1j} \sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}^2 \\ [\lambda_2 + 2(1-\gamma)b(\lambda_2)]\kappa = (1-\gamma) \left[\sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}^2 - \frac{\eta^2}{3}g(\lambda_1) \right] \\ \left[f_s(\lambda_2) - (1+\gamma\kappa^2)b(\lambda_2) \right] \rho_{2j} = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{2i}\alpha_{ij} - \gamma\kappa \left[\sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}\alpha_{ij} - \frac{\rho_{1j}\eta}{3}g(\lambda_1) \right] \\ \left[\lambda_2 + a(\lambda_2) + (1-\gamma\kappa^2)b(\lambda_2) - \frac{\gamma\kappa}{3}g(\lambda_1) \right] \eta = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{1i}\rho_{2i} - \gamma\kappa \sum_{i=1}^2 \beta_i\rho_{1i}^2\rho_{2i} \end{array} \right.$$

where $\alpha_{ij} = \alpha$ if $i \neq j$ and 1 otherwise, and denote $f_s(z) = z + s(z)$.

Case $\gamma = 1$. The above system reduces to

$$\left\{ \begin{array}{l} f_g(\lambda_2) = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{2i}^2 \\ h_g(\lambda_2)\theta_{2j} - \frac{\eta^2}{3}g(\lambda_1)\rho_{1j} = \sum_{i=1}^2 \beta_i\alpha_{ij}\rho_{2i}^2 - \rho_{1j} \sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}^2 \\ h_g(\lambda_2)\rho_{2j} = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{2i}\alpha_{ij} \\ h_g(\lambda_2)\eta = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{1i}\rho_{2i} \end{array} \right.$$

since $\kappa = 0$.

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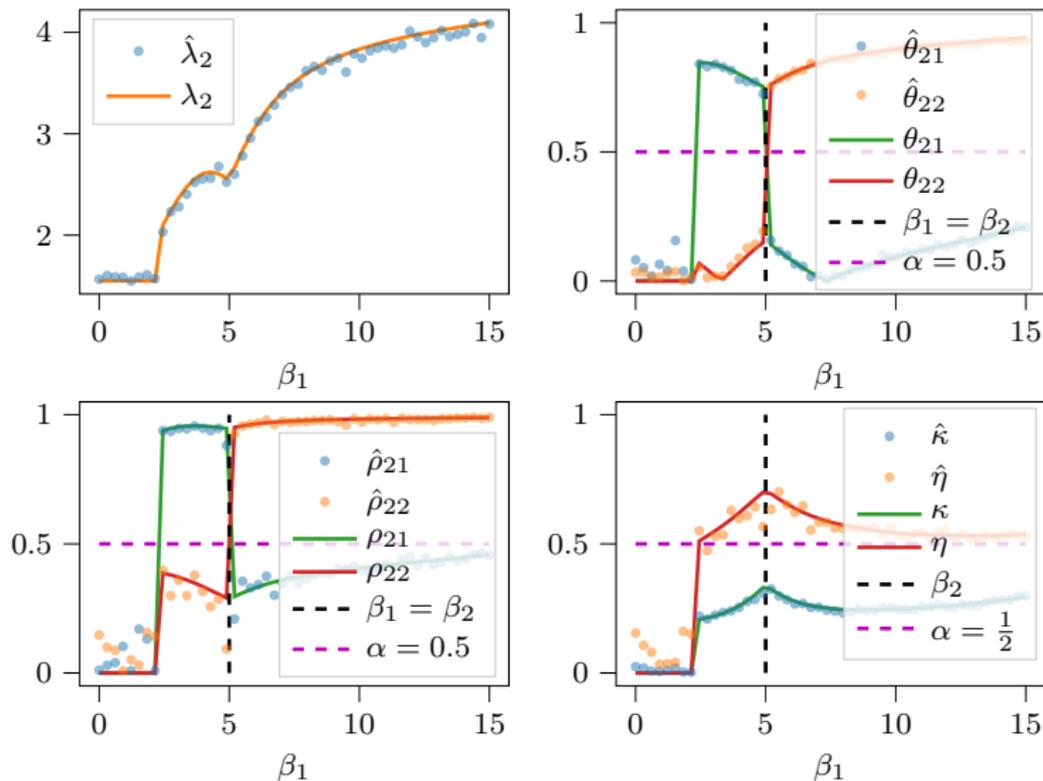


Figure: $\beta_2 = 5$, $\alpha = 0.5$, $p = 100$, $\gamma = 0.8$ and varying $\beta_1 \in [0, 15]$.

