Applying free probability to black holes

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The generalized entropy of a black hole

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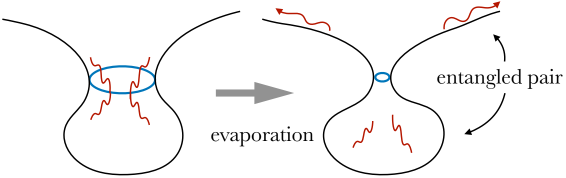
Hawking: a black hole does radiate quantum mechanically, and putting this fact together with the other known laws of black hole mechanics confirms Bekenstein's conjecture,

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{horizon}}}{4G_N\hbar}$$



The black hole information paradox

If the radiation is thermal and featureless as shown by Hawking, the radiation entropy, $S(R) := -\mathrm{Tr}\rho_R \log \rho_R$, shall keep increasing. Where is the information carried by the collapsing star after the black hole evaporation? It seems that we end up having a non-unitary evolution. This is a version of the black hole information paradox.

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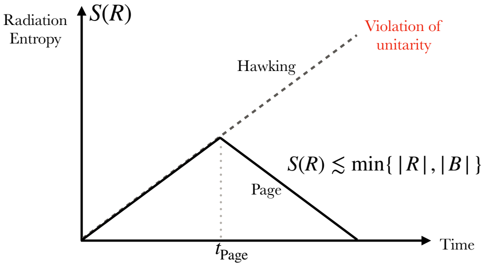
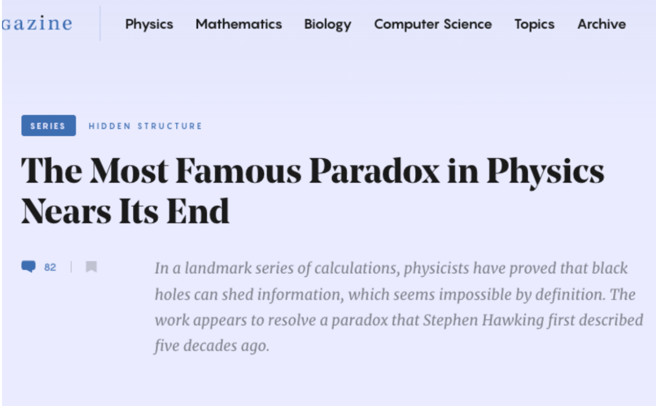
Page asked, "How would the entropy behave if we demand unitarity?" If black holes were to evolve unitarily as demanded by quantum theory, say under some typical random unitary, then the radiation entropy should follow the Page curve.



The challenge is to obtain the Page curve using the first-principled gravity calculations.

Recent progress on deriving the Page curve



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SERIES HIDDEN STRUCTURE

The Most Famous Paradox in Physics Nears Its End

82

In a landmark series of calculations, physicists have proved that black holes can shed information, which seems impossible by definition. The work appears to resolve a paradox that Stephen Hawking first described five decades ago.

[Penington, Almheiri, Engelhardt, Marolf, Maxfield, Maldacena, Mahajan, Zhao, Hartman, Shaghoulian, Tajdini, Shenker, Stanford, Yang ...]

Gravitational replica trick

The standard formula for the vN entropy seems to give us a non-unitary answer, so let's try a different way to compute the radiation entropy. [Cardy, Calabrese, Hawking, Gibbons, Hartle, Lewkowycz, Maldacena, Faulkner]

$$S(R) = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr} \rho_R^n = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr} \rho_R^{\otimes n} \tau_n.$$

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We can use the gravitational path integral (GPI) to capture $\langle \tau_n \rangle_\rho = \mathrm{Tr} \rho_R^{\otimes n} \tau_n$ by setting up appropriate boundary conditions.

$$S(R) := \lim_{n \to 1} \frac{1}{1-n} \log \langle \tau_n \rangle := \lim_{n \to 1} \frac{1}{1-n} \log \frac{Z[\mathcal{B}^{\times n}, \tau_n]}{Z[\mathcal{B}]^n} .$$

