Abstract

We study the stochastically forced system of isentropic Euler equations of gas dynamics with a $\gamma$-law for the pressure. We show the existence of martingale solutions; we also discuss the existence of invariant measure in the concluding section.

Keywords: Stochastic partial differential equations, isentropic Euler equations, kinetic formulation, entropy solutions.

MSC: 60H15, 35R60, 35L65, 76N15

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In this paper, we study the stochastically forced system of isentropic Euler equations of gas dynamics with a $\gamma$-law for the pressure. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis, let $T$ be the one-dimensional torus, let $T > 0$ and set $Q_T := T \times (0, T)$. We study the system

\begin{align*}
d\rho + (\rho u)_x dt &= 0, \quad \text{in } Q_T, \quad (1.1a) \\
(d\rho u + (\rho u^2 + p(\rho))_x) dt &= \Phi(\rho, u)dW(t), \quad \text{in } Q_T, \quad (1.1b) \\
\rho = \rho_0, \quad \rho u = \rho_0 u_0, \quad \text{in } T \times \{0\}, \quad (1.1c)
\end{align*}

where $p$ follows the $\gamma$-law

$$p(\rho) = \kappa \rho^{\gamma}, \quad \kappa = \frac{\theta^2}{\gamma}, \quad \theta = \frac{\gamma - 1}{2}, \quad (1.2)$$

for $\gamma > 1$, $W$ is a cylindrical Wiener process and $\Phi(0, u) = 0$. Therefore the noise affects the momentum equation only and vanishes in vacuum regions. Our aim is to prove the existence of solutions to (1.1) for general initial data (including vacuum). When $\gamma \leq 2$, we will manage to show the existence of martingale solutions, cf. Theorem 2.7 below.
There are to our knowledge no existing results on stochastically forced systems of first-order conservation laws\textsuperscript{1}. The question of existence of solutions for (1.1) is one of the first problem to be solved, and what we will do here for $1 < \gamma \leq 2$, but there are some other ones to be considered (see the conclusion part, Section 6 on this subject). In the deterministic case, the existence of (entropy) solutions has been proved by Lions, Perthame, Souganidis in [LPS96] (let us mention also the anterior papers by Di Perna [DiP83a], Ding, Chen, Luo [DCL85], Chen [Che86], Lions, Perthame, Tadmor [LPT94a]). The uniqueness of entropy solutions is still an open question nowadays.

For scalar non-linear hyperbolic equations with a stochastic forcing term, the theory has recently known a lot of developments. Well-posedness has been proved in different contexts and under different hypotheses and also with different techniques: by Lax-Oleinik formula (E, Khanin, Mazel, Sinai [EKMS00]), Kruzhkov doubling of variables for entropy solutions (Kim [Kim03], Feng, Nualart [FN08], Vallet, Wittbold [VW09], Chen, Ding, Karlsen [CDK12], Baudet, Vallet, Wittbold [BVW12], kinetic formulation (Debussche, Vovelle [DV10, DV], the more general results being the resolution in $L^1$ given in Debussche, Vovelle [DV13]. Let us also mention the works of Hofmanová in this fields (extension to second-order scalar degenerate equations, convergence of the BGK approximation [Hof13b, DHV13, Hof13a]) and the recent works by Lions, Perthame, Souganidis [LPS12, LPS13] on scalar conservation laws with quasilinear stochastic terms.

We will show existence of martingale solutions to (1.1), see Theorem 2.7 below, under the restriction $\gamma \leq 2$. The procedure is standard: we prove the convergence of (subsequence of) solutions to the parabolic approximation to (1.1). For this purpose we have to adapt the concentration compactness technique (cf. [DiP83a, LPS96]) of the deterministic case to the stochastic case. Such an extension has already been done for scalar conservation laws by Feng and Nualart [FN08] and what we do is quite similar. The mode of convergence for which there is compactness, if we restrict ourselves to the alea, is the convergence in law. That is why we obtain martingale solutions. There is a usual trick (the Gyöngy-Krylov characterization of convergence in probability) that allow to recover pathwise solutions once pathwise uniqueness of solutions is known (cf. [Hof13b, Section 4.5]). However for the stochastic problem (1.1) (as it is already the case for the deterministic one), no such results of uniqueness are know and we will say nothing else about pathwise solutions.

Besides the proof of convergence of the parabolic approximation to Problem (1.1) (cf. Problem (3.1)) which needs serious adaptation due to the loss of $L^\infty$ bounds with respect to the deterministic case, a large part of our

\textsuperscript{1}see however the works by Tornare and Fujita [TFY97] and by Feireisl, Maslowski, Novotny [FMN13] on stochastically forced flows with viscosity
analysis is devoted to the proof of existence of martingale solutions to the parabolic approximation (3.1) which is not a classical point here. What is challenging, and differ from the deterministic case, is the issue of positivity of the density. We solve this problem by using the uniformization effects of parabolic equations with drifts and a bound given by the entropy, in the spirit of Mellet, Vasseur [MV09], cf. Theorem A.1. We will give more details about the main problematic of the paper in Section 2.4, after our framework has been introduced more precisely.

2 Notations and main result

2.1 Stochastic forcing

Our hypotheses on the stochastic forcing term $\Phi(\rho,u)W(t)$ are the following ones. We assume that $\mathcal{W} = \sum_{k \geq 1} \beta_k \mathbf{e}_k$ where the $\beta_k$ are independent brownian processes and $(\mathbf{e}_k)_{k \geq 1}$ is a complete orthonormal system in a Hilbert space $\mathcal{H}$. For each $\rho \geq 0$, $u \in \mathbb{R}$, $\Phi(\rho,u) : \mathcal{H} \to L^2(\mathbb{T})$ is defined by

$$\Phi(\rho,u)e_k = \sigma_k(\cdot,\rho,u) = \rho \sigma^*_k(\cdot,\rho,u), \quad (2.1)$$

where $\sigma^*_k(\cdot,\rho,u)$ is a 1-periodic continuous function on $\mathbb{R}$. More precisely, we assume $\sigma^*_k \in C(T \times \mathbb{R}_+ \times \mathbb{R})$, with the bound

$$G^*(x,\rho,u) := \left( \sum_{k \geq 1} |\sigma^*_k(x,\rho,u)|^2 \right)^{1/2} \leq D_0 \left[ 1 + u^2 + \rho^{\gamma-1} \right]^{1/2}, \quad (2.2)$$

and $A$ revoir

$$G^*_j(x,\rho,u) := \left( \sum_{k \geq 1} |\nabla^j \rho_u \sigma^*_k(x,\rho,u)|^2 \right)^{1/2} \leq \zeta_0 (1 + \rho^{2+1-j}), \quad (2.3)$$

where $D_0$, $\zeta_0$ are some non-negative constants, $x \in \mathbb{T}$, $\rho \geq 0$, $u \in \mathbb{R}$ and $j \in \{1,2\}$.

We define the auxiliary space $\mathcal{H}_0 \supset \mathcal{H}$ via

$$\mathcal{H}_0 = \left\{ v = \sum_{k \geq 1} \alpha_k \mathbf{e}_k; \sum_{k \geq 1} \frac{\alpha_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$\|v\|^2_{\mathcal{H}_0} = \sum_{k \geq 1} \frac{\alpha_k^2}{k^2}, \quad v = \sum_{k \geq 1} \alpha_k \mathbf{e}_k.$$
2.2 Notations

We denote by

\[ \mathbf{U} = \left( \begin{array}{c} \rho \\ q \end{array} \right), \quad \mathbf{A}(\mathbf{U}) = \left( \begin{array}{c} q \\ \frac{q^2}{\rho} + p(\rho) \end{array} \right), \quad q = \rho u, \quad (2.4) \]

the 2-dimensional unknown and flux of the conservative part of the problem.

We also set

\[ \Phi^*(\mathbf{U}) = \rho^{-1}\Phi(\mathbf{U}), \quad \psi_k(\mathbf{U}) = \left( \begin{array}{c} 0 \\ \sigma_k(\mathbf{U}) \end{array} \right), \quad \Psi(\mathbf{U}) = \left( \begin{array}{c} 0 \\ \Phi(\mathbf{U}) \end{array} \right). \]

Note that \( \Phi^* \) is well defined by (2.1) and that, with the notation above, (1.1) can be more concisely rewritten as the following stochastic first-order system

\[ d\mathbf{U} + \partial_x \mathbf{A}(\mathbf{U})dt = \Psi(\mathbf{U})dW(t). \quad (2.5) \]

We will also set, for \( \rho \geq 0 \),

\[ \mathbf{G}(\mathbf{U}) = \rho \Phi^*(\mathbf{U}). \quad (2.6) \]

If \( E \) is a space of real-valued functions on \( T \), we will denote \( \mathbf{U}(t) \in E \) instead of \( \mathbf{U}(t) \in E \times E \) when this occur (with \( E = W^{1,\infty}(T) \) or \( E = L^\infty(T) \) for example in Definition ??, Theorem ??, etc.)

We denote by \( \mathcal{P} \) the predictable \( \sigma \)-algebra on \( \Omega \times [0,T] \) generated by \( (\mathcal{F}_t) \) and we denote by \( \mathcal{P}_2 \) the completion of \( \mathcal{P} \otimes \mathcal{B}(T) \), where \( \mathcal{B}(T) \) is the Borel \( \sigma \)-algebra on \( T \).

We will also use the following notation in various estimates below:

\[ A = O(1)B, \]

where \( A, B \in \mathbb{R}_+ \), with the meaning \( A \leq CB \) for a constant \( C \geq 0 \). In general, the dependence of \( C \) over the data and parameters at stake will be detailed, see for instance Proposition 3.7 below.

2.3 Entropy Solution

In relation with the kinetic formulation for (1.1) in [LPT94b], there is a family of entropies

\[ \eta(\mathbf{U}) = \int \mathbf{g}(\xi)\chi(\rho, \xi - u)d\xi, \quad \text{with } q = \rho u, \quad (2.7) \]

for (1.1), where

\[ \chi(\mathbf{U}) = c_\lambda(\rho^2 - u^2)^\lambda, \quad \lambda = \frac{3 - \gamma}{2(\gamma - 1)}, \quad c_\lambda = \left( \int_{-1}^{1} (1 - z^2)^\lambda dz \right)^{-1}, \]

\[ 5 \]
Indeed, if \( g \in C^2(\mathbb{R}) \) is a convex function, then \( \eta \) is of class \( C^2 \) on the set
\[
\mathcal{U} := \left\{ \mathbf{U} = (\rho, q) \in \mathbb{R}^2; \rho > 0 \right\}
\]
and \( \eta \) is a convex function of the argument \( \mathbf{U} \). Formally, by the Itô Formula, solutions to (1.1) satisfy
\[
d\mathbb{E} \eta(\mathbf{U}) + \mathbb{E} H(\mathbf{U}) \, dt = \frac{1}{2} \mathbb{E} \partial_{qq} \eta(\mathbf{U}) \mathbf{G}^2(\mathbf{U}) \, dt,
\]
where the entropy flux \( H \) is given by
\[
H(\mathbf{U}) = \int_{\mathbb{R}} g(\xi) [\theta \xi + (1 - \theta) u] \chi(\rho, \xi - u) \, d\xi,
\]
with \( q = \rho u \).

Note that, by a change of variable, we also have
\[
\eta(\mathbf{U}) = \rho c_\lambda \int_{-1}^{1} g \left( \frac{q}{\rho} + z \rho^{(\gamma - 1)/2} \right) \left( 1 - z^2 \right) \lambda \, dz
\]
and
\[
H(\mathbf{U}) = \rho c_\lambda \int_{-1}^{1} g \left( \frac{q}{\rho} + z \rho^{(\gamma - 1)/2} \right) \left( \frac{q}{\rho} + \frac{\gamma - 1}{2} z \rho^{(\gamma - 1)/2} \right) \left( 1 - z^2 \right) \lambda \, dz.
\]

In particular, for \( g(\xi) = 1 \) we obtain the density \( \eta_0(\mathbf{U}) = \rho \). To \( g(\xi) = \xi \) corresponds the impulsion \( \eta(\mathbf{U}) = q \) and to \( g(\xi) = \frac{1}{2} \xi^2 \) corresponds the energy
\[
\eta_E(\mathbf{U}) = \frac{1}{2} \rho u^2 + \frac{\kappa}{\gamma - 1} \rho^\gamma.
\]

Note the form of the energy, in particular the fact that the hypothesis (2.2) on the noise gives a bound
\[
\mathbf{G}^2(\mathbf{U}) = \sum_{k \geq 1} |\Phi(\rho, u)e_k|^2 \leq \rho D_0^2 (\eta_0(\mathbf{U}) + \eta_E(\mathbf{U})),
\]
for a constant \( D_0^2 \) depending on \( D_0 \) and \( \gamma \) (recall that \( \eta_0(\mathbf{U}) := \rho \)). If (2.8) is satisfied with an inequality \( \leq \), then formally (2.2) and the Gronwall Lemma give a bound on \( \mathbb{E} \int_T (\eta_0 + \eta_E)(\mathbf{U})(t) \, dx \) in terms of \( \mathbb{E} \int_T (\eta_0 + \eta_E)(\mathbf{U})(0) \, dx \).

Indeed, we have \( \partial_{qq} \eta_E(\mathbf{U}) = \frac{1}{\rho} \), which gives
\[
\mathbb{E} \partial_{qq} \eta_E(\mathbf{U}) \mathbf{G}^2(\mathbf{U}) \leq D_0^2 \mathbb{E} (\eta_0(\mathbf{U}) + \eta_E(\mathbf{U})),
\]
where \( C \) is a constant.

We will prove rigorously uniform bounds for approximate parabolic solutions in Section 3.1.2. The above formal computations are however sufficient for the moment to introduce the following definition.
Definition 2.1 (Entropy solution). Let $\rho_0, u_0 \in L^\infty(T)$ with $\rho_0 \geq 0$ a.e.. A $\mathcal{P}_2$-measurable function $U : \Omega \times (0, T) \times T \to \mathbb{R}_+ \times \mathbb{R}$ is said to be an entropy solution to (1.1) with initial datum $U_0$ if

1. we have

$$\Phi(U) \in L^2(\Omega \times [0, T], \mathcal{P}, d\mathbb{P} \times dt; L_2(\mathcal{U}; L^2(T))),$$

where $L_2(\mathcal{U}; L^2(T))$ is the space of Hilbert-Schmidt operators from $\mathcal{U}$ into $L^2(T)$,

2. the bound

$$\mathbb{E} \sup_{0 \leq t \leq T} \int_\Omega \eta(U(t, x)) dx < +\infty,$$

is satisfied for $\eta = \eta_E$, the energy defined in (2.12),

3. for any $(\eta, H)$ given by (2.7)-(2.9), where $g \in C^2(\mathbb{R})$ is convex and subquadratic\(^2\), for all $t \in (0, T]$, for all nonnegative $\varphi \in C^1(T)$, and nonnegative $\alpha \in C^1_c([0, t])$, we have

$$\int_0^t \langle \eta(U)(s), \varphi \rangle \alpha'(s) + \langle H(U)(s), \partial_x \varphi \rangle \alpha(s) \, ds$$

$$+ \int_0^t \langle G(U) \partial^2 q \eta(U), \varphi \rangle \alpha(s) \, ds + \langle \eta(U_0), \varphi \rangle \alpha(0)$$

$$+ \sum_{k \geq 1} \int_0^t \langle \sigma_k(U) \partial_q \eta(U), \varphi \rangle \alpha(s) \, d\beta_k(s) \geq 0,$$

$\mathbb{P}$-almost surely.