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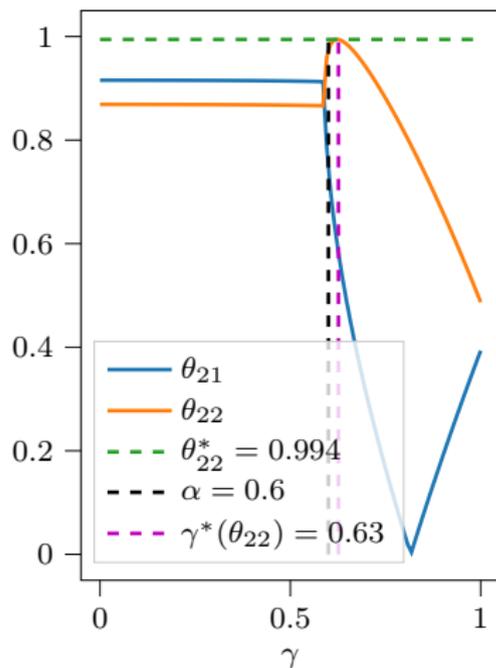
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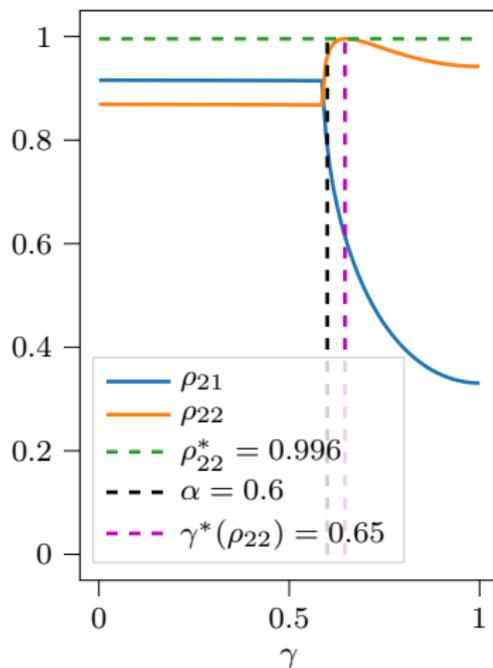
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First mode alignments



Other modes alignments

Figure: $\beta_1 = 10$, $\beta_2 = 9$, $\alpha = 0.6$, $p = 100$, $\alpha = 0.6$ and varying $\gamma \in [0, 1]$.

Model Parameters Estimation

- ▶ Compute $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\eta} = \langle \hat{\mathbf{u}}_{21}, \hat{\mathbf{u}}_{22} \rangle$ by orthogonalized deflation for $\gamma = 1$.
- ▶ Solve the previous systems (for $\gamma = 1$) in β_1 , β_2 , α and the alignments.

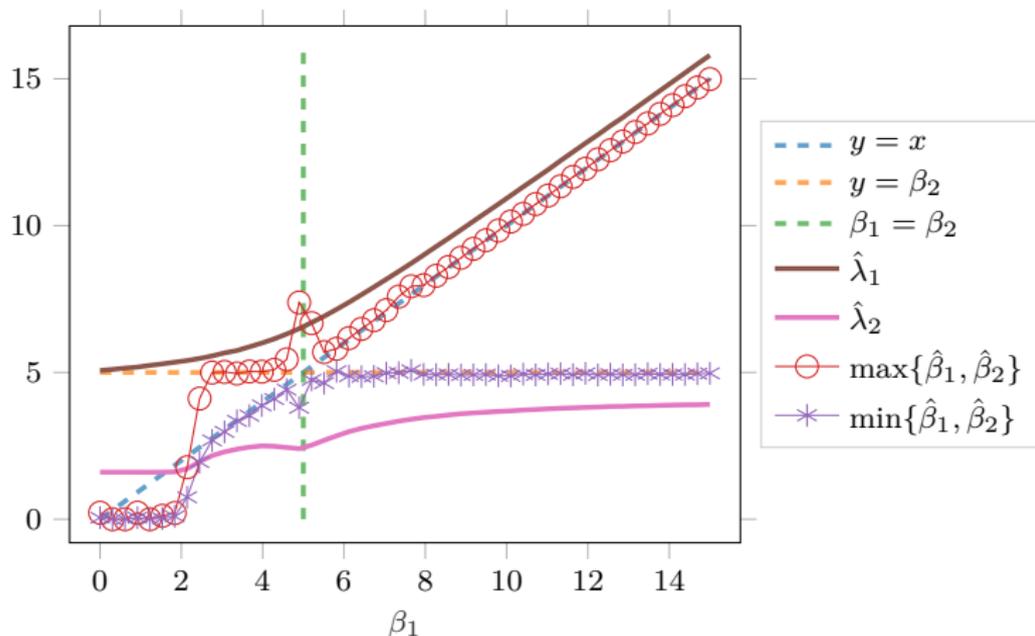


Figure: $\beta_2 = 5$, $\alpha = 0.5$, $p = 150$ and $\gamma = 1$ while varying β_2 . The curves are averaged over 100 realizations of \mathbf{T}_1 .