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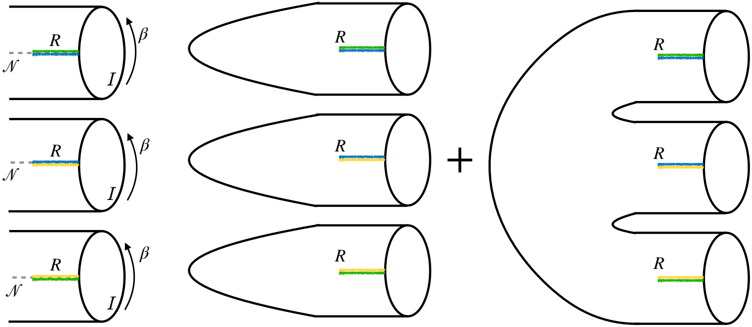
The full GPI is tricky to evaluate, so we use the *saddlepoint approximation.* There are two most important saddles. A fully disconnected one and a fully connected one.

Two saddles

For example, we compute $Z[\mathcal{B}^{\times 3}, \tau_3]$,



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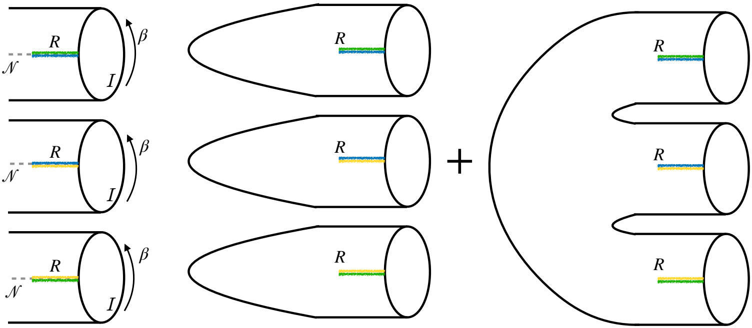
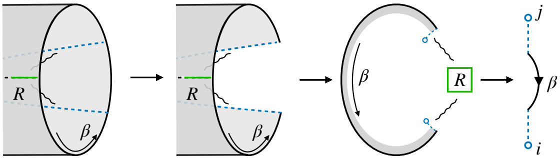
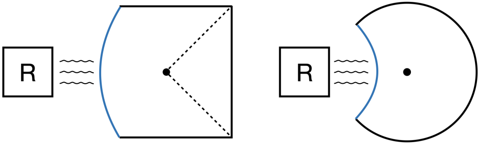


The former is known by Gibbons and Hawking which results in increasing entropy; whereas the latter, known as the replica wormhole, is responsible for the decreasing entropy in the second half of the Page curve. They swap dominance after the Page time. The radiation entropy is then given by the *island formula* that accounts for the effect of the saddle switch.

A concrete problem: the Penington-Shenker-Stanford-Yang model



The solution to this boundary condition in JT gravity is given by an "End of the World" (EOW) brane (geodesic) that acts as the black hole interior which hosts all the d.o.f. that are entangled with the emitted radiation collected in some auxiliary system.



Page curve in the PSSY model

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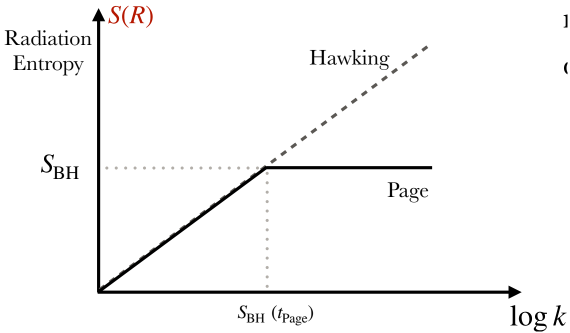

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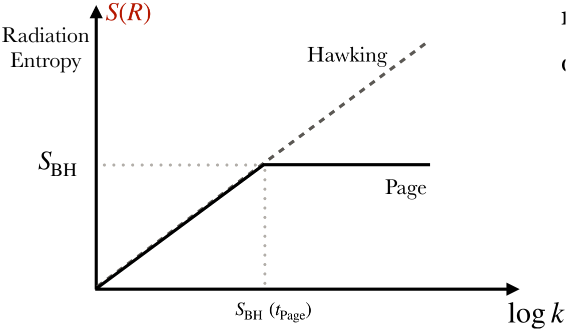
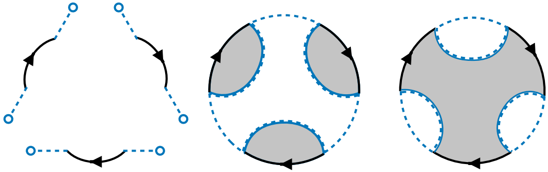


We can understand the Hawking curve as the entanglement entropy the maximally entangled state that one naively thinks to describe the black hole-radiation systems, $\frac{1}{k}\sum_{i=1}^{k}|\psi_i\rangle_B|i\rangle_R$. However, the GPI calculation shall show that $\{|\psi_i\rangle_B\}$ are slightly overlapping rendering a "capping-off" of the radiation entropy as predicted by Page. This is consistent with the fact that B is really only has a finite amount of d.o.f.

The island formula

We consider the replica trick calculation of the radiation entropy $S(R)$ in the PSSY model (JT+ brane). For example, consider the calculation for three replicas again,



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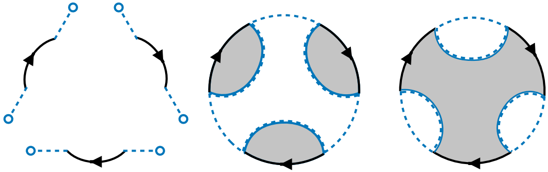

We want to compute Z_3/Z_1^3. The disconnected saddle roughly evaluates to $ke^{3S_{\rm BH}}$, and the wormhole saddle evaluates to $k^3e^{S_{\rm BH}}$. We have

$$Z_3 = ke^{3S_{\rm BH}} + k^3e^{S_{\rm BH}}.$$

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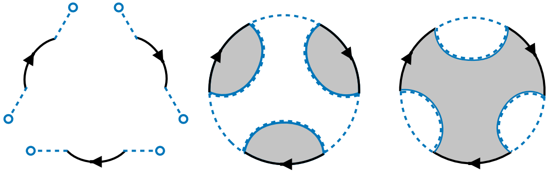

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The normalization factor is given by $Z_1^3 = (ke^{S_{\rm BH}})^3$.Using the replica trick formula, the end result is given by the (island) formula:

$$S(R) = \min\{\log k, S_{\rm BH}\}.$$

Breakdown of the island formula "near" the transition

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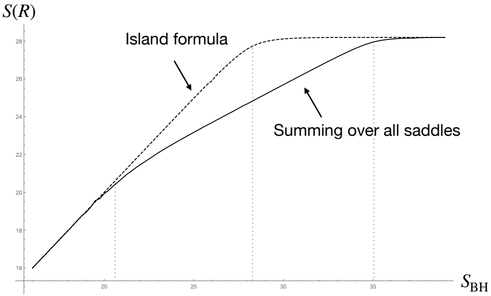
For example, consider a black hole-radiation system BR in a superposition of two branches, where one has much more entanglement than the other. The spectrum is “L”-shaped, and the island formula $S(R) = \min\{\log k, S_{\mathrm{BH}}\}$ is largely different from the answer obtained from summing over all geometries.

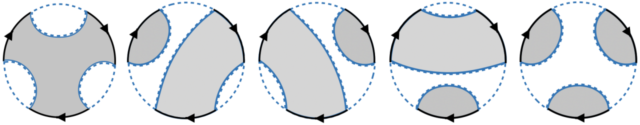


PSSY model with non-flat bulk entanglement spectrum

Consider now a more general "naive" state that describes the black hole-radiation system. We want to know what's the corresponding Page curve.

$$|\rho\rangle_{\mathrm{BR}} = \sum_{i=1}^{k} \sqrt{c_i} |\psi_i\rangle_{\mathrm{B}} |i\rangle_{\mathrm{R}}$$

We'd like to sum over all saddles in the planar limit ($k \gg 1$ no crossings, $S_0 \gg 1$ no higher genus), e.g.



From each saddle, there are contributions from the gravitational sector (Z_n from an n-connected disk/wormhole) and the matter sector (brane index loops). Unlike in PSSY, the brane index loops are now weighed by c_i and an n-connected loop evaluates to $\mathrm{Tr}(k\rho_{\mathrm{R}})^n$, which always gives k for a flat $\rho_{\mathrm{R}} \propto I$.

The precise formula for the GPI

The partition function for an n-connected disk/wormhole reads

$$Z_n = \int_0^\infty \mathrm{d}E\, \rho(E) y(E)^n \,,$$
$$y(E) := 2^{-\beta E} 2^{1-2\mu} |\Gamma(\mu - 1/2 + i\sqrt{2E})|^2, \rho(E) := 2^{S_0} \sinh(2\pi\sqrt{2E})/(2\pi^2)$$

where $\rho(E)$ is the **density of states**. Z_1 is the thermal partition function (for a thermal circle of length β disrupted by the brane of tension μ). The constant S_0 is the topological entropy. Therefore, Z_n/Z_1^n computes the moments of a thermal state,

$$Z_n/Z_1^n = \int_0^{y(0)} \frac{\rho(E(y))}{-y'(E(y))} (y/Z_1)^n \mathrm{d}y \overset{x=y/Z_1}{=} \int_0^{x(0)} \nu_b(x) x^n \mathrm{d}x \,.$$

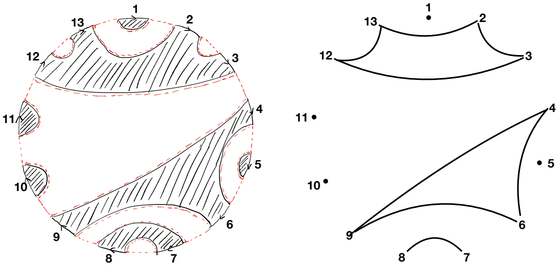
Bekenstein-Hawking entropy is then given by

$$S_{\mathrm{BH}} = -\int \nu_b(x)\, x \log x \overset{\mu \gg 1/\beta}{=} S_0 + \frac{4\pi^2}{\beta} + \mathcal{O}(1) \,.$$

The moments Z_n/Z_1^n are packaged in the distribution ν_b, which is **NOT** a probability distribution! (it has a divergent zeroth moment.)

Summing over all saddles: sorting out the combinatorics

In the planar limit ($k \gg 1$ no crossings, $S_0 \gg 1$ no higher genus), the saddles are organized by non-crossing partitions. e.g. $(1)(2\ 3\ 12\ 13)(4\ 6\ 9)(5)(7\ 8)(10)(11)$

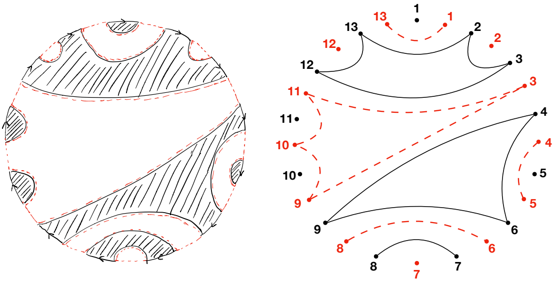


So we have

$$\tilde{Z}_n = \sum_{\pi \in \mathrm{NC}_n} \left(\prod_{V \in \pi} Z_{|V|} \cdot \text{(matter sector)} \right).$$

Summing over all saddles: sorting out the combinatorics

Key observation: the brane index loops define another set of cycles/partitions corresponding to a unique NC permutation $\bar{\pi} \in \mathrm{NC}_n$ dual to $\pi \in \mathrm{NC}_n$. This is called the **Kreweras complement** of π, defined by the following diagram. (Rule of the game: replicate another set of $[n]$ (in red) and interlace them; then link up as many red dots as possible without crossing the black dots.)



e.g. $\bar{\pi} = (1\ 13)(2)(3\ 9\ 10\ 11)(4\ 5)(6\ 8)(7)(12) \in \mathrm{NC}_{13}$.

A formula for $\tilde{Z}_n$

We thus obtain the general formula for the replica trick partition functions,

$$\tilde{Z}_n = \sum_{\pi \in \mathrm{NC}_n} \left(\prod_{V \in \pi} Z_{|V|} \cdot \prod_{\bar{V} \in \bar{\pi}} \mathrm{Tr}(k\rho_{\mathrm{R}})^{|\bar{V}|} \right) .$$

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Note that this formula resembles the combinatorial formula describing **free multiplicative convolution**, provided we treat Z_n's as cumulants.

$$m_n(\nu_a \boxtimes \nu_b) = \sum_{\pi \in \mathrm{NC}_n} \left(\prod_{V \in \pi} \kappa_{|V|}(\nu_a) \cdot \prod_{\bar{V} \in \bar{\pi}} m_{|\bar{V}|}(\nu_b) \right) .$$

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The main hypothesis of the replica trick is that the $\tilde{Z}_n$'s (with appropriate normalization) give the moments (w.r.t. an appropriate trace) of a density operator $\tilde{\rho}_R$ with spectrum density μ_r. We now show that this is indeed true here.

Regularize the density of states

Recall that the density of states $\rho(E) \sim \sinh(\sqrt{2E})$ is not integrable and so is ν_b which diverges at $x \to 0$. Therefore, ν_b is not a probability distribution :(

The way out is to work with a sequence of regularized density of states $(\rho_N(E))_N$ parameterized by N that are integrable. For example,

$$\rho_N(E) := \frac{2^{S_0}}{2\pi^2} \sinh\left(2\pi\sqrt{\frac{E(E_N - E)}{E_N}}\right), \quad 0 \leq E \leq E_N$$

where E_N is determined via $\int_0^{E_N} \mathrm{d}E \rho_N(E) = N$.

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where E_N is determined via $\int_0^{E_N} \mathrm{d}E\, \rho_N(E) = N$. Then we define a regularized probability distribution ν_{b_N} supported on $[y(E_N)/Z_1, y(0)/Z_1]$ such that

$$(Z_n/Z_1^n =)\ m_n(\nu_b) = \lim_{N\to\infty} N^{1-n} m_n(\nu_{b_N})\ .$$

We then work at finite N with ν_{b_N} and take the large N limit in the end. In the end, the result is independent of how we regularize $\rho_N(E)$.

Making the convolution work

Recall the formula for $\tilde{Z}_n$,

$$\tilde{Z}_n = \sum_{\pi \in \mathrm{NC}_n} \left(\prod_{V \in \pi} Z_{|V|} \cdot \prod_{\bar{V} \in \bar{\pi}} \mathrm{Tr}(k\rho_{\mathrm{R}})^{|\bar{V}|} \right).$$

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Aside: free Poisson law

- $\mu_{N/k,\nu_{b_N}}$ is defined by the cumulants $(N/k)m_n(\nu_{b_N})$, which looks like the rescaled moments of another distribution ν_{b_N}. This is known as the **compound free Poisson law** $\mu_{\lambda,\nu}$, characterized by the jump rate λ and the jump distribution ν.
- A free Poisson law is when $\nu = \delta_\alpha$, which is more commonly known as the Marchenko–Pastur distribution in RMT.
- Free Poisson has some nice properties: free infinite divisibility ($\mu = \mu_k^{\boxplus k}, \forall k \in \mathbb{Z}_+$), free Levy-Khintchine theorem, etc.

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$$S_{\nu_r}(x) = \mathfrak{z}(x)(1+x)/x \tag{0.1}$$

where $\mathfrak{z}(x)$ is defined via

$$\mathfrak{z}(x) R_{\mu_{1,\nu_r}}(\mathfrak{z}(x)) = x \ , \tag{0.2}$$

and

$$S_{\mu_{b;N}}(x) = z(x)/x \tag{0.3}$$

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$$z(x) R_{\mu_{b;N}}(z(x)) = x \ . \tag{0.4}$$

Then it follows that

$$\begin{cases} R_{\mu_{1,\nu_r}}(\mathfrak{z}(x)) & = x z(x) \\ R_{\mu_{b;N}}(z(x)) & = x \mathfrak{z}(x) \end{cases} . \tag{0.5}$$

Then we solve for $\mathfrak{z}(x)$ and take the large N limit. Using $\mathfrak{z}(x)$, the Cauchy transform is given by

$$G_{\nu_r \boxtimes \mu_b}(x) = x^{-1} + \mathfrak{z}(x) R_{\mu_{1,\nu_r}}(\mathfrak{z}(x))/x \ . \tag{0.6}$$

Evaluating the convolution

Consider any y in the upper complex plane, $y \in \mathbb{C}^+$, solve the following equation for $\mathfrak{z}(y)$,

$$\int_{\mathbb{R}^+} \frac{\nu_b(x)}{z/x - \int_{\mathbb{R}^+} \frac{x'}{1 - x'\mathfrak{z}(y)} \nu_r(x')\mathrm{d}x'} \mathrm{d}x = \mathfrak{z}(y)$$

This is a fixed-point equation for $\mathfrak{z}(y)$. Then plug the solution into

$$G_{\mu_r}(z) = \frac{1}{z} \int_{\mathbb{R}^+} \frac{1}{1 - x\mathfrak{z}(y)} \nu_r(x)\mathrm{d}x \,,$$

which gives the Cauchy transform of μ_r. The spectral distribution can be extracted using the Stieltjes inversion,

$$\mu_r(x) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \operatorname{Im} G_{\mu_r}(x + i\epsilon) \,.$$

There is no closed form solution in generally but it is numerically tractable.

Page curve

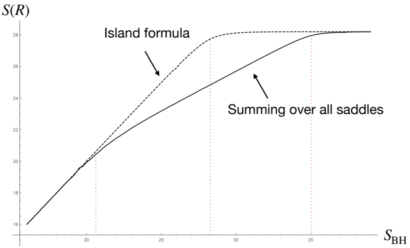
We can then compute the radiation entropy and obtain the accurate Page curve.

$$S(R) = -\int \mu_r(x)\, x \log x \ .$$

This is to be contrasted with the other two entropies

$$S_{\mathrm{BH}} = -\int \mu_b(x)\, x \log x \ , \quad S(\rho_{\mathrm{R}}) = -\int \nu_r(x)\, x \log x \ .$$

With this, we obtain the more accurate Page curve that the island formula fails to capture.



The island formula holds when the convolution **factorizes** if a particular term dominates the sum.

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- The important physics entailed in resolving the information paradox has already been captured by the replica wormholes and the island formula. Free probability theory only serves as a technical tool to help with the calculations.
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- The hope is to find some physical problems where notions from FPT like free independence or free entropy, etc are conceptually indispensable, which in turn can help illustrate the obscure meanings of these ideas.

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- This also seems to be the case for almost all applications of FPT in physics.
- The hope is to find some physical problems where notions from FPT like free independence or free entropy, etc are conceptually indispensable, which in turn can help illustrate the obscure meanings of these ideas.
- Given other examples such as the well-known interplay between 2D gravity and matrix models and the more recently discussed connection between the double-scaled SYK model and the q-Gaussians, quantum gravity could potentially provide a set of such problems that are most naturally perceived through the lens of free probability.

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Thank you!

GPI as a free multiplicative convolution

Recall the formula for $\tilde{Z}_n$,

$$\tilde{Z}_n = \sum_{\pi \in \mathrm{NC}_n} \left(\prod_{V \in \pi} Z_{|V|} \cdot \prod_{\bar{V} \in \bar{\pi}} \mathrm{Tr}(k\rho_{\mathrm{R}})^{|\bar{V}|} \right) .$$

Let's use $\tilde{Z}_n / kZ_1^n$ as moments to define a probability distribution μ_r, (such that our radiation density operator is correctly normalized). Also, the moments $\frac{1}{k}\mathrm{Tr}\,(k\rho_{\mathrm{R}})^n$ define a probability distribution ν_{r}.

$$m_n(\mu_r) := \frac{\tilde{Z}_n}{kZ_1^n} = k^n \sum_{\pi \in \mathrm{NC}_n} \left(\prod_{V \in \pi} \frac{1}{k} m_{|V|}(\nu_b) \cdot \prod_{\bar{V} \in \bar{\pi}} m_{|\bar{V}|}(\nu_{\mathrm{r}}) \right) .$$

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We can write it in terms of finite N,

$$m_n(\mu_r) := \lim_{N \to \infty} \sum_{\pi \in \mathrm{NC}_n} \left(\prod_{V \in \pi} \frac{N^{1-n}}{k^{1-n}} m_{|V|}(\nu_{b_N}) \cdot \prod_{\bar{V} \in \bar{\pi}} m_{|\bar{V}|}(\nu_{\mathrm{r}}) \right),$$

To match it with the free multiplicative convolution, we need to treat $\frac{N^{1-n}}{k^{1-n}} m_n(\nu_{b_N})$ as **cumulants**, and they are in fact a legit set of cumulants.