Remark 2.2. An entropy solution $U$ is not defined as a process. It is indeed difficult to specify a Banach space in which the process $U(t)$ would evolve. Indeed the natural bounds for $q$ are in $L^p(\nu)$ spaces where the measure $\nu$ depends on $\rho$. Actually, even if we had found such a Banach space $X$, we would still have to consider entropy solutions $U$ as equivalence classes in, say, $L^1(\Omega \times [0, T], \mathcal{P}, d\mathbb{P} \times dt; X)$, for the simple reason that the entropy solutions that we obtain are issued from a limiting process that does not give convergence for every time $t$. Certainly, if we knew that a $U \in L^1(\Omega \times [0, T], \mathcal{P}, d\mathbb{P} \times dt; X)$ had a representative with some continuity property in time, this one may be considered as a stochastic process in its own. However, we do not know how to obtain such continuity properties for entropy solutions to (1.1); this is related to the lack of techniques for proving uniqueness.

\(^2\)in the sense that $g$ satisfies (3.1)
Remark 2.3. By (2.14), the stochastic integral $t \mapsto \int_0^t \Phi(U)(s)dW(s)$ is a well defined process taking values in $L^2(\mathbb{T})$ (see [DPZ92] for detailed construction). There is a little redundancy here in the definition of entropy solutions since, apart from the predictability, the integrability property (2.14) will follow from (2.2) and the bounds (2.15), cf. (2.13).

Remark 2.4. We will construct (martingale) entropy solutions which satisfy the entropy bound (2.15) for all entropy $\eta$ of the form (2.7) with $g$ convex and $C^2$ satisfying a polynomial bound

$$|g(\xi)| \leq C(1 + |\xi|^m), \quad m \in \mathbb{N},$$

for some constant $C \geq 0$. In particular the solutions will satisfy the following bounds (with weight $\rho$):

$$E \esssup_{0 \leq t \leq T} \int_{\mathbb{T}^1} (|u|^\alpha + \rho^\beta) \rho dx < +\infty,$$

for all $\alpha, \beta \geq 0$. We will also obtain $L^p$ bounds of the form

$$\esssup_{0 \leq t \leq T} E\|u(t)\|_{L^p(T)} + \esssup_{0 \leq t \leq T} E\|\rho(t)\|_{L^p(T)}^{\frac{\alpha + \beta}{\alpha}} < +\infty. \quad (2.17)$$

**PRECISIONS about this bound, refer to where it is proved**

Remark 2.5. If we take (for $g(\xi) = 1$ and $\xi$ respectively)

$$(\eta(U), H(U)) = (\rho, q), \quad (\eta(U), H(U)) = (q, \frac{q}{\rho} + p(\rho)),$$

we can deduce from (2.16) the weak formulation of (1.1).

**Definition 2.6 (Martingale solution).** Let $\rho_0, u_0 \in L^\infty(\mathbb{T})$ with $\rho_0 \geq 0$ a.e. A martingale solution to (1.1) with initial datum $U_0$ is a multiplet

$$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), W, U),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, with filtration $(\mathcal{F}_t)$ satisfying the usual conditions, and $W$ a $(\mathcal{F}_t)$-cylindrical Wiener process, and $(U(t))$ defines, according to Definition 2.1, an entropy solution to (1.1) with initial datum $U_0$.

**Theorem 2.7 (Main result).** Assume that the structure and growth hypotheses (2.2), (2.3) on the noise are satisfied. Let $\rho_0, u_0 \in L^\infty(\mathbb{T})$ with $\rho_0 \geq 0$ a.e. Then there exists a martingale solution to (1.1) with initial datum $U_0$. 
2.4 Organization of the paper and main problematic

The paper is organized as follows. In Section 3, we prove the existence of martingale solutions to the parabolic approximation of Problem (1.1). This is done by a splitting method. One difficulty here is to obtain gradient estimates on high moments of the solution (cf. Section 3.1.4) since no $L^\infty$ bounds are satisfied due to the stochastic perturbation. An other difficulty is to obtain some positivity results on the density: we need quantitative estimates, cf. Section 3.1.5. To that purpose we use De Giorgi type estimates in a way developed by Mellet and Vasseur in [MV09]: this is the subject of Appendix A. Once the existence of martingale solution to the parabolic approximation of Problem (1.1) has been proved, we want to take the limit on the regularizing parameter to obtain a martingale solution to (1.1). As in the deterministic case [DiP83a, DiP83b, LPS96], we use the concept of measure-valued solution (Young measure) to achieve this. In Section 4 we develop the tools on Young measure (in our stochastic framework) which are required. This is taken in part (but quite different) from Section 4.3 in [FN08]. We also use the probabilistic version of Murat’s Lemma from [FN08, Appendix A], to identify the limiting Young measure. This is the content of Section 5, which requires two other fundamental tools: the analysis of the consequences of the div-curl lemma in [LPS96, Section I.5] and an identification result for densely defined martingales from [Hof13b, Appendix A]. We obtain then the existence of a martingale solution to (1.1). In Section 6 we conclude on some open questions about Problem (1.1) (uniqueness inlaw, existence of an invariant measure). As explained above, we need at some point some bounds from below on solutions to (1-dimensional here) parabolic equations, which are developed in Appendix A. We also need some regularity results, with few variations, on the (1-dimensional) heat semi-group, and those are given in Appendix B.

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3 Parabolic Approximation

Let $\varepsilon > 0$. In this section, we prove the existence of a solution to the following problem.
\[
dU^\varepsilon + \partial_x A(U^\varepsilon) dt = \varepsilon \partial^2_x U^\varepsilon dt + \Psi(U^\varepsilon) dW(t),
\]
\[
U^\varepsilon_{t=0} = U^\varepsilon_0.
\]

Recall that \( U \) and \( A(U) \) are defined by
\[
U = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad A(U) = \begin{pmatrix} q^2 + p(\rho) \\ \frac{q}{\rho} \end{pmatrix}.
\]

**Definition 3.1** (Pathwise solution to the parabolic approximation). Let \( U_0^\varepsilon \in (W^{1,\infty}(\mathbb{T}))^2 \) and \( T > 0 \). A process \((U^\varepsilon(t))_{t \in [0,T]}\) with values in \((L^2(\mathbb{T}))^2\) is said to be a pathwise solution to \((3.1)\) if it is a predictable process such that

1. there exists an adapted random process \((c^\varepsilon(t))\) with values in \((0, +\infty)\) such that, almost surely,
   \[
   \rho^\varepsilon, q^\varepsilon \in C([0,T]; W^{1,2}(\mathbb{T})), \quad \rho^\varepsilon \in L^\gamma(Q_T), \quad \rho^\varepsilon \geq c^\varepsilon \text{ a.e. in } Q_T,
   \]
2. for all test function \( \varphi \in C^\infty(\mathbb{T}; \mathbb{R}^2) \), almost surely, the following equation is satisfied
   \[
   \langle U^\varepsilon(t), \varphi \rangle = \langle U_0, \varphi \rangle - \int_0^t \langle \partial_x A(U^\varepsilon) - \partial^2_x U^\varepsilon, \varphi \rangle \, ds
   + \int_0^t \langle \Psi(U^\varepsilon) dW^\varepsilon(s), \varphi \rangle, \quad t \in [0,T].
   \]

We will prove the existence of pathwise solutions to the parabolic stochastic problem \((3.1)\) satisfying uniform (or weighted) estimates with respect to \( \varepsilon \).

If \( \eta \) is an entropy function given by \((2.7)\) with a convex function \( g \) of class \( C^2 \), we denote by
\[
\Gamma_\eta(U) = \int_T \eta(U(x)) dx,
\]
the total entropy of a function \( U: T \to \mathbb{R}^2 \).

**Theorem 3.2** (Existence of pathwise solution to \((3.1)\)). Let \( U_0^\varepsilon \in (W^{1,\infty}(\mathbb{T}))^2 \) satisfy \( \rho_0^\varepsilon \geq c_0 \) a.e. in \( T \), for a positive constant \( c_0 \). Then the problem \((3.1)\) admits a pathwise solution \( U^\varepsilon \), which has the following property:

1. it satisfies the following \( L^p \) estimates: for all \( p \geq 1 \),
   \[
   \sup_{0 \leq t \leq T} \mathbb{E} \left\| u^\varepsilon(t) \right\|_{L^p(\mathbb{T})} + \sup_{0 \leq t \leq T} \mathbb{E} \left\| |\rho^\varepsilon(t)|^\gamma \right\|_{L^p(\mathbb{T})} = O(1),
   \]
   where \( O(1) \) depends on \( \gamma \), on the constant \( D_0 \) in \((2.2)\), on \( p \), \( \|u_0\|_{L^\infty} \), \( \|\rho_0\|_{L^\infty} \) and \( T \).
2. it satisfies the following gradient estimates: for all $\alpha, \beta \geq 0$,

$$
\varepsilon \mathbb{E} \int_{Q_T} \left( |u^\varepsilon|^\alpha |\rho^\varepsilon|^{7-2} + |\rho^\varepsilon|^{7-2+\beta} \right) |\partial_x \rho^\varepsilon|^2 dx dt = O(1),
$$

(3.4)

and

$$
\varepsilon \mathbb{E} \int_{Q_T} \left( |u^\varepsilon|^\alpha |\rho^\varepsilon|^{1+\beta} \right) |\partial_x u^\varepsilon|^2 dx dt = O(1),
$$

(3.5)

where $O(1)$ depends on $T$, $\gamma$, on the constant $D_0$ in (2.2) and on a finite number of quantities $E \Gamma_\eta(U_0^\varepsilon)$, for entropies $\eta_j$ associated to various power-like functions $|\xi|^{r_j}$, for given $r_j$’s depending on $\alpha, \beta$.

Besides, $U^\varepsilon$ satisfies the Itô formula

$$
d\eta(U^\varepsilon) + \partial_x H(U^\varepsilon) dt = -\varepsilon \partial_x^2 \eta(U^\varepsilon) dt + \partial_x \eta(U^\varepsilon) \Phi(U^\varepsilon) dW(t) - \varepsilon \eta''(U^\varepsilon) \cdot (U_x^\varepsilon, U_x^\varepsilon) dt + \frac{1}{2} |G(U^\varepsilon)|^2 \partial_{qq} \eta(U^\varepsilon) dt.
$$

(3.6)

for all entropy - entropy flux couple $(\eta, H)$ where $\eta$ is of the form (2.7) with a convex function $g$ of class $C^2$ such that $g''$ is bounded on $\mathbb{R}$.

Note that (3.6) has the following meaning: for all test-function $\varphi \in C^\infty(T)$, we have

$$
\langle \eta(U^\varepsilon), \varphi \rangle(t) = \langle \eta(U_0^\varepsilon), \varphi \rangle(t) + \int_0^t \langle H(U^\varepsilon(s), \partial_x \varphi) - \varepsilon \langle \eta(U^\varepsilon(s), \partial_x^2 \varphi) ds
$$

$$
- \int_0^t \varepsilon \langle \eta''(U^\varepsilon), (U_x^\varepsilon, U_x^\varepsilon), x \varphi \rangle ds + \sum_{k \geq 1} \int_0^t \langle \partial_x \eta(U^\varepsilon) \sigma_k(U^\varepsilon), \varphi \rangle dW(s)
$$

$$
+ \frac{1}{2} \int_0^t \langle |G(U^\varepsilon)|^2 \partial_{qq} \eta(U^\varepsilon), \varphi \rangle ds,
$$

for all $t \in [0, T]$, $\mathbb{P}$-almost surely.

To prove the existence of such pathwise solutions, we will prove first the existence of martingale solution and then use a Yamada-Watanabe argument (see [?]) to conclude. This means that we have to prove a result of pathwise uniqueness, which is the content of the following theorem.

**Theorem 3.3** (Uniqueness of pathwise solution to (3.1)). Let $U_0^\varepsilon \in (W^{1,\infty}(T))^2$ satisfy $\rho_0^\varepsilon \geq c_0$ a.e. in $T$, for a positive constant $c_0$

**Remark 3.4.**
3.1 Solution to the parabolic problem

3.1.1 Time splitting

Assume, without loss of generality, that \( \varepsilon = 1 \). To prove the existence of a solution to (3.1), we perform a splitting in time. Let \( \tau > 0 \). Set \( t_k = k\tau, \quad k \in \mathbb{N} \). We solve alternatively the deterministic, parabolic part of (3.1) on time intervals \([t_{2k}, t_{2k+1})\) and the stochastic part of (3.1) on time intervals \([t_{2k+1}, t_{2k+2})\), i.e.

- for \( t_{2k} \leq t < t_{2k+1} \),
  \[
  \partial_t U^\tau + 2\partial_x A(U^\tau) = 2\partial^2_x U^\tau \quad \text{in } Q_{t_{2k}, t_{2k+1}}, \quad (3.7a)
  \]
  \[
  U^\tau(t_{2k}) = U^\tau(t_{2k}^-) \quad \text{in } \mathbb{T}, \quad (3.7b)
  \]
- for \( t_{2k+1} \leq t < t_{2k+2} \),
  \[
  dU^\tau = \sqrt{2}\Psi^\tau(U^\tau)dW(t) \quad \text{in } Q_{t_{2k+1}, t_{2k+2}}, \quad (3.8a)
  \]
  \[
  U^\tau(t_{2k+1}) = U^\tau(t_{2k+1}^-) \quad \text{in } \mathbb{T}. \quad (3.8b)
  \]

Note that we took care to speed up the deterministic equation (3.7a) by a factor 2 and the stochastic equation (3.8a) by a factor \( \sqrt{2} \), this rescaling procedure should yield a solution \( U^\tau \) consistent with the solution \( U^{\varepsilon=1} \) to (3.1) when \( \tau \to 0 \). In (3.8) we have also regularized the coefficient \( \Phi \) into a coefficient \( \Phi^\tau \). Precisely, we assume that, for a finite integer \( K^\tau \geq 1 \), for each \( \rho \geq 0, u \in \mathbb{R} \), we have

\[
\Phi^\tau(\rho, u)e_k = \sigma^\tau_k(\cdot, \rho, u)1_{k \leq K^\tau} = \rho\sigma^{\tau,*}_k(\cdot, \rho, u)1_{k \leq K^\tau}, \quad (3.9)
\]

where \( \sigma^{\tau,*}_k(\cdot, \rho, u) \) is a 1-periodic continuous function on \( \mathbb{R} \) which is bounded together with all its derivatives. We also assume that the equivalent of the bounds (2.2)-(2.3) are satisfied uniformly in \( \tau \):

\[
G^{\tau,*}(x, \rho, u) := \left( \sum_{k=1}^{K^\tau} |\sigma^{\tau,*}_k(x, \rho, u)|^2 \right)^{1/2} \leq D_0 \left[ 1 + u^2 + \rho^{\gamma-1} \right]^{1/2}, \quad (3.10)
\]

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\[
G_j^{\tau,*}(x, \rho, u) := \left( \sum_{k=1}^{K^\tau} |\nabla_{\rho,u} \sigma^{\tau,*}_k(x, \rho, u)|^2 \right)^{1/2} \leq \zeta_0(1 + \rho^{\frac{\gamma-1}{2}-j}), \quad (3.11)
\]

where \( x \in \mathbb{T}, \rho \geq 0, u \in \mathbb{R} \) and \( j \in \{1, 2\} \). In what follows we often do as if \( \Phi^\tau = \Phi \)...

Let us define the following indicator functions

\[
1_{\text{det}} = \sum_{k=0}^{K^\tau} 1_{[t_{2k}, t_{2k+1})}, \quad 1_{\text{sto}} = 1 - 1_{\text{det}}, \quad (3.12)
\]

which will be used to localize various estimates below.
Definition 3.5 (Pathwise solution to the splitting approximation). Let \( U_0 \in (W^{1,2}(\mathbb{T}))^2 \) and \( T > 0 \). A process \( (U(t))_{t \in [0,T]} \) with values in \((L^2(\mathbb{T}))^2\) is said to be a pathwise solution to (3.7)-(3.8) if it is a predictable process such that

1. there exists an adapted random process \((c^\tau(t))\) with values in \((0, +\infty)\) such that, almost surely,
   \[
   \rho, q \in C([0, T]; W^{1,2}(\mathbb{T})), \quad \rho \in L^\gamma(Q_T), \quad \rho \geq c^\tau \text{ a.e. in } Q_T, \quad (3.13)
   \]
2. for all test function \( \varphi \in C^\infty(\mathbb{T}; \mathbb{R}^2) \), the following equation is satisfied: \( \mathbb{P}\)-almost surely, for all \( t \in [0, T] \),
   \[
   \langle U(t), \varphi \rangle = \langle U_0, \varphi \rangle - 2 \int_0^t 1_{\text{det}}(s) \langle \partial_x A(U(s)) - \partial_x^2 U(s), \varphi \rangle \, ds 
   + \sqrt{2} \int_0^t 1_{\text{sto}}(s) \langle \Psi(U(s)) \rangle \, dW(s), \varphi \rangle. \quad (3.14)
   \]

Proposition 3.6 (Existence of pathwise solution to the splitting approximation). Let \( T > 0 \), let \( U_0 \in (W^{2,2}(\mathbb{T}))^2 \) such that \( \rho_0 \geq c_0 \) a.e. in \( T \) for a given constant \( c_0 > 0 \). Then there exists a pathwise solution \( U^T \) to (3.7)-(3.8), which has the following additional property:

1. almost surely, \( U^T \in C([0, T]; W^{2,1}(\mathbb{T})) \),
2. for all \( p \geq 1 \), \( \mathbb{E} \sup_{t \in [0, T]} \| U^T(t) \|_{W^{1,2}(\mathbb{T})}^p < +\infty \),
3. \( U^T \) satisfies the following Itô Formula.

Proof. The deterministic problem (3.7) is solved in [LPS96]: for Lipschitz continuous initial data \( (\rho_0, q_0) \) with an initial density \( \rho_0 \) uniformly positive, say \( \rho_0 \geq c_0 > 0 \) on \( T \), the Problem (3.7) admits a unique solution \( U \) in the class of functions

\[
U \in L^\infty(0, \tau; W^{1,\infty}(\mathbb{T}))^2 \cap C([0, \tau]; L^2(\mathbb{T}))^2; \quad \rho \geq c_1 \text{ on } T \times [0, \tau].
\]

Here \( c_1 > 0 \) is a constant depending continuously on \( \tau \) and on \( c_0 \), \( \| \rho_0 \|_{L^\infty(\mathbb{T})} \), \( \| q_0 \|_{L^\infty(\mathbb{T})} \) (see Theorem A.1 and Remark A.3 in this paper for more detail about this positivity result). By [DiP83b, Section 4], we have \( U \in L^\infty(0, \tau; W^{2,2}(\mathbb{T})) \). It follows that \( \partial_t U \in L^\infty(0, \tau; L^2(\mathbb{T})) \) and, therefore, \( U \in C([0, \tau]; W^{1,2}(\mathbb{T})) \). This gives the properties 1. and 2. of the Proposition for \( T = t_1 \). Note, besides, that, by Theorem 3.3 and Remark 3.4 below, the map \( U_0 \mapsto U \) is Lipschitz continuous from \( L^2(\mathbb{T})^2 \) into \( C([0, \tau]; L^2(\mathbb{T})^2) \).

To finish with the first step of resolution of (3.7) for \( k = 0 \), let us set

\[
c^\tau(t) = c_1, \quad t \in [0, t_1).
\]
On \([t_1, t_2)\) we solve now in a second step the stochastic problem (3.8). It is an ordinary stochastic differential equation that does not act on the density. Recall that the coefficients of the noise in (3.9) are assumed to be functions with bounded derivatives at all orders. Since \(x \mapsto \rho^\tau(t_1, x)\) is smooth, we may rewrite the second equation of (3.8) as

\[
 dq = \sum_{k=1}^{K^\tau} g_k(x, q) d\beta_k(t), \tag{3.15}
\]

where \(g_k \in C^\infty(T \times \mathbb{R})\). The existence of a solution to (3.15) on \((t_1, t_2)\) with initial datum \(q(t_1, x)\) at \(t = t_1\) is ensured by a classical fixed point theorem, in the space of adapted functions

\[
 q \in C([t_1, t_2]; L^2(\Omega \times T)).
\]

Since \(\rho\) is unchanged on the time interval \([t_1, t_2)\), we set

\[
 c^\tau(t) = c^\tau(t_1), \quad t \in [t_1, t_2).
\]

Note that, as such, \((c^\tau(t))_{t \in [0, t_2)}\) is \(\mathcal{F}_t\)-measurable. In particular, this is an adapted process. By differentiating once with respect to \(x\) in (3.15), we obtain

\[
 d(\partial_x q) = \sum_{k=1}^{K^\tau} \left( \partial_x g_k(x, q) + \partial_q g_k(x, q)(\partial_x q) \right) d\beta_k(t).
\]

By the Itô Formula and the Gronwall Lemma, it follows that

\[
 \sup_{t \in [t_1, t_2]} \mathbb{E}\|\partial_x q\|^p_{L^p(T)} \leq C \mathbb{E}\|\partial_x q(t_1)\|^p_{L^p(T)}, \quad p \geq 2, \tag{3.16}
\]

where the constant \(C\) depends on the function \(g_k\)’s, on \(p\) and on \(\tau\). By differentiating again in (3.15), we have

\[
 d(\partial^2_x q) = \sum_{k=1}^{K^\tau} \left( \partial^2_x g_k(x, q) + 2\partial^2_{x,q} g_k(x, q)(\partial_x q) + \partial^2_q g_k(x, q)(\partial_x q)^2 
 + \partial_q g_k(x, q)(\partial^2_x q) \right) d\beta_k(t). \tag{3.17}
\]

Using (3.16) with \(p = 2\) and \(p = 4\), the Itô Formula and the Gronwall Lemma, we obtain

\[
 \sup_{t \in [t_1, t_2]} \mathbb{E}\|\partial^2_x q\|^2_{L^2(T)} \leq C \left( \mathbb{E}\|\partial^2_x q(t_1)\|^2_{L^2(T)} + \mathbb{E}\|\partial_x q(t_1)\|^2_{L^2(T)} + \mathbb{E}\|\partial_x q(t_1)\|^4_{L^4(T)} \right), \tag{3.18}
\]

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where the constant $C$ depends on the function $g_k$’s and on $\tau$. By the Doob’s Martingale Inequality, we have therefore

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \left\| \int_{t_1}^{t} \partial_q g_k(x, q(s)) \partial_{q}^2 q(s) d\beta_k(s) \right\|_{L^2(T)}^2 \leq 2 \mathbb{E} \left\| \int_{t_1}^{t_2} \partial_q g_k(x, q(s)) \partial_{q}^2 q(s) d\beta_k(s) \right\|_{L^2(T)}^2 \leq C(\mathbb{E} \|\partial_{q}^2 q(t_1)\|_{L^2(T)}^2 + \mathbb{E} \|\partial_x q(t_1)\|_{L^2(T)}^3 + \mathbb{E} \|\partial_x q(t_1)\|_{L^1(T)}^3).$$

Returning to (3.17), we deduce that

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \|\partial_{q}^2 q\|_{L^2(T)}^2 \leq C(\mathbb{E} \|\partial_{q}^2 q(t_1)\|_{L^2(T)}^2 + \mathbb{E} \|\partial_x q(t_1)\|_{L^2(T)}^3 + \mathbb{E} \|\partial_x q(t_1)\|_{L^1(T)}^3).$$

By a similar argument, using Doob’s Martingale Inequality, we can similarly improve (3.16) into

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \|\partial_x q\|_{L^p(T)}^p \leq C \mathbb{E} \|\partial_x q(t_1)\|_{L^p(T)}^p, \quad p \geq 2. \quad (3.20)$$

Note that differentiation in (3.15) has to be justified. The argument is standard: to obtain a solution to (3.15) which satisfies (3.20) and (3.19), we simply prove existence by using a fixed-point in a smaller space, incorporating the bounds (3.20) and (3.19). By (3.19), we have $\mathbf{U}(t_2) \in W^{2,2}(T)$. The initial datum $U(t_2)$ is therefore admissible with regard to the resolution of the deterministic problem (3.7) on $Q_{t_2, t_3}$ using identical arguments to those given at the beginning of the proof for the resolution on the strip $Q_{0, t_1}$. Furthermore, the function $U'(t_2)$ is $\mathcal{F}_{t_2}$-measurable. Since $U'(t_2) \mapsto (U'(t))_{t \in (t_2, t_3)}$ is Lipschitz continuous from $L^2(T)^2$ into $C([t_2, t_3]; L^2(T)^2)$, the random variable $U'(t)$ is $\mathcal{F}_{t_2}$-measurable for every $t \in [t_2, t_3]$. In particular, $U'(t)$ is adapted on $[t_2, t_3]$. Moreover, there exists $c_2 > 0$ depending continuously on $\tau$ and $U'(t_2)$ such that $\rho' \geq c_2$ a.e. on $Q_{t_2, t_3}$. In particular, $c_2$ is $\mathcal{F}_{t_2}$-measurable and continuing $c'$ on $[t_2, t_3]$ by setting

$$c'(t) = c_2, \quad t \in [t_2, t_3),$$

we obtain an adapted process. We can then go on with the resolution of the Problem 3.8 on $Q_{t_2k+1, t_{2k+2}}$, $k = 1$, as was done in the case $k = 0$, and so on. In this way we construct an adapted process $(U'(t))$ satisfying the first item of Definition 3.5. Besides $U' \in C([0, T]; W^{1,2}(T))$ almost surely and satisfies item 2. of the Proposition. In particular, since $U'$ has almost sure continuous trajectories in $L^2(T)^2$, this is a predictable process. Eventually, to show that (3.14) is satisfied, observe that it is satisfied on each interval $[t_j, t_{j+1}]$, $[k, t]$, where $K$ is such that $t_K \leq t < t_{K+1}$. By adding these mild formulations, we obtain (3.14). Itô. ■
3.1.2 Entropy bounds

If \( \eta \in C(\mathbb{R}^2) \) is an entropy and \( U: T \to \mathbb{R}^2 \), let

\[
\Gamma_\eta(U) := \int_T \eta(U(x)) \, dx
\]
denote the total entropy of \( U \).

**Proposition 3.7** (Entropy bounds). Let \( m \in \mathbb{N} \). Let \( \eta_m \) be the entropy given by (2.7) with \( g(\xi) = \xi^{2m} \). Then the solution \( U^\tau \) to (3.7)-(3.8) satisfies the estimate

\[
E \sup_{t \in [0,T]} \Gamma_\eta(U^\tau(t)) + 2E \int_{Q_T} 1_{\text{det} \eta''(U^\tau)} \cdot (U^\tau_x, U^\tau_y) \, dx \, dt = O(1),
\]

(3.21)

where the quantity denoted by \( O(1) \) depends on \( T, \gamma \), on the constant \( D_0 \) in (2.2), on \( m \) and on the initial quantities \( E \Gamma_\eta(U_0) \) for \( \eta \in \{ \eta_0, \eta_m, \eta_{2m} \} \).

**Proof.** To prove Proposition 3.7 we will use the following result.

**Lemma 3.8.** Let \( m, n \in \mathbb{N} \). We have

\[
\rho(u^{2m} + \rho^{2m-1} - \gamma^2) = O(1) \eta_m(U), \quad \eta_m(U) = O(1) \left[ \rho(u^{2m} + \rho^{2m-1} - \gamma^2) \right],
\]

(3.22)

where \( O(1) \) depends on \( m \); we have

\[
\eta_m(U) \cdot \eta_n(U) = O(1) \left[ \rho \eta_{m+n}(U) \right],
\]

(3.23)

where \( O(1) \) depends on \( m \) and \( n \) and

\[
\eta_n(U) = O(1) \left[ \rho \eta_0(U) + \eta_m(U) \right],
\]

(3.24)

where \( O(1) \) depends on \( m \) and \( n \) if \( 0 \leq n \leq m \).

Lemma 3.8 is obtained simply by repeated applications of the Young Inequality. Using an entropy identity for (3.7) and Itô Formula for the evolution by (3.8), we have

\[
E \Gamma_{\eta_m}(U^\tau(t)) + 2E \int_{Q_T} 1_{\text{det} \eta''(U^\tau)} \cdot (U^\tau_x, U^\tau_y) \, dx \, dt = E \Gamma_{\eta_m}(U^\tau_0) + E R_{\eta_m}(t),
\]

where

\[
R_{\eta_m}(t) := \int_{Q_t} 1_{\text{sto}} G^\tau(U^\tau)^2 \partial_{qq} \eta_m(U^\tau) \, dx \, ds
\]
is the Itô correction. If \( m = 0 \), then \( \partial_{qq} \eta = 0 \) and we obtain (note the difference with (3.21) in the first term)

\[
\sup_{t \in [0,T]} E \Gamma_{\eta_0}(U^\tau(t)) + 2E \int_{Q_T} 1_{\text{det} \eta''(U^\tau)} \cdot (U^\tau_x, U^\tau_y) \, dx \, dt = O(1). \quad (3.25)
\]
To get an estimate on $R_{\eta m}$ in the case $m \geq 1$, we compute, by (2.10),

$$\partial^2_{qq} \eta_m(U) = \frac{1}{\rho} c_\lambda \int_{-1}^{1} g''(u + z\rho^{(\gamma - 1)/2})(1 - z^2)\lambda dz.$$ \hspace{1cm} (3.26)

Since $g(\xi) = \xi^{2m}$, and $(a + b)^\alpha \leq 2^\alpha (a^\alpha + b^\alpha)$, $a, b \in \mathbb{R}$, $\alpha = 2m - 2$, we obtain, by (3.22),

$$\partial^2_{qq} \eta_m(U) = O(1) \left[ 1 - \frac{\eta_{m-1}(U)}{\rho} \right].$$

Then we deduce from (2.13) and by (3.23) (note that for the energy $\eta E$, we have $\eta E = \eta_1$ with our notations), the estimate

$$G^\tau(U^\tau)^2 \partial^2_{qq} \eta_m(U) = O(1) \left[ \eta_{m-1}(U) + \eta_m(U) \right].$$

By (3.24), it follows that

$$G^\tau(U^\tau)^2 \partial^2_{qq} \eta_m(U) = O(1) \left[ \eta_0(U) + \eta_m(U) \right].$$

We deduce that

$$E R_{\eta m}(t) = O(1) \left[ \int_{0}^{t} E(\Gamma_{\eta m}(U^\tau(s))) + \Gamma_{\eta m}(U^\tau(s))ds \right]$$ \hspace{1cm} (3.27)

and, by Gronwall’s Lemma and (3.25), we obtain

$$\sup_{t \in [0, T]} \frac{1}{\eta m}(U^\tau(t)) + 2E \int_{Q_T} 1_{\det \eta''_{m}}(U^\tau) \cdot (U^\tau_x, U^\tau_x) dx dt = O(1).$$ \hspace{1cm} (3.28)

To prove (3.21), we have to take into account the noise term: we have

$$0 \leq \Gamma_{\eta m}(U^\tau(t)) = \Gamma_{\eta m}(U^\tau_0) + M_{\eta m}(t) + R_{\eta m}(t) - D_{\eta m}(t)$$ \hspace{1cm} (3.29)

where

$$M_{\eta m}(t) = \sqrt{2} \sum_{k \geq 1} \int_{0}^{t} 1_{\sigma_k}(s) \langle \sigma_k(U^\tau(s)), \partial_q \eta_m(U^\tau(s)) \rangle_{L^2(\mathbb{T})} d\beta_k(s)$$

and

$$D_{\eta m}(t) = 2 \int_{Q_T} 1_{\det \eta''_{m}}(U^\tau) \cdot (U^\tau_x, U^\tau_x) dx ds.$$ 

Since $D_{\eta m} \geq 0$, we obtain

$$0 \leq \Gamma_{\eta m}(U^\tau(t)) \leq \Gamma_{\eta m}(U^\tau_0) + M_{\eta m}(t) + R_{\eta m}(t).$$

Similarly as for (3.27), we have

$$\mathbb{E} \sup_{t \in [0, T]} |R_{\eta m}(t)| = O(1) \left[ \int_{0}^{T} \mathbb{E}(\Gamma_{\eta m}(U^\tau(s))) + \Gamma_{\eta m}(U^\tau(s))ds \right].$$
and therefore, by (3.28), the last term $R_{\eta m}$ satisfies the bound
\[ E \sup_{t \in [0,T]} |R_{\eta m}(t)| = O(1). \]
By the Doob’s Martingale Inequality, we also have
\[ E \sup_{t \in [0,T]} |M_{\eta m}(t)| \leq C \mathbb{E} \left( \int_0^T \sum_{k \geq 1} \langle \sigma_k^\tau(U^\tau(s)), \partial_q \eta_m(U^\tau(s)) \rangle_{L^2(T)}^2 ds \right)^{1/2} \]
\[ \leq C \mathbb{E} \left( \int_{Q_T} G^\tau(U^\tau)^2 |\partial_q \eta_m(U^\tau)|^2 \, dx \, ds \right)^{1/2} \]
for a given constant $C$. Now we have
\[ \partial_q \eta_m(U) = 2m c_\lambda \int_{-1}^1 \left( u + z \rho^{(\gamma - 1)/2} \right)^{2m - 1} (1 - z^2)^{\frac{\gamma}{2}} \, dz. \]
From (3.22) follows $|\partial_q \eta_m(U)| = O(1) \left[ \frac{\eta_2m}{\rho^2(U)} \right]$ and by (3.23), (2.13) and (3.24), we deduce
\[ G^\tau(U)^2 |\partial_q \eta_m(U)|^2 = O(1) \left[ \frac{\rho^{-1}(\eta_0(U) + \eta_1(U)) \eta_{2m - 1}}{\rho(U)} \right] \]
\[ = O(1) \left[ \eta_0(U) + \eta_{2m}(U) \right]. \]
Using (3.25), we obtain then
\[ E \sup_{t \in [0,T]} |M_{\eta}(t)| = O(1). \]
This concludes the proof of the proposition. ■

**Corollary 3.9** (Bounds on the moments). Let $m \in \mathbb{N}$. Let $\eta_m$ be the entropy given by (2.7) with $g(\xi) = \xi^{2m}$. Then, the solution $U^\tau$ to (3.7)-(3.8) satisfies:
\[ E \sup_{t \in [0,T]} \int_T \left( |u^\tau|^{2m} + |\rho^\tau|^{m(\gamma - 1)} \right) \rho^\tau dx = O(1), \quad (3.30) \]
where $O(1)$ depends on $T$, $\gamma$, on the constant $D_0$ in (2.2), on $m$ and on the initial quantities $E \Gamma_\eta(U_0)$ for $\eta \in \{\eta_0, \eta_{2m}\}$.

If we introduce the measure
\[ \mu^\tau_\varphi : \varphi \mapsto \int_T \varphi(x) \rho^\tau(t,x) dx, \]
(which depend on the solution), we thus obtain estimates on
\[ E \int_T |u^\tau(t,x)|^\alpha d\mu^\tau_\varphi(x) \quad \text{and} \quad E \int_T |\rho^\tau(t,x)|^\beta d\mu^\tau_\varphi(x), \]
for all $\alpha, \beta \geq 0$. It is then straightforward, using Young’s inequality and Hölder’s inequality to obtain some bounds on the moments of the entropy and entropy flux:
Proposition 3.10 (Bounds on the moments of the entropy). Let \((\eta, H)\) be given by (2.7)-(2.9), where \(g \in C^2(\mathbb{R})\) is convex and subquadratic. Let \(\eta_m\) be the entropy given by (2.7) with \(g(\xi) = \xi^{2m}\). Then, the solution \(U^\tau\) to (3.7)-(3.8) satisfies the estimate

\[
\mathbb{E} \sup_{t \in [0,T]} \int T \left[ |\eta(U^\tau(t))|^p + |H(U^\tau(t))|^p \right] dx = O(1),
\]

for all \(p \geq 1\), where \(O(1)\) depends on \(T, \gamma, \) on the constant \(D_0\) in (2.2), on \(p\) and on the initial quantities \(\mathbb{E} \Gamma_\eta(U_0)\) for \(\eta \in \{\eta_0, \eta_\rho\}\).

3.1.3 \(L^p\) estimates

The entropy estimates give \(L^p\) estimates on \(u\) and \(\rho\) with a weight \(\rho\), see Equation (3.30). The following estimates (\(L^p\) estimates without the weight \(\rho\)) are an analog to the classical \(L^\infty\) estimate for the deterministic equation (cf. [DiP83b, Section 4.]).

Proposition 3.11 (\(L^p\) estimates). Let \(U^\tau\) be the solution to (3.7)-(3.8). Let \(p \geq 1\). Then

\[
\sup_{0 \leq t \leq T} \mathbb{E} \|u^\tau(t)\|_{L^p(T)} + \sup_{0 \leq t \leq T} \mathbb{E} \left\| \left| \rho^\tau(t) \right|^\frac{2-1}{2} \right\|_{L^p(T)} = O(1),
\]

where \(O(1)\) depends on \(\gamma, \) on the constant \(D_0\) in (2.2), on \(p, \|u_0\|_{L^\infty}, \|\rho_0\|_{L^\infty}\) and \(T\).

Proof. Let us introduce the Riemann invariants

\[
z^\tau = u^\tau - |\rho^\tau|^\theta, \quad w^\tau = u^\tau + |\rho^\tau|^\theta.
\]

By [DiP83b, Section 4.], we have

\[
z^\tau(t) \geq z(t_{2k}), \quad w^\tau \leq w(t_{2k}),
\]

for each \(t \in [t_{2k}, t_{2k+1}]\) and \(k \geq 0\).

Let \(m \in 2\mathbb{N}, m \geq 4\). For \(a \in \mathbb{R}\), set \(g_+(\xi) = \xi^m\) and

\[
\eta_+(U, a) = \int_{\mathbb{R}} g_+(\xi - a) \chi_{[\rho, \rho + u]} d\xi = c_\lambda \rho \int_{-1}^{1} (u + z \rho^{\frac{m-1}{2}} - a)^m (1 - z^2) dz.
\]

Let \(\Gamma(t, a)\) be defined by

\[
\Gamma(t, a) = \int T \eta_+(U^\tau(t, x), a) dx.
\]
By the Itô formula, we have

\[ \Gamma(t, a) = \Gamma(0, a) + \sum_{\alpha=1}^{3} \sum_{k=1}^{K^\tau} \int_0^t h^\alpha(s) \zeta^{\alpha,k}(s, a) dY^{\alpha,k}(s), \]

almost surely, where the continuous semi-martingales \( Y^{\alpha,k}(s) \) are defined by

\[ Y^{1,k}(t) = Y^{2,k}(t) = t, \quad Y^{3,k}(t) = \beta_k(t), \]

and the integrands \( \zeta^{\alpha,k}(s, a) \) are given by

\[ \zeta^{1,k}(s, a) = -2 \int_T \mathcal{D}_t \mathcal{U}_t^r(\mathcal{U}_t^r(s), a) \cdot (\mathcal{U}_t^r(s), \mathcal{U}_t^r(s)) dx \delta_{k,1}, \]
\[ \zeta^{2,k}(s, a) = \int_T \sigma_k^r(\mathcal{U}_t^r(s))^2 \partial q \eta_+ (\mathcal{U}_t^r(s), a) dx, \]
\[ \zeta^{3,k}(s, a) = \sqrt{\frac{2}{\mathcal{T}}} \int_T \sigma_k^r(\mathcal{U}_t^r(s)) \partial q \eta_+ (\mathcal{U}_t^r(s), a) dx, \]

while

\[ h^1(s) = 1_{\text{det}}(s), \quad h^2(s) = h^3(s) = 1_{\text{sto}}(s). \]

**Lemma 3.12.** Almost surely, the function \( \Gamma(t, a) \) is twice continuously differentiable with respect to \( a \), with derivatives continuous in \((t, a)\); the functions \( \zeta^{\alpha,k}(t, a) \) are twice continuously differentiable in \( a \) and continuous in \((t, a)\). Besides, for each \( a \), \( \zeta^{\alpha,k}(t, a) \) is an adapted process.

**Proof.** The lemma follows from the a.s. regularity \( \mathcal{U}_t^r \in C([0, T]; \mathcal{W}^{1,2}(T)) \), cf. item 1. in Proposition (3.6).

Consider now the martingale

\[ a(t) = a_0 + \sum_{k=1}^{K^\tau} \int_0^t 1_{\text{sto}}(s) a_k^k(s) d\beta_k(s). \]

We assume that, for each \( k \geq 1, a_k^k \) is an adapted process with a.s. continuous trajectories. By the Itô-Wentzell formula [?, Theorem 8.1], we have

\[ \Gamma(t, a(t)) = \Gamma_0(a(0)) + \sum_{\alpha=1}^{3} \sum_{k=1}^{K^\tau} \int_0^t h^\alpha(s) \zeta^{\alpha,k}(s, a(s)) dY^{\alpha,k}(s) \]
\[ + \int_0^t \frac{\partial \Gamma}{\partial a}(s, a(s)) da(s) \]
\[ + \sum_{k \geq 1} \int_0^t h^\alpha(s) \frac{\partial \zeta^{3,k}}{\partial a}(s, a(s)) a_k^k(s) ds \]
\[ + \frac{1}{2} \sum_{k \geq 1} \int_0^t \frac{\partial^2 \Gamma}{\partial a^2}(t, a(t)) |a_k^k(s)|^2 ds, \]
and we remark that 

\[ y \text{ such values of } a \text{ trajectories (T Wentzell formula on each interval } [t_j, t_{j+1}], t_j \leq t, \text{ and by summing the result over } j. \text{ Now, we use the following formula} \]

\[ \partial_t \eta_+(U, a) = \frac{1}{\rho} \partial_u \eta_+(U, a), \quad \sigma_k^\tau(\mathbf{U}) = \rho \sigma_k^{\tau,*,*} \mathbf{U}, \quad \partial_u \eta_+(U, a) = -\partial_a \eta_+(U, a) \]

and we remark that \( \zeta_{1,k}(s, a) \) is non-positive to obtain

\[
\Gamma(t, a(t)) \leq \Gamma_0(a(0)) + \sum_{k=1}^{K^\tau} \int_0^t \int_{\mathbb{T}} 1_{\text{sto}}(s) \partial^2_{uu} \eta_+(U^\tau(s, x), a(s)) \times \left[ \sigma_k^{\tau,*,*}(U^\tau(s, x))^2 + \frac{1}{2} |a^k(s)|^2 - \sqrt{2} \sigma_k^{\tau,*,*}(U^\tau(s, x)) a^k(s) \right] dxds \tag{3.38}
\]

\[ + \sum_{k=1}^{K^\tau} \int_0^t \int_{\mathbb{T}} 1_{\text{sto}}(s) \partial_u \eta_+(U^\tau(s, x), a(s)) \times \left[ \sqrt{2} \sigma_k^{\tau,*,*}(U^\tau(s, x)) - a^k(s) \right] dxds \beta_k(s) \tag{3.39} \]

almost surely. The term (3.38) is deduced from the terms (3.34) (the part of the sum for \( \alpha = 2 \)), (3.36) and (3.37). The term (3.39) is deduced from (3.34) (the part of the sum for \( \alpha = 3 \)) and (3.35). Let us set now

\[ a^k(s) = \sqrt{2} \sigma_k^{\tau,*,*}(U^\tau(y, s)), \quad a_0 = \|u_0^+\|_{L^\infty} + \|\rho_0^+\|_{L^\infty}^{\frac{1}{2}}, \tag{3.40} \]

where \( y \in \mathbb{T} \) is fixed and let us denote by

\[ a_+(t, y) = \|u_0^+\|_{L^\infty} + \|\rho_0^+\|_{L^\infty}^{\frac{1}{2}} + \sqrt{2} \sum_{k=1}^{K^\tau} \int_0^t 1_{\text{sto}}(s) \sigma_k^{\tau,*,*}(U^\tau(y, s)) d\beta_k(s) \]

the resulting martingale. Let also \( (\rho_\varepsilon) \) be an approximation of the unit on \( \mathbb{T} \). Note that, for all \( y \in \mathbb{T}, a^k \) is an adapted process with a.s. continuous trajectories (cf. item 1. in Proposition (3.6)). We apply (3.38)-(3.39) with such values of \( a^k \) and \( a_0 \), multiply the result by \( \rho_\varepsilon(x - y) \) and sum over \( y \in \mathbb{T} \); we have then

\[
\int_{\mathbb{T}} \Gamma(t, a_+(t, y)) \rho_\varepsilon(x - y) dy \leq A_\varepsilon(t) + B_\varepsilon(t), \tag{3.41}
\]

almost surely, where

\[
A_\varepsilon(t) = \sum_{k=1}^{K^\tau} \int_0^t \int_{\mathbb{T} \times \mathbb{T}} 1_{\text{sto}}(s) \partial^2_{uu} \eta_+(U^\tau(s, x), a_+(s, y)) \times \left[ \sigma_k^{\tau,*,*}(U^\tau(s, x)) - \sigma_k^{\tau,*,*}(U^\tau(s, y)) \right]^2 \rho_\varepsilon(x - y) dxdyds,
\]

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and

$$B_{\varepsilon}(t) = \sqrt{2} \sum_{k=1}^{K^*} \int_0^t \int_{\mathbb{T} \times \mathbb{T}} 1_{\text{sto}}(s) \partial_u \eta_+ \left( U_t(s, x), a_+(s, y) \right)$$

$$\times \left[ \sigma_{k}^{T,*}(U_t(s, x)) - \sigma_{k}^{T,*}(U_t(s, y)) \right] \rho_{\varepsilon}(x - y) dx dy d\beta_k(s).$$

Indeed, with $a_0$ given by (3.40), the first term $\Gamma(0, a(0))$ in (3.38)-(3.39) vanishes. Note that we have used the Fubini Theorem to rewrite $A_{\varepsilon}(t)$ and the stochastic Fubini Theorem to rewrite $B_{\varepsilon}(t)$. Let us give some details for the latter: we apply [DPZ92, Theorem 4.18]. Note that $B_{\varepsilon}(T) = \int_T \left( \int_0^T b(t, y) dW(t) \right) dy,$

where

$$b(t, y) = \sqrt{2} \sum_{k \leq K^*} \int_{\mathbb{T}} 1_{\text{sto}}(t) \partial_u \eta_+ \left( U_t(t, x), a_+(t, y) \right)$$

$$\times \left[ \sigma_{k}^{T,*}(U_t(t, x)) - \sigma_{k}^{T,*}(U_t(t, y)) \right] \rho_{\varepsilon}(x - y) dx.$$

Then $b$ is a measurable process from $(\Omega \times (0, T) \times \mathbb{T}, \mathcal{P}_T \times \mathcal{B}(\mathbb{T}))$ into $(L_2^0, \mathcal{B}(L_2^0))$, where $L_2^0$ is the space of Hilbert-Schmidt operators on $\mathcal{U}$. Indeed, since only a finite number of the quantities $b(t, y) e_k$ do not vanish, it is sufficient to prove that each of them is a measurable process from $(\Omega \times (0, T) \times \mathbb{T}, \mathcal{P}_T \times \mathcal{B}(\mathbb{T}))$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. This follows from, first, the fact that $$(U, \tilde{U}, a, y) \mapsto \int_{\mathbb{T}} \partial_u \eta_+ \left( U(x), a(x) \right) \left[ \sigma_{k}^{T,*}(U(x)) - \sigma_{k}^{T,*}(\tilde{U}) \right] \rho_{\varepsilon}(x - y) dx$$
is continuous from $X := C_2 \times \mathcal{U} \times C(\mathbb{R}) \times \mathbb{T}$ into $\mathbb{R}$, where, we recall,

$$\mathcal{U} = \left\{ U = \left( \begin{matrix} \rho \\ q \end{matrix} \right) \in \mathbb{R}^2; \rho > 0 \right\},$$

and where we have set

$$C_2 = \left\{ U: \mathbb{T} \rightarrow \mathcal{U}; \rho, u \in C(\mathbb{T}) \right\}.$$ Secondly, by Proposition (3.6), the map

$$(t, y) \mapsto (U_t(t), \tilde{U}_t(t, y), a_+(t, y), y)$$
is measurable from $(\Omega \times (0, T) \times \mathbb{T}, \mathcal{P}_T \times \mathcal{B}(\mathbb{T}))$ into $(X, \mathcal{B}(X))$. To apply the stochastic Fubini Theorem, we have also to check the integrability condition

$$\int_{\mathbb{T}} \| b(\cdot, \cdot, y) \|_T dy < +\infty,$$ (3.42)
where
\[ \|b(\cdot, \cdot, y)\|_T^2 = 2 \sum_{k=1}^{K^r} \mathbb{E} \int_0^T 1_{st0}(t) \left| \int_{\mathbb{T}} \partial_u \eta_+ (U^r(t,x), a_+(t,y)) \right| \| \sigma_k^{r^+} (U^r(t,x)) - \sigma_k^{r^+} (U^r(t,y)) \| \rho_\varepsilon (x-y) dx \right|^2 dt. \]

Condition (3.42) follows from the estimate
\[
\int_{\mathbb{T}} \|b(\cdot, \cdot, y)\|_T^2 dy \leq 4 \sum_{k=1}^{K^r} \mathbb{E} \int_0^T 1_{st0}(t) \left| \int_{\mathbb{T} \times \mathbb{T}} |\partial_u \eta_+ (U^r(t,x), a_+(t,y))|^2 \right| \left[ |\sigma_k^{r^+} (U^r(t,x))|^2 + |\sigma_k^{r^+} (U^r(t,y))|^2 \right] \rho_\varepsilon^2 (x-y) dx dy \]
\[ = \mathcal{O}(1), \]

Indeed, we have
\[ |\partial_u \eta_+ (U, a)| \leq m c_{\lambda \rho} \int_{-1}^{1} (|u| + |z| \rho^{\frac{m-1}{2}} (1 - z^2)^{\lambda} dz, \]
and therefore
\[
\int_{\mathbb{T}} \|b(\cdot, \cdot, y)\|_T^2 dy = \mathcal{O}(1) \int_{\mathbb{T} \times \mathbb{T}} \rho_\varepsilon^2 (x-y) dx dy = \varepsilon^{-1} \mathcal{O}(1),
\]
where the term \( \mathcal{O}(1) \) depends on \( K^r, \|\sigma_k^{r^+}\|_{L^\infty} \) (finite by hypothesis) and \( \mathbb{E}\|\rho^r, u^r\|_{L^\infty(Q_T)} \). Now we want to pass to the limit \( \varepsilon \to 0 \) in (3.41). We will show that, up to extraction of subsequence in \( \varepsilon \),
\[ \lim_{\varepsilon \to 0} A_\varepsilon (t) = \lim_{\varepsilon \to 0} B_\varepsilon (t) = 0, \]
almost surely. Since \( U^r \) is almost surely bounded and continuous on \( \mathbb{T} \times (0,T) \)(cf. item 1. in Proposition (3.6)), we have,
\[ |\partial_u \eta_+ (U^r, a)| \leq m (m-1) c_{\lambda \rho} \int_{-1}^{1} (|u| + |z| \rho^{\frac{m-1}{2}} (1 - z^2)^{\lambda} dz = \mathcal{O}(1), \]
almost surely, where \( \mathcal{O}(1) \) depends on particular on \( \|\rho^r, u^r\|_{L^\infty(Q_T)} \). It follows that
\[ A_\varepsilon (t) = \mathcal{O}(1) \sum_{k=1}^{K^r} \int_0^t \int_{\mathbb{T} \times \mathbb{T}} |\sigma_k^{r^+} (U^r(s,x)) - \sigma_k^{r^+} (U^r(s,y))|^2 \rho_\varepsilon (x-y) dxdyds \]
\[ = o(1). \]
Furthermore, the Itô isometry formula gives
\[ \mathbb{E} |B_\varepsilon (t)|^2 = \sum_{k=1}^{K^r} \mathbb{E} \int_0^t 1_{st0}(s) c_\varepsilon^k (s) ds, \]

Where
with

\[ c^k_\varepsilon(s) = \left| \int_{T \times T} \partial_\nu \eta_+ (U^\tau(s, x), a_+(s, y)) \times [\sigma^{*, \tau}_k(U^\tau(s, x)) - \sigma^{*, \tau}_k(U^\tau(s, y))] \rho_\varepsilon(x - y) dx dy \right|^2. \]

Since \( \|\sigma^{*, \tau}_k\|_{L^\infty} = \mathcal{O}(1) \) and

\[ \int_{T \times T} \rho_\varepsilon(x - y) dx dy = 1, \]

we have \( c^k_\varepsilon(s) = \mathcal{O}(1 + \|\rho^\tau, u^\tau\|_{L^m(Q_T)}) \), a quantity that is integrable with respect to \((\omega, s)\) by item 2. in Proposition (3.6). At fixed \((\omega, s)\), we also have

\[ \lim_{\varepsilon \to 0} c^k_\varepsilon(s) = 0 \]

since \( \partial_\nu \eta_+ (U^\tau(t, x), a_+(s, y)) \) is bounded and \( U^\tau \) continuous. By the dominated convergence theorem, we deduce that \( \lim_{\varepsilon \to 0} \mathbb{E}|B_\varepsilon(t)|^2 = 0 \). In particular, up to a subsequence, \( B_\varepsilon(t) \to 0 \) almost surely. The convergence of the left hand-side in (3.41) can be analysed in a similar way. In conclusion, at the limit \([\varepsilon \to 0]\) in (3.41), we obtain

\[ \int_{T} \eta_+ (U^\tau(t, x), a_+(t, x)) dx \leq 0 \]

almost surely, that is to say

\[ u^\tau(t, x) + \rho^\tau(t, x)^{\frac{2}{1 - z}} \leq a_+(t, x) \]

almost surely, for all \((x, t) \in T \times [0, T]\) and all \(z \in [-1, 1]\). In particular, summing over \(z \in [0, 1]\) gives

\[ u^\tau(t, x) + \frac{1}{2} \rho^\tau(t, x)^{\frac{2}{1 - z}} \leq a_+(t, x) \]

almost surely, for all \((x, t) \in T \times [0, T]\). A similar work based on the entropy

\[ \eta_-(U, a) = c_\lambda \rho \int_{-1}^{1} (u + z \rho^{\frac{2}{1 - z}} - a)^m(1 - z^2)^{\lambda} dz \]

then gives

\[ |u^\tau(t, x)| + \frac{1}{2} \rho^\tau(t, x)^{\frac{2}{1 - z}} \leq a(t, x) \] (3.44)

almost surely, for all \((x, t) \in T \times [0, T]\), where

\[ a(t, x) = \|u_0\|_{L^\infty} + \|\rho_0\|_{L^\infty}^{\frac{1}{1 - z}} + \sqrt{2} \sum_{k=1}^{K^*} \int_{0}^{t} 1_{s(t)}(s) \sigma^{*, \tau}_k(U^\tau(s, x)) d\beta_k(s). \]
Let \( h(t, x) = |u^\tau(t, x)| + \frac{1}{2} \rho^\tau(t, x)^{\frac{\gamma - 1}{2}} \) denote the left hand-side of (3.44) and let \( p \geq 2 \). Note that (3.10) implies

\[
\sum_{k=1}^{K^\tau} |\sigma_k^\tau(U^\tau(t, x))|^2 = \mathcal{O}(1)(1 + |h(t, x)|^2),
\]

where \( \mathcal{O}(1) \) depends on \( D_0 \). By the Burkholder-Davis-Gundy Inequality, we deduce from (3.44) and (3.45) the following estimate:

\[
E \int_T |h(t, x)|^p dx = \mathcal{O}(1) \left[ 1 + E \int_T \left( \sum_{k=1}^{K^\tau} \int_0^t |\sigma_k^\tau(U^\tau(s, x))|^2 ds \right)^{p/2} dx \right]
\]

\[
= \mathcal{O}(1) \left[ 1 + E \int_T \left( \int_0^t |h(s, x)|^2 ds \right)^{p/2} dx \right]
\]

\[
= \mathcal{O}(1) \left[ 1 + E \int_T \int_0^t |h(s, x)|^p ds dx \right], \tag{3.46}
\]

where \( \mathcal{O}(1) \) depends on \( D_0, p, \|u_0\|_{L^\infty}, \|\rho_0\|_{L^\infty} \) and \( T \). Using the Gronwall Lemma, (3.32) then follows from (3.46).

### 3.1.4 Gradient estimates

In Proposition 3.7 above, we have obtained an estimate on \( U^\tau_x \). In the case where \( \eta = \eta_E \) is the energy (this corresponds to \( g(\xi) = \frac{1}{2} \xi^2 \)), some computations show that

\[
\eta''_E(U) \cdot (U_x, U_x) = \kappa \gamma |\rho|^{\gamma - 2} |\rho_x|^2 + \rho |u_x|^2. \tag{3.47}
\]

More generally, we have the following weighted estimates (see Corollary 3.14 below).

**Proposition 3.13** (Gradient bounds). Let \( m \in \mathbb{N} \). Let \( \eta_m \) be the entropy given by (2.7) with \( g(\xi) = \xi^{2m} \). Then, the solution \( U^\tau \) to (3.7)-(3.8) satisfies the estimate

\[
\mathbb{E} \int_{Q_T} 1_{\text{det}(t)} G^{[2]}(\rho^\tau, u^\tau)[\partial^2 |\rho|^\gamma \partial_x \rho^\tau |^2 + \rho^\tau |\partial_x u^\tau|^2] dx dt \leq \mathbb{E} \int_{Q_T} 1_{\text{det}(t)} G^{[1]}(\rho^\tau, u^\tau)[2\partial |\rho|^\gamma \partial^2 \rho^\tau | \cdot |\rho^\tau|^{1/2} \partial_x u^\tau|] dx dt + \mathcal{O}(1), \tag{3.48}
\]

where

\[
G^{[2]}(\rho, u) = c_\lambda \int_{-1}^{1} g''(u + z \rho^{\gamma - 1}) (1 - z^2) \lambda^\lambda dz,
\]

\[
G^{[1]}(\rho, u) = c_\lambda \int_{-1}^{1} |z| g''(u + z \rho^{\gamma - 1}) (1 - z^2) \lambda^\lambda dz,
\]

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and $O(1)$ depends on $T$, $\gamma$, on the constant $D_0$ in (2.2) and on the initial quantities $E\Gamma_0(U_0)$ for $\eta \in \{\eta_0, \eta_m, \eta_m^2\}$.

Proof. We introduce the probability measure
\[ dm_\lambda(z) = c_\lambda(1-z^2)^{\lambda/2}dz \]
and the $2 \times 2$ matrix
\[ S = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \]
which satisfies
\[ \partial_x U = SW, \quad W := \begin{pmatrix} \partial_x \rho \\ \partial_x u \end{pmatrix}. \]
(3.49)

By (3.21), we then have
\[ \int_0^T \mathbb{E} \int T_0 \det(t)^{\langle S'\eta''(U^\tau)SW, W\rangle} dx dt = O(1), \]
(3.50)
where $\langle \cdot, \cdot \rangle$ is the canonical scalar product on $\mathbb{R}^2$ and $S^*$ is the adjoint of $S$ for this scalar product. We compute (recall that $\theta = \frac{\gamma - 1}{2}$)
\[ \eta''(U) = \frac{1}{\rho} \int \mathbb{R} \left[ A(z)g'\left( u + z\rho^\theta \right) + B(z)g''\left( u + z\rho^\theta \right) \right] dm_\lambda(z), \]
where
\[ A(z) = \begin{pmatrix} \gamma^2 - 1 & z\rho^\theta \\ 0 & 0 \end{pmatrix}, \quad B(z) = \begin{pmatrix} -u + \theta z\rho^\theta & -u + \theta z\rho^\theta \\ -u + \theta z\rho^\theta & 1 \end{pmatrix}. \]
In particular
\[ S^* AS(z) = \begin{pmatrix} \gamma^2 - 1 & z\rho^\theta \\ 0 & 0 \end{pmatrix}, \quad S^* BS(z) = \begin{pmatrix} \theta^2 z^2\rho^{2\theta} & \theta z\rho^\theta \\ \theta z\rho^\theta & 1 \end{pmatrix}, \]
and (3.49)-(3.50) give
\[ \mathbb{E} \int_{Q_T} \mathbb{E} \int_{T_0} 1_{\text{det}}(t) \left( I|\partial_x \rho^\tau|^2 + J|\partial_x \rho^\tau|^{1/2} |\partial_x u^\tau + K| \partial_x u^\tau|^2 \right) dx dt = O(1), \]
(3.51)
where
\[ I = |\rho^\tau|^{2n-1} \int_\mathbb{R} \theta^2 z^2 g''\left( u^\tau + z|\rho^\tau|^\theta \right) dm_\lambda(z) \]
\[ + |\rho^\tau|^{\theta - 1} \int_\mathbb{R} \frac{\gamma^2 - 1}{4} z g'\left( u^\tau + z|\rho^\tau|^\theta \right) dm_\lambda(z), \]
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and
\[ J = 2|\rho|^{\theta - \frac{1}{2}} \int_{R} \theta z g'' \left( u^\tau + z |\rho|^\theta \right) \, dm_\lambda(z), \]
\[ K = \int_{R} g'' \left( u^\tau + z |\rho|^\theta \right) \, dm_\lambda(z). \]

We observe that \( 2z \, dm_\lambda(z) = -\frac{c}{\lambda + 1} d(1 - z^2)^{\lambda+1} \). By integration by parts, the second term in \( I \) can therefore be written
\[ I = |\rho|^2 \, g'' \left( u^\tau + z |\rho|^\theta \right) \, dm_\lambda(z). \]

Since \( \frac{\gamma^2 - 1}{8 (1 + \lambda)^2} \), we have
\[ I = |\rho|^2 \, g'' \left( u^\tau + z |\rho|^\theta \right) \, dm_\lambda(z). \]

This gives (3.48). □

We apply (3.48) with \( g(\xi) := |\xi|^{2m+2} \) and \( \eta = \eta_{m+1} \) given by (2.7). Then
\[ |u|^{2m} + \rho^m (\gamma - 1) = O(1) G[2](\rho, u) \]
and since the higher order terms in \( G[1] \) are (strictly) dominated by the higher order terms in \( G[2] \), our estimate (3.48) gives
\[ E \int_{Q_T} \left[ |u|^2 + \rho^\gamma |\partial_x u|^2 \right] \, dx \, dt = O(1). \]

By letting \( m \) vary, we obtain the following

**Corollary 3.14.** The solution \( U^\tau \) to (3.7)-(3.8) satisfies the estimates
\[ E \int_{Q_T} \left[ |u|^\alpha |\rho|^\gamma - 2 + |\rho|^\gamma - 2 + \beta \right] |\partial_x u|^2 \, dx \, dt = O(1), \quad (3.52) \]
and
\[ E \int_{Q_T} \left[ |u|^\alpha + |\rho|^1 + \beta \right] |\partial_x u|^2 \, dx \, dt = O(1), \quad (3.53) \]
for all \( \alpha, \beta \geq 0 \), where \( O(1) \) depends on \( T, \gamma \), on the constant \( D_0 \) in (2.2) and on a finite number of quantities \( E \Gamma_{n_i}(U_0) \), for entropies \( n_i \) associated to various power-like functions \( |\xi|^\gamma \), for given \( r_j \)'s depending on \( \alpha, \beta \).
3.1.5 Positivity of the density

**Proposition 3.15** (Positivity). Let $U^\tau$ be the solution to (3.7)-(3.8) with initial datum $U_0 = (\rho_0, q_0)$ and assume that $\rho_0$ is uniformly positive: there exists $c_0 > 0$ such that $\rho_0 \geq c_0$ a.e. on $\mathbb{T}$. Let $m > 3$. Then, a.s., there exists $c > 0$ depending on $c_0$, $T$, such that

$$\int \int_{Q_T} 1_{\text{det}(t) \rho^\tau |\partial_x u^\tau|^2} dt dx$$

and

$$\int \int_{Q_T} |u^\tau|^m dt dx$$

only, such that

$$\rho^\tau \geq c$$

a.e. in $T \times [0, T]$.

**Proof.** We apply Theorem (A.1) (and Remark A.2) proved in Appendix A. 

**Remark 3.16.** Note that the constant $c$ in (3.55) depends on $\omega$, but only through the values of the random variables involved in (3.54).

3.1.6 Bounds on higher derivatives of $U^\tau$

Assume, by adjusting $\tau$ if necessary, that $T = t_{2K}$, $K \in \mathbb{N}^*$. Let $R > 0$ and let us introduce the stopping time $T_R$ defined as the infimum of the times $t_{2k}$ such that

$$\sup_{t \in [0, t_{2k}]} \Gamma_\eta(U^\tau(t)) + 2 \int \int_{Q_{t_{2k}}} 1_{\text{det}} \eta''(U^\tau(t)) \cdot (U^\tau_x, U^\tau_x) dt dx \leq R,$$

where the infimum over the empty set is taken to be $t_{2K} = T$ by definition. Here we take for $\eta$ the energy, i.e. the entropy associated to $g(\xi) = \frac{1}{2} |\xi|^2$. In that case, we have by (3.47) the control

$$\int \int_{Q_{t_{2K}}} 1_{\text{det}}(t) \rho^\tau |\partial_x u^\tau|^2 dt dx \leq R.$$

Let then $U^\tau_R$ denote the process deduced from $U^\tau$ by following a determinist evolution after the time $T_R$: $U^\tau_R$ is defined like $U^\tau$ by (3.7) where $\Phi_{\tau}$ is replaced with $\Phi_{\tau} 1_{t < T_R}$. Since $\Gamma_\eta(U^\tau(T_R)) \leq R$, we can adapt the proof of Proposition 3.7 (it is now purely deterministic) to obtain

$$\int_{T_R}^T \int_{\mathbb{T}} 1_{\text{det}}(t) \rho^\tau_R |\partial_x u^\tau_R|^2 dt dx \leq M(R),$$

where the deterministic constant $M(R)$ depends on $T$, $\gamma$, $D_0$ and $R$ only. In particular, we have

$$\int \int_{Q_T} 1_{\text{det}}(t) \rho^\tau_R |\partial_x u^\tau_R|^2 dt dx \leq M(R) + R.$$
Proposition 3.15 in the previous section then asserts that there exists \( c(R) > 0 \) depending on \( c_0, T \) and \( R \) such that

\[
\rho^*_R \geq c(R) \text{ a.e. on } \mathbb{T} \times [0, T].
\]

(3.57)

In this section we will use this uniform estimate (3.57) and the entropy estimates from Section 3.1.2 to obtain some bounds independent on \( \tau \) on the higher derivatives of \( U_R^\tau \). Note in particular that the proof of Corollary 3.9 and Proposition 3.13 apply readily to \( U_R^\tau \) as well so that we have the following

**Remark 3.17.** The estimates (3.30), (3.52), (3.53) are satisfied by \( U_R^\tau \).

We will prove the following result.

**Proposition 3.18** (Bounds on higher derivatives). Let \( U_0 = (\rho_0, q_0) \in W^{1,\infty}(\mathbb{T})^2, U_0 \) deterministic. Let \( U^\tau \) be the solution to (3.7)-(3.8) with initial datum \( U_0 = (\rho_0, q_0) \). Assume that \( \rho_0 \geq c_0 \) a.e. on \( \mathbb{T} \) where \( c_0 > 0 \). Let \( p > 2 \) and \( \alpha \in (0, 1/2) \). Then for all \( R > 0 \), we have

\[
\mathbb{E}\|\rho^*_R\|_{C^0([0,T];L^p(\mathbb{T}))}, \mathbb{E}\|\rho^*_R\|_{L^p(0,T;W^{3/2,p}(\mathbb{T}))} = \mathcal{O}(1),
\]

(3.58)

and

\[
\mathbb{E}\|u^*_R\|_{C^0([0,T];L^p(\mathbb{T}))}, \mathbb{E}\|u^*_R\|_{L^p(0,T;W^{3/2,p}(\mathbb{T}))} = \mathcal{O}(1),
\]

(3.59)

where \( \mathcal{O}(1) \) depends on \( R, \) on \( c_0, \) on \( p, \alpha, \gamma, T, \) on the constants \( D_0 \) and \( D_1 \) in (2.2), (2.3) and on the moments of \( U_0 \) and \( \partial_x U_0 \) up to a given order depending on \( p \) and \( \alpha \).

**Proof.** For \( 0 \leq s < t \leq T \), let \( Q_{s,t} \) denote the cylinder \( \mathbb{T} \times (s, t) \). We work on the variables \( \rho, u = q/\rho \) and rewrite (3.7) as a system of two heat equations with source terms in the domain \( Q_{12k,42k+1} \):

\[
\left( \frac{1}{2} \partial_t - \partial_x^2 \right) \rho^\tau = -u^\tau \partial_x \rho^\tau - \rho^\tau \partial_x u^\tau =: f^\tau_1,
\]

(3.60)

and

\[
\left( \frac{1}{2} \partial_t - \partial_x^2 \right) u^\tau = -u^\tau \partial_x u^\tau - \kappa \gamma |\rho^\tau|^{-2} \partial_x \rho^\tau + 2 \frac{\partial_x \rho^\tau}{\rho^\tau} \partial_x u^\tau =: f^\tau_2.
\]

(3.61)

Let us extend \( f^\tau_1 \) and \( f^\tau_2 \) by 0 in the strips \( Q_{12k+1,42k+2} \), and set

\[
\mathbf{F}^\tau = \begin{pmatrix}
  f^\tau_1 \\
  f^\tau_2
\end{pmatrix},
\]

and let \( S(t) \) denote the semi-group associated to \( -\partial_x^2 \) on \( L^p(\mathbb{T}) \). For \( f \in L^1(Q_T), z_0 \in L^1(\mathbb{T}) \), the solution to the equation

\[
\left( \frac{1}{2} \partial_t - \partial_x^2 \right) z(t, x) = f(t, x), \quad x \in \mathbb{T}, t > 0,
\]

(3.62)
with initial datum $z_0$ is
\[ z(t) = S(2t)z_0 + 2 \int_0^t S(2(t-s))f(s)ds. \]

By iteration, it follows that $U^γ$ is given by
\[
U^γ(t) = S(t^γ)U^γ(0) + \int_0^{t^γ} S(t^γ - s)F^γ(s_γ)ds + \sum_{k \geq 1} \sqrt{2} \int_0^{t^γ} S(t^γ - s_γ)1_{sto}(s)\psi_k(U^γ(s))dβ_k(s),
\]
where we have defined
\[
t^γ = \min(2t - t_{2n}, t_{2n+2}), \quad t_0 = \frac{t + t_{2n}}{2}, \quad t_{2n} \leq t < t_{2n+2}.
\]

For the “stopped” process $U^γ_R$, we have, similarly, a decomposition
\[
U^γ_R(t) = U^γ_{0,R} + U^γ_{det,R} + U^γ_{sto,R}, \tag{3.62}
\]
where
\[
U^γ_{0,R}(t) = S(t^γ)U^γ(0),
U^γ_{det,R}(t) = \int_0^{t^γ} S(t^γ - s)F^γ_R(s_γ)ds,
U^γ_{sto,R}(t) = \sum_{k \geq 1} \sqrt{2} \int_0^{t^γ∧T_R} S(t^γ - s_γ)1_{sto}(s)\psi_k(U^γ(s))dβ_k(s),
\]
and where $F^γ_R$ is defined by similarity with $F^γ$, i.e.
\[
F^γ_R = \begin{pmatrix}
        f^γ_{1,R} \\
        f^γ_{2,R}
\end{pmatrix} = \begin{pmatrix}
        -u^γ_R\partial_xρ^γ_R - ρ^γ_R\partial_xu^γ_R \\
        -u^γ_R\partial_xu^γ_R - κγ|ρ^γ_R|^{γ-2}\partial_xρ^γ_R + 2\frac{∂xρ^γ_R}{ρ^γ_R}\partial_xu^γ_R
\end{pmatrix}.
\]

**Higher derivatives in $x$:** to obtain some estimates on the higher derivatives in $x$ of $ρ^γ_R$, $u^γ_R$, we need to use a bootstrap method. Actually we will use improved estimates in Equation (3.62) three times. For a better readability of this part of the proof, we will adopt the following convention of notation: a constant $C_R$, which may vary from lines to lines, is a constant depending on $T$, $γ$, on the constant $D_0$ in (2.2) and on $R$. A number $ε > 0$ being given, we use the notation $ε$ for quantities small with $ε$ but possibly different from $ε$. Therefore in what follows, $L^{r-ε}$ may well be actually $L^{r-\overline{ε}}$. Also, $L^{mε}$ is $L^{m}$ for a $m$ (quite high) depending on $ε$, which we do not specify and again, the value of $mε$ may vary from line to line. Eventually, we denote by $M_ε(U_0)$ a constant depending only on the moments
\[
\int_T (u^{m_ε}_0 + ρ^{m_ε(γ-1)}_0)ρ_0dx, \quad \int_T |∂_xu_0|^{m_ε} + |∂_xρ_0|^{m_ε}dx,
\]

where the exponent \( m_\varepsilon \) depends on \( \varepsilon \).

We begin by showing that

\[
\| \partial_x \rho^\tau_R \|_{L^{6-\varepsilon}(Q_T)} \leq M_\varepsilon(U_0). (3.63)
\]

Rewriting

\[
f^\tau_{1,R} = -|\rho^\tau_R|^{2/2} \cdot |\rho^\tau_R|^{2/2} u^\tau_R \partial_x \rho^\tau_R - \rho^\tau_R \partial_x u^\tau_R,
\]

we deduce from Remark 3.17, from the moment estimate (3.30), and from the gradient estimates (3.52), (3.53) and from the Hölder inequality that

\[
\| f^\tau_{1,R} \|_{L^{6-\varepsilon}(Q_T)} \leq M_\varepsilon(U_0).
\]

By (B.7) this implies \( \| \partial_x \rho^\tau_{\text{det},R} \|_{L^{6-\varepsilon}(Q_T)} \leq M_\varepsilon(U_0) \). We have besides \( \rho^\tau_{\text{sto},R} = 0 \) and thus (3.63).

Then, in the equation satisfied by \( u^\tau_R \), we use the expression

\[
f^\tau_{2,R} = -\frac{1}{|\rho^\tau_R|^{1/2}} \cdot |\rho^\tau_R|^{1/2} u^\tau_R \partial_x u^\tau_R - \kappa \gamma \frac{1}{|\rho^\tau_R|^{2/2}} \cdot |\rho^\tau_R|^{2/2} \partial_x \rho^\tau_R
\]

\[+ 2 \frac{1}{|\rho^\tau_R|^2} \cdot \partial_x \rho^\tau_R \cdot \rho^\tau_R \partial_x u^\tau_R.
\]

We deduce from Remark 3.17, from (3.30), (3.52), (3.53), (3.63), from (3.57) and from the Hölder inequality that \( \| f^\tau_{2,R} \|_{L^{3/2-\varepsilon}(Q_T)} \leq C_R M_\varepsilon(U_0) \), and thus, by (B.7),

\[
\| \partial_x u^\tau_{\text{det},R} \|_{L^{3/2-\varepsilon}(Q_T)} \leq C_R M_\varepsilon(U_0). (3.64)
\]

To get a bound on \( u^\tau_{\text{sto},R} \), we use the following lemma.

**Lemma 3.19.** Let \( q \geq 2 \). For \( k \in \mathbb{N}^* \), let \( h_k \in L^2(\Omega \times (0,T);L^q(\mathbb{T})) \) be some predictable processes such that

\[
H := \left( \sum_{k \geq 1} |h_k|^2 \right)^{1/2}
\]

satisfies \( H \in L^q(\Omega \times (0,T) \times \mathbb{T}) \). Then we have the estimate

\[
E \left\| \sum_{k \geq 1} \int_0^{t_o} S(t_o - s)h_k(s)d\beta_k(s) \right\|_{L^q(\mathbb{T})}^q \leq C_{\text{BDG}}(q)t_o^{q-1/q} E\|H\|_{L^q(\Omega \times (0,T))}^q, (3.66)
\]

for a given constant \( C_{\text{BDG}}(q) \) depending on \( q \) only which is bounded for bounded values of \( q \in [2, +\infty) \).
Proof of Lemma 3.19: we apply the Burkholder-Davis-Gundy Inequality in the 2-smooth Banach space \( L^q(\mathbb{T}) \) to obtain

\[
\mathbb{E} \sup_{0 \leq \sigma \leq t} \left\| \sum_{k \geq 1} \int_0^\sigma S(t - s)\rho_k(s) d\beta_k(s) \right\|_{L^q(\mathbb{T})}^q \\
\leq C_{\text{BDG}}(q) \mathbb{E} \left( \int_0^{t_s} \|S(t_2 - s)\mathbf{H}(s)\|^2_{L^2(\mathbb{T})} ds \right)^{q/2}.
\]

Since \( S(t) \) is a contraction from \( L^2(\mathbb{T}) \) into \( L^2(\mathbb{T}) \), we obtain (3.66) by Jensen’s inequality. 

We apply Lemma 3.19 with

\[
h_k(t) = \partial_x \left( 1_{\text{sto}}(t) \sigma_k(\mathbf{U}_R(t)) \right) 1_{t < T_R}
\]

and \( q = 3 - \varepsilon \). By the growth hypothesis (2.3) and (3.57) we obtain, for \( T \geq 1 \),

\[
\mathbb{E}\|\partial_x u_{\text{sto}, R}(t)\|_{L^{3-\varepsilon}(\mathbb{T})}^{3-\varepsilon} \leq C_T C_R \left( \mathbb{E}\|\partial_x u_R\|_{L^{3-\varepsilon}(Q_T)}^{3-\varepsilon} + \mathbb{E}\|\partial_x \rho_R\|_{L^{3-\varepsilon}(Q_T)}^{3-\varepsilon} \right).
\]

Here \( C := \sup_{2 \leq q \leq 3} C_{\text{BDG}}(q) \) is an absolute constant.

From the identity (3.62), from (3.63) and the estimates (3.64), (3.67) and the Gronwall Lemma, we obtain

\[
\mathbb{E}\|\partial_x u_R\|_{L^{3-\varepsilon}(Q_T)}^{3-\varepsilon} \leq C_R M_\varepsilon(U_0). \tag{3.68}
\]

Now, Remark 3.17, the estimates (3.30), (3.57), (3.63), (3.68) and the identity

\[
f_{1,R} = -|\rho_R|^{-\frac{1}{2}} \cdot |\rho_R|^\frac{1}{2} u_R \cdot \partial_x \rho_R - \rho_R^\gamma \cdot \partial_x u_R,
\]

give

\[
\mathbb{E}\|f_{1,R}\|_{L^{3-\varepsilon}(Q_T)}^{3-\varepsilon} \leq C_R M_\varepsilon(U_0).
\]

Notice that for example, a bound for the term \( \rho_R^\gamma \cdot \partial_x u_R \) in \( L^p \) is given by

\[
\|\rho_R^\gamma \|_{L^m} \|\partial_x u_R\|_{L^q} \quad \text{with} \quad q = 3 - \varepsilon, \quad m = (3 - \varepsilon)(3 - 2\varepsilon)/\varepsilon, \quad \text{and the obtained space is} \quad L^p \quad \text{with} \quad p = 3 - 2\varepsilon \quad \text{which is also denoting by} \quad L^{3-\varepsilon}.
\]

By (B.7) and (B.4b), it follows that

\[
\mathbb{E}\|\partial_x \rho_R\|_{L^{m_\varepsilon}(Q_T)}^{m_\varepsilon} \leq C_R M_\varepsilon(U_0). \tag{3.69}
\]

Eventually, it follows from the decomposition

\[
f_{2,R} = -\frac{1}{|\rho_R|^{1/(2m)}} \cdot |\rho_R|^{1/(2m)} u_R \cdot \partial_x u_R - \kappa \gamma |\rho_R|^\gamma \cdot \partial_x \rho_R \\
+ \frac{1}{|\rho_R|^2} \cdot \partial_x \rho_R^\gamma \cdot \partial_x u_R,
\]

Eventually, it follows from the decomposition

\[
f_{2,R} = -\frac{1}{|\rho_R|^{1/(2m)}} \cdot |\rho_R|^{1/(2m)} u_R \cdot \partial_x u_R - \kappa \gamma |\rho_R|^\gamma \cdot \partial_x \rho_R \\
+ \frac{1}{|\rho_R|^2} \cdot \partial_x \rho_R^\gamma \cdot \partial_x u_R,
\]
from Remark 3.17 and from estimates (3.30), (3.57), (3.68) and (3.69) that
\[ \mathbb{E}\|f_{T, R}^\tau\|_{L^{3-\varepsilon}(Q_T)}^{3-\varepsilon} \leq C_R M_{\varepsilon}(U_0), \]
which gives by (B.4a) the estimate
\[ \mathbb{E}\|\partial_x u_{\text{det}, R}^\tau\|_{L^{m_3}(Q_T)}^{m_3} \leq C_R M_{\varepsilon}(U_0). \]  
(3.70)

We also apply Lemma 3.19 to the martingale part in (3.62). Let \( m > 3 \) be fixed (quite large). If \( 2 \leq m \varepsilon \leq m \), we have
\[ \mathbb{E}\|\partial_x u_{\text{sto}, R}^\tau(t)\|_{L^{m_3}(Q_T)}^{m_3} \parallel \leq \tilde{C} \mathbb{E}\|\partial_x u_{R}^\tau\|_{L^{m_3}(Q_T)}^{m_3}, \]
(3.71)
where \( \tilde{C} = \sup_{2 \leq q \leq m} C_{\text{BDG}}(q) \). By Gronwall Lemma it follows that
\[ \mathbb{E}\|\partial_x u_{\text{det}, R}^\tau\|_{L^{m_3}(Q_T)}^{m_3} \leq C_{R, M_{\varepsilon}}(U_0). \]  
(3.72)

Now the “near \( L^\infty \)” estimates (3.69), (3.72) together with the positivity estimate (3.57) give \( f_{T, R}^\tau \in L^p(Q_T) \) where \( p \) is arbitrary. We use the estimate
\[ \|D^{3/2} \int_0^{t_1} S(t_2 - s)f(s)ds\|_{L^p(Q_T)} \leq C\|f\|_{L^p(Q_T)} \quad \text{if} \quad \frac{1}{q} > \frac{1}{p} - \frac{1}{6}, \]
(3.73)
which can be derived similarly to (B.7) by using the estimate
\[ \|D^{3/2} S(t)\|_{L^p_q \rightarrow L^p} \leq C t^{-\frac{1}{2} \left(\frac{1}{p} - \frac{1}{q}\right) - \frac{3}{4}}. \]  
(3.74)

Here, \( |D| \) denotes the square-root of the Laplacian: \( |D| = (-\partial_x^2)^{1/2} \). We also deduce from (3.74) that
\[ \|D^{3/2} S(t_2)u_0\|_{L^p(Q_T)} \leq C\|u_0\|_{L^p(Q_T)}. \]

Therefore (3.62) and (3.73) give
\[ \mathbb{E}\|D^{3/2}(u_{0, R}^\tau + u_{\text{det}, R}^\tau)\|_{L^p(Q_T)} = O(1). \]  
(3.75)

By (2.3) and by interpolation, we have
\[ \|H\|_{L^p(Q_T)} \leq C_R (1 + \|D^{3/2} u_{R}^\tau\|_{L^p(Q_T)}), \]
where \( H \) is defined by (3.65) with \( h_k = |D^{3/2}(1_{\text{sto}}(t)\sigma_k(U_R(t)))1_{t<T_R}. \) By Lemma 3.19, we deduce that
\[ \mathbb{E}\|D^{3/2} u_{\text{sto}, R}^\tau\|_{L^p(Q_T)} = C(1 + \mathbb{E}\|D^{3/2} u_{R}^\tau\|_{L^p(Q_T)}). \]
By the Gronwall Lemma and (3.75) it follows that $E\|u^\tau_R\|_{L^p(0,T;W^{3/2,p}(\mathbb{T}))} = O(1)$. To obtain time estimates on $u^\tau_R$ now, we use the Kolmogorov criterion and the a priori bounds of Section 3.1.2. Let $p > 2$. Set

$$h_k(s) = 1_{s_1}(s)\rho^\tau_R(s)\sigma_k(U^\tau_R(s)).$$

By the Burkholder-Davis-Gundy Inequality in the 2-smooth Banach space $L^p(\mathbb{T})$ we have, for $0 \leq t \leq t' \leq T$,

$$E\left\| \sum_{k \geq 1} \int_{t}^{t'} S(t'^\ast - s)h_k(s)d\beta_k(s) \right\|_{L^p(\mathbb{T})}^p \leq C_{BDG}(p)E\left( \int_{t}^{t'} \|S(t'^\ast - s)H(s)\|_{L^2(\mathbb{T})}^2 dx ds \right)^{p/2},$$

where $H$ is defined as in (3.65). Since $S(t)$ is a contraction from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$, (2.2), Remark 3.17 and the moment estimate (3.30) show that

$$E\left\| \sum_{k \geq 1} \int_{t}^{t'} S(t'^\ast - s)h_k(s)d\beta_k(s) \right\|_{L^p(\mathbb{T})}^p \leq C_R M_e(U_0)|t' - t|^{p/2}.$$

The Kolmogorov continuity Theorem ([DPZ92], Theorem 3.3) therefore gives

$$E\|u^\tau_{sto,R}\|_{C^\alpha([0,T];L^p(\mathbb{T}))} = O(1),$$

where $0 < \alpha < \frac{1}{2} - \frac{1}{p}$. We can use a similar argument to deal with the term $u^\tau_{det,R}$: we have

$$E\left\| \int_{t}^{t'} S(t'^\ast - s)\rho^\tau_{2,R}(s)d\beta_k(s) \right\|_{L^p(\mathbb{T})}^p \leq C_R M_e(U_0)|t' - t|^{p-1},$$

and thus

$$E\|u^\tau_{det,R}\|_{C^\alpha([0,T];L^p(\mathbb{T}))} = O(1),$$

where $0 < \alpha < 1 - \frac{2}{p}$. By the identity

$$S(t'^\ast)u_0 - S(t)u_0 = \int_{t}^{t'} \partial_x S(t)\partial_x u_0 dt$$

and (B.3), we obtain also $\|u^\tau_{0,R}\|_{C^{1/2}([0,T];L^p(\mathbb{T}))} = O(1)$. This concludes the proof of (3.59). The proof of (3.58) is similar and simpler since $\rho^\tau_{sto,R} = 0$. 

\[\Box\]
3.1.7 Compactness argument

For \( k \in \mathbb{N}^* \), we introduce the independent processes

\[ X_k(t) = \sqrt{2} \int_0^t 1_{\mathcal{S}_0}(s) d\beta_k(s) \]

and set \( W^\tau(t) = \sum_{k \geq 1} X_k(t)e_k \). When \( \tau \to 0 \), each \( X_k \) converges in law to \( \beta_k \). Let

\[ \mathcal{X}_W = C([0, T]; \mathcal{U}_0) \]

Since the embedding \( \mathcal{U} \hookrightarrow \mathcal{U}_0 \) is Hilbert-Schmidt, the \( \mathcal{X}_W \)-valued process \( W^\tau \) converges in law to \( W \):

\[ \mu_{W^\tau} \to \mu_W, \quad (3.76) \]

where \( \mu_{W^\tau} \) denote the the law of \( W^\tau \) and \( \mu_W \) is the the law of \( W \) on \( \mathcal{X}_W \).

Let us fix \( p > 2 \). Define the path space \( \mathcal{X} = \mathcal{X}_U \times \mathcal{X}_W \), where

\[ \mathcal{X}_U = \left[ C([0, T]; W^{1,p}(T)) \right]^2. \]

Let us denote by \( \mu_U^\tau \) the law of \( U^\tau \) on \( \mathcal{X}_U \). The joint law of \( U^\tau \) and \( W^\tau \) on \( \mathcal{X} \) is denoted by \( \mu^\tau \).

**Proposition 3.20.** The set \( \{ \mu^\tau; \tau \in (0, 1) \} \) is tight and therefore relatively weakly compact in \( \mathcal{X} \).

**Proof.** First, we prove tightness of \( \{ \mu_U^\tau; \tau \in (0, 1) \} \) in \( \mathcal{X}_U \). Let \( \alpha \in (0, 1/2) \) and \( M > 0 \). Then

\[ K_M := \{ U \in \mathcal{X}_U; \|U\|_{C^\alpha([0, T]; L^p(T))} + \|U\|_{L^p([0, T]; W^{3/2, p}(T))} \leq M \} \]

is compact in \( \mathcal{X}_U \). Recall that the stopping time \( T_R \) is defined by (3.56). By the Markov inequality and the entropy estimate (3.21), we have

\[ \mathbb{P}(T_R < T) \leq \frac{C}{R}, \]

where the constant \( C \) depends on \( \gamma, T, D_0 \) and on the moments of \( U_0 \) and \( \partial_x U_0 \) up to a given order. We have therefore, by Proposition 3.18 and the Markov inequality,

\[ \mathbb{P}(U^\tau \notin K_M) \leq \frac{C}{R} + \mathbb{P}(U_R^\tau \notin K_M) \leq \frac{C}{R} + \frac{C_R}{M}, \]

by possibly augmenting the constant \( C \). Therefore, given \( \eta > 0 \) there exists \( R, M > 0 \) such that

\[ \mu_U^\tau(K_M) \geq 1 - \eta. \]

Besides, the law \( \mu_{W^\tau} \) is tight by (3.76). Consequently the set of the joint laws \( \{ \mu^\tau; \tau \in (0, 1) \} \) is tight. By Prokhorov’s theorem, it is relatively weakly compact. \( \blacksquare \)
Let now \((\tau_n)\) be a sequence decreasing to 0. For simplicity we will keep the notation \(\tau\) for \((\tau_n)\) and possible subsequences of \((\tau_n)\). Let \(\mu\) be an adherence value, for the weak convergence, of \(\mu^\tau\). By the Skorokhod Theorem we can assume a.e. convergence of the random variables by changing the probability space.

Proposition 3.21. There exists a probability space \((\~\Omega, \~\mathcal{F}, \~P)\) with a sequence of \(X\)-valued random variables \((\~U^\tau, \~W^\tau)\), \(\tau \in (0, 1)\), and \((\~U, \~W)\) such that

1. the laws of \((\~U^\tau, \~W^\tau)\) and \((\~U, \~W)\) under \(\~P\) coincide with \(\mu^\tau\) and \(\mu\) respectively,

2. \((\~U^\tau, \~W^\tau)\) converges \(\~P\)-almost surely to \((\~U, \~W)\) in the topology of \(X\).

3.1.8 Identification of the limit

Our aim in this section is to identify the limit \((\~U, \~W)\) given by Proposition 3.21. Let \((\~\mathcal{F}_t)\) be the \(\~P\)-augmented canonical filtration of the process \((\~U, \~W)\), i.e.

\[
\~\mathcal{F}_t = \mathcal{F} \cup \{ N \in \mathcal{F}; \~P(N) = 0 \}, \quad t \in [0, T],
\]

where \(\rho_t\) is the operator of restriction to the interval \([0, t]\) defined as follows: if \(E\) is a Banach space and \(t \in [0, T]\), then

\[
\rho_t : C([0, T]; E) \rightarrow C([0, t]; E)
\]

\[k \mapsto k|_{[0,t]}.
\]

(3.77)

Clearly, \(\rho_t\) is a continuous mapping. The following statement is given for \(\~U^\varepsilon\) (we reintroduce the factor \(\varepsilon\) here), which is built like \(\~U\).

Proposition 3.22 (Martingale solution to (3.1)). The sextuplet

\[(\~\Omega, \~\mathcal{F}, (\~\mathcal{F}_t), \~P, \~W, \~U^\tau)\]

is a weak martingale solution to (3.1).

The proof of Proposition 3.22 uses a method of construction of martingale solutions of SPDEs that avoids in part the use of representation Theorem. This technique has been developed in Ondreját [Ond10], Brzeźniak, Ondreját [BO11] and used in particular in Hofmanová, Seidler [HS12] and in [Hof13b, DHV13].

Recall that \(A\) (the flux of Equation (1.1)) is defined by (2.4). Let us define for all \(t \in [0, T]\) and a test function \(\varphi = (\varphi_1, \varphi_2) \in C^\infty(\mathbb{T}; \mathbb{R}^2),

\[
M^\tau(t) = \langle U^\tau(t), \varphi \rangle - \langle U_0, \varphi \rangle + 2 \int_0^t 1_{\text{det}(s)} \langle \partial_x A(U^\tau) - \partial_x^2 U^\tau, \varphi \rangle ds,
\]

\[
\~M^\tau(t) = \langle \~U^\tau(t), \varphi \rangle - \langle \~U_0, \varphi \rangle + 2 \int_0^t 1_{\text{det}(s)} \langle \partial_x A(\~U^\tau) - \partial_x^2 \~U^\tau, \varphi \rangle ds,
\]

\[
\~M(t) = \langle \~U(t), \varphi \rangle - \langle \~U_0, \varphi \rangle + \int_0^t \langle \partial_x A(\~U) - \partial_x^2 \~U, \varphi \rangle ds.
\]

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In what follows, we fix some times $s, t \in [0, T], s \leq t$, and a continuous function
$$\gamma : C([0, s]; L^p(T)) \times C([0, s]; \mathcal{U}_0) \longrightarrow [0, 1].$$

The proof of Proposition 3.22 will be a consequence of the following two lemmas.

**Lemma 3.23.** The process $\tilde{W}$ is a $(\tilde{\mathcal{F}}_t)$-cylindrical Wiener process, i.e. there exists a collection of mutually independent real-valued $(\tilde{\mathcal{F}}_t)$-Wiener processes $\{\tilde{\beta}_k\}_{k \geq 1}$ such that $\tilde{W} = \sum_{k \geq 1} \tilde{\beta}_k e_k$.

**Proof.** By (3.76), $\tilde{W}$ is a $\mathcal{U}_0$-valued cylindrical Wiener process and is $(\tilde{\mathcal{F}}_t)$-adapted. According to the Lévy martingale characterization theorem, it remains to show that it is also a $(\tilde{\mathcal{F}}_t)$-martingale. We have
$$\mathbb{E} \gamma(\theta_s \tilde{U}, \theta_s \tilde{W}) \left[ \tilde{W}(t) - \tilde{W}(s) \right] = \mathbb{E} \gamma(\theta_s U, \theta_s W) \left[ W(t) - W(s) \right] = 0$$
since $W^\tau$ is a martingale and the laws of $(\tilde{U}^\tau, \tilde{W}^\tau)$ and $(U^\tau, W^\tau)$ coincide. Next, the uniform estimate
$$\sup_{t \in (0, 1)} \mathbb{E} \|\tilde{W}^\tau(t)\|_{\mathcal{U}_0}^2 = \sup_{t \in (0, 1)} \mathbb{E} \|W^\tau(t)\|_{\mathcal{U}_0}^2 < \infty$$
and the Vitali convergence theorem yield
$$\mathbb{E} \gamma(\theta_s \tilde{U}, \theta_s \tilde{W}) \left[ \tilde{W}(t) - \tilde{W}(s) \right] = 0$$
which finishes the proof. $\blacksquare$

**Lemma 3.24.** The processes
$$\tilde{M}, \tilde{M}^2 - \sum_{k \geq 1} \int_0^t \langle \psi_k(\tilde{U}), \varphi \rangle^2 \, dr, \quad \tilde{M}\tilde{\beta}_k - \int_0^t \langle \psi_k(\tilde{U}), \varphi \rangle \, dr \quad (3.78)$$
are $(\tilde{\mathcal{F}}_t)$-martingales.

**Proof.** Here, we use the same approach as in the previous lemma. Let us denote by $\tilde{X}^\tau_k, k \geq 1$ the real-valued Wiener processes corresponding to $\tilde{W}^\tau$, that is $\tilde{W}^\tau = \sum_{k \geq 1} \tilde{X}^\tau_k e_k$. For all $\tau \in (0, 1)$, the process
$$M^\tau = \sum_{k \geq 1} \int_0^t \langle \sigma_k^\tau(U^\tau), \varphi_2 \rangle \, dX^\tau_k(r)$$
is a square integrable $(\mathcal{F}_t)$-martingale and therefore
$$M^\tau_2 := (M^\tau)^2 - \sum_{k \geq 1} \int_0^t \langle \sigma_k^\tau(U^\tau), \varphi_2 \rangle^2 \, d\langle X^\tau \rangle(r),$$
and
\[ M_t^3 := M^\tau \beta_k - \int_0^t \langle \sigma_k(\tilde{U}^\tau), \varphi_2 \rangle \, d\langle X^\tau \rangle(r) \]
are \((\mathcal{F}_t)\)-martingales, where we have denoted by
\[ \langle \langle X^\tau \rangle \rangle(t) = 2 \int_0^t 1_{\text{sto}}(r) \, dr \]
the quadratic variation of \(X_t^\tau\). Besides, it follows from the equality of laws that
\[ \tilde{\mathbb{E}} \gamma(\varphi_s \tilde{U}^\tau, \varphi_s \tilde{W}^\tau) [\tilde{M}(t) - \tilde{M}(s)] = \mathbb{E} \gamma(\varphi_s U^\tau, \varphi_s W^\tau) [M^\tau(t) - M^\tau(s)]. \]

hence \( \tilde{\mathbb{E}} \gamma(\varphi_s \tilde{U}^\tau, \varphi_s \tilde{W}^\tau) [\tilde{M}(t) - \tilde{M}(s)] = 0 \). We can pass to the limit in this equation, due to the moment estimates (3.30) and the Vitali convergence theorem. We obtain
\[ \tilde{\mathbb{E}} \gamma(\varphi_s \tilde{U}^\tau, \varphi_s \tilde{W}^\tau) [\tilde{M}(t) - \tilde{M}(s)] = 0, \]
i.e. \( \tilde{M} \) is a \((\mathcal{F}_t)\)-martingale. We proceed similarly to show that
\[ \tilde{M}_2 := \tilde{M}^2 - \sum_{k \geq 1} \int_0^t \langle \psi_k(\tilde{U}), \varphi \rangle^2 \, dr \]
is a \((\mathcal{F}_t)\)-martingale, by passing to the limit in the identity
\[ \tilde{\mathbb{E}} \gamma(\varphi_s \tilde{U}^\tau, \varphi_s \tilde{W}^\tau) [\tilde{M}_2^\tau(t) - \tilde{M}_2^\tau(s)] = 0, \]
and again similarly for
\[ \tilde{M}_3 := \tilde{M} \beta_k - \int_0^t \langle \psi_k(\tilde{U}), \varphi \rangle \, dr. \]

\[ \blacksquare \]

**Proof of Proposition 3.22.** Once the above lemmas are established, we infer that
\[ \left\langle \tilde{M} - \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{U}) \, d\tilde{\beta}_k, \varphi_2 \rangle \right\rangle = 0, \]
where \( \langle \cdot \rangle \) denotes the quadratic variation process. Accordingly, we have
\[ \langle \tilde{U}(t), \varphi \rangle = \langle \tilde{U}_0, \varphi \rangle - \int_0^t \langle \partial_x A(\tilde{U}) - \partial_x^2 \tilde{U}, \varphi \rangle \, ds \]
\[ + \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{U}) \, d\tilde{\beta}_k, \varphi_2 \rangle, \quad t \in [0, T], \quad \tilde{\mathbb{P}}\text{-a.s.}, \]
and the proof is complete. \( \blacksquare \)
Proof of Theorem ??.

We have just obtained the existence of a martingale solution $$\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}, \tilde{U}^\varepsilon\right)$$ to (3.1). By passing to the limit on the approximation $$U^\varepsilon$$, taking care to the occurrence of the factor $$\varepsilon$$ now, we also obtain the various estimates of Theorem ?? and the Itô formula (3.6).

4 Probabilistic Young measures

Let $$\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, \mathbb{W}, \mathbb{U}^\varepsilon\right)$$ be a martingale solution to (3.1). Our aim is to prove the convergence of $$\mathbb{U}^\varepsilon$$. The standard tool for this is the notion of measure-valued solution (introduced by Di Perna, [DiP83a]). In this section we give some precisions about it in our context of random solutions. More precisely, we know that, almost surely, $$(\mathbb{U}^\varepsilon)$$ defines a Young measure $$\nu^\varepsilon$$ on $$\mathbb{R}_+ \times \mathbb{R}$$ by the formula

$$\langle \nu^\varepsilon_{x,t}, \varphi \rangle := \langle \delta_{\mathbb{U}^\varepsilon(t,x)}, \varphi \rangle = \varphi(\mathbb{U}^\varepsilon(t,x)), \quad \forall \varphi \in C^b(\mathbb{R}_+ \times \mathbb{R}). \tag{4.1}$$

Our aim is to show that $$\nu^\varepsilon \rightharpoonup \nu$$ (in the sense to be specified), where $$\nu$$ has some specific properties. To that purpose, we will use the probabilistic compensated compactness method developed in the Appendix of [FN08] and some results on the convergence of probabilistic Young measures that we introduce here.

4.1 Young measures embedded in a space of Probability measures

Let $$(Q, \mathcal{A}, \lambda)$$ be a finite measure space. Without loss of generality, we will assume $$\lambda(Q) = 1$$. A Young measure on $$Q$$ (with state space $$E$$) is a measurable map $$Q \rightarrow \mathcal{P}_1(E)$$, where $$E$$ is a topological space endowed with the $$\sigma$$-algebra of Borel sets, $$\mathcal{P}_1(E)$$ is the set of probability measures on $$E$$, itself endowed with the $$\sigma$$-algebra of Borel sets corresponding to the topology defined by the weak\(^3\) convergence of measures, i.e. $$\mu_n \rightharpoonup \mu$$ in $$\mathcal{P}_1(E)$$ if

$$\langle \mu_n, \varphi \rangle \rightarrow \langle \mu, \varphi \rangle, \quad \forall \varphi \in C_b(E).$$

As in (4.1), any measurable map $$w: Q \rightarrow E$$ can be viewed as a Young measure $$\nu$$ defined by

$$\langle \nu_z, \varphi \rangle = \langle \delta_{w(z)}, \varphi \rangle = \varphi(w(z)), \quad \forall \varphi \in C_b(E), \quad \forall z \in Q.$$

\(^3\)actually, weak convergence of probability measures, also corresponding to the tight convergence of finite measures
A Young measure \( \nu \) on \( Q \) can itself be seen as a probability measure on \( Q \times E \) defined by

\[
\langle \nu, \psi \rangle = \int_Q \int_E \psi(p,z) d\nu_z(p) d\lambda(z), \quad \forall \psi \in C_b(E \times Q),
\]

We then have, for all \( \psi \in C_b(Q) \) (\( \psi \) independent on \( p \in E \)), \( \langle \nu, \psi \rangle = \langle \lambda, \psi \rangle \), that is to say

\[
\pi_* \nu = \lambda, \quad (4.2)
\]

where \( \pi \) is the projection \( E \times Q \to Q \) and the push-forward of \( \nu \) by \( \pi \) is defined by \( \pi_* \nu(A) = \nu(\pi^{-1}(A)) \), for all Borel subset \( A \) of \( Q \). Assume now that \( Q \) is a subset of \( \mathbb{R}^s \) and \( E \) is a closed subset of \( \mathbb{R}^m \), \( m, s \in \mathbb{N}^* \), and, conversely, let \( \mu \) is a probability measure on \( E \times Q \) such that \( \pi_* \mu = \lambda \): by the Slicing Theorem (cf. Attouch, Buttazzo, Michaille [ABM06, Theorem 4.2.4]), we have: for \( \lambda \)-a.e. \( z \in Q \), there exists \( \mu_z \in \mathcal{P}_1(E) \) such that,

\[
z \mapsto \langle \mu_z, \varphi \rangle
\]

is measurable from \( Q \) to \( \mathbb{R} \) for every \( \varphi \in C_b(Q) \), and

\[
\langle \mu, \psi \rangle = \int_Q \int_E \psi(p,z) d\mu_z(p) d\lambda(z),
\]

for all \( \psi \in C_b(E \times Q) \), which precisely means that \( \mu \) is a Young measure on \( Q \). We therefore denote by

\[
\mathcal{Y} = \{ \nu \in \mathcal{P}_1(E \times Q); \pi_* \nu = \lambda \}
\]

the set of Young measures on \( Q \).

We use now the Prohorov’s Theorem, cf. Billingsley [Bil99], to give a compactness criteria in \( \mathcal{Y} \). We assume that \( Q \) is a compact subset of \( \mathbb{R}^s \) and \( E \) is a closed subset of \( \mathbb{R}^m \). We also assume that the \( \sigma \)-algebra \( \mathcal{A} \) of \( Q \) is the \( \sigma \)-algebra of Borel sets of \( Q \).

**Proposition 4.1** (Bound against a Lyapunov functional). Let \( \eta: E \to \mathbb{R}_+ \) satisfy the growth condition

\[
\lim_{p \in E, |p| \to +\infty} \eta(p) = +\infty.
\]

Let \( C > 0 \) be a positive constant. Then the set

\[
K_C = \left\{ \nu \in \mathcal{Y}; \int_{Q \times E} \eta(p) d\nu(z,p) \leq C \right\} \quad (4.3)
\]

is a compact subset of \( \mathcal{Y} \).
Proof. The condition \( \pi_* \nu = \lambda \) being stable by weak convergence, \( \mathcal{Y} \) is closed in \( \mathcal{P}_1(E \times Q) \). By Prohorov’s Theorem, [Bil99], \( K_C \) is relatively compact in \( \mathcal{Y} \) if, and only if it is tight. Besides, \( K_C \) is closed since

\[
\int_{Q \times E} \eta(p) d\nu(z, p) \leq \liminf_{n \to +\infty} \int_{Q \times E} \eta(p) d\nu_n(z, p)
\]

if \( (\nu_n) \) converges weakly to \( \nu \). It is therefore sufficient to prove that \( K_C \) is tight, which is classical: let \( \epsilon > 0 \). For \( R \geq 0 \), let

\[
V(R) = \inf_{|p| > R} \eta(p).
\]

Then \( V(R) \to +\infty \) as \( R \to +\infty \) by hypothesis and, setting \( M_R = \overline{B}(0, R) \cap E \times Q \), we have

\[
V(R) \nu(M^c_R) \leq \int_{Q \times E} \eta(p) d\nu(z, p) \leq C,
\]

for all \( \nu \in K \), whence \( \sup_{\nu \in K} \nu(M^c_R) < \epsilon \) for \( R \) large enough. \( \blacksquare \)

4.2 A compactness criterion for probabilistic Young measures

As above, we assume that \( Q \) is a compact subset of \( \mathbb{R}^s \) and \( E \) is a closed subset of \( \mathbb{R}^m \).

Definition 4.2. A random Young measure is a \( \mathcal{Y} \)-valued random variable.

Proposition 4.3. Let \( \eta: E \to \mathbb{R}_+ \) satisfy the growth condition

\[
\lim_{p \in E, |p| \to +\infty} \eta(p) = +\infty.
\]

Let \( M > 0 \) be a positive constant. If \( (\nu_n) \) is a sequence of random Young measures on \( Q \) satisfying the bound

\[
\mathbb{E} \int_{Q \times E} \eta(p) d\nu_n(z, p) \leq M,
\]

then, up to a subsequence, \( (\nu_n) \) is converging in law.

Proof. We endow \( \mathcal{P}_1(E \times Q) \) with the Prohorov’s metric \( d \). Then \( \mathcal{P}_1(E \times Q) \) is a complete, separable metric space, weak convergence coincides with \( d \)-convergence, and a subset \( A \) is relatively compact if, and only if it is tight, [Bil99]. Let \( \mathcal{L}(\nu_n) \in \mathcal{P}_1(\mathcal{Y}) \) denote the law of \( \nu_n \). To prove that it is tight, we use the Prohorov’s Theorem. Let \( \epsilon > 0 \). For \( C > 0 \), let \( K_C \) be the compact set defined by (4.3). For \( \nu \in \mathcal{Y} \), we have

\[
\mathbb{P}(\nu \notin K_C) = \mathbb{P} \left( 1 < \frac{1}{C} \int_{Q \times E} \eta(p) d\nu(z, p) \right) \leq \frac{1}{C} \mathbb{E} \int_{Q \times E} \eta(p) d\nu(z, p),
\]
hence
\[ \sup_{n \in \mathbb{N}} \mathcal{L}(\nu_n)(\mathcal{Y} \setminus K_C) = \sup_{n \in \mathbb{N}} \mathbb{P}(\nu_n \notin K_C) \leq \frac{M}{C} < \varepsilon, \]
for \( C \) large enough, which proves the result. \( \blacksquare \)

Let \( \mathcal{X}_W = C([0, T]; \mathcal{U}_0) \) be the path space for the Wiener process \( W \). If \( (\nu_n) \) is as in Proposition 4.3, then the law of \( (\nu_n, W) \) is tight in \( \mathcal{Y} \times \mathcal{X}_W \) since the law of \( W \) is tight in \( \mathcal{X}_W \). We obtain the following extension:

**Proposition 4.4.** Under the hypothesis of Proposition 4.3, there exists a subsequence of \( (\nu_n, W) \) converging in law on \( \mathcal{Y} \times \mathcal{X}_W \).

**Remark 4.5** (Almost-sure convergence). Assume that \( (\nu_n, W) \) is converging in law on \( \mathcal{Y} \times \mathcal{X}_W \). Then we can apply the Skorokhod Theorem [Bil99] to \( (\nu_n, W) \): there exists a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \), some random Young measures \( \tilde{\nu}_n, \tilde{\nu} : \tilde{\Omega} \rightarrow \mathcal{Y}, \) some random variable \( \tilde{W} : \tilde{\Omega} \rightarrow \mathcal{X}_W \), such that \( \text{Law}(\tilde{\nu}_n) = \text{Law}(\nu_n), \text{Law}(\tilde{\nu}) = \text{Law}