RTT-improved Deflation Algorithm

- ▶ Perform orthogonalized deflation with $\gamma = 1$.
- ▶ Model estimation $(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})$.
- ▶ Estimate optimal γ^* to maximize ρ_{22} (solve systems and update $\gamma \leftarrow \gamma - \epsilon$ for $\epsilon > 0$).
- ▶ Perform orthogonalized deflation with γ^* .
- ▶ Re-estimate first component as best rank-one approximation of $\mathbf{T}_2 - \min\{\hat{\beta}_1, \hat{\beta}_2\} \hat{u}_2^* \otimes \hat{v}_2^* \otimes \hat{w}_2^*$.

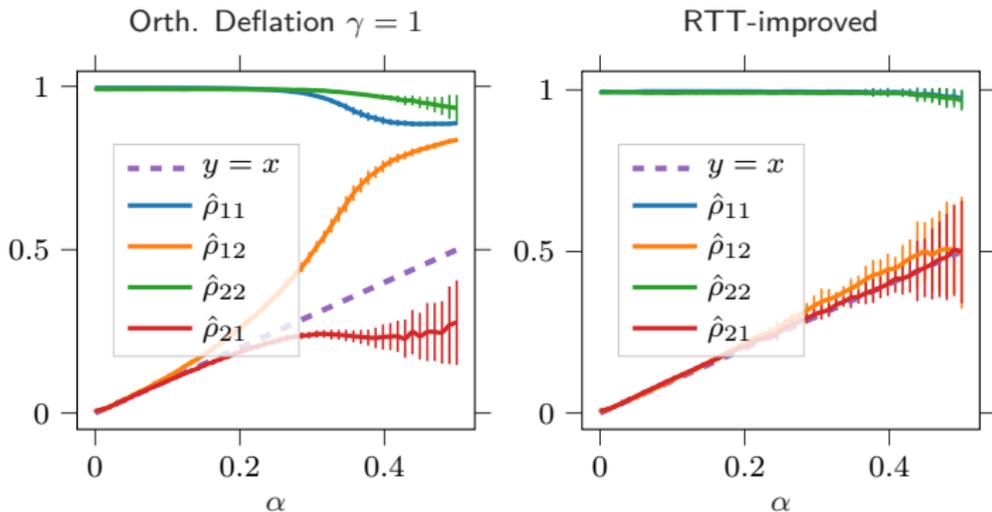


Figure: $\beta_1 = 6$, $\beta_2 = 5.7$ and $p = 150$. The curves are obtained by averaging over 200 realizations of \mathbf{T}_1 .

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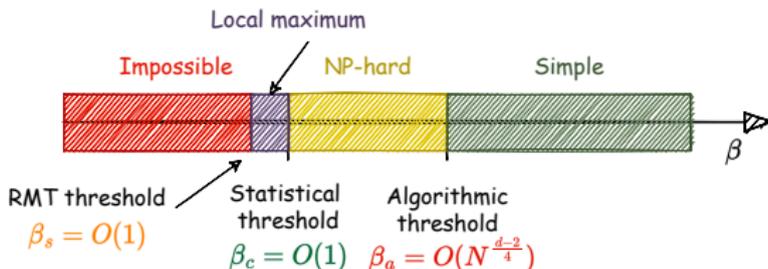
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Take Away Messages

- ▶ The RMT approach allows the study of asymmetric spiked tensor models.
- ▶ The obtained results characterize the performance of the MLE for β large enough (i.e., $\beta \geq \beta_c$).



Open questions:

- ▶ Still unclear how to characterize the **phase transition** of the MLE with the RMT approach.
- ▶ Is it possible to find a **polynomial time algorithm** that is consistent below the computational threshold β_a ?
- ▶ Study of higher order statistics and **fluctuations**?
- ▶ Proof of **consistency** of model estimation?
- ▶ Study the **existence** and **uniqueness** of the solutions of the deflation cases.
- ▶ **Universality** and generalization to other decomposition methods.

Thank you for your attention!

[melaseddik.github.io](https://github.com/melaseddik)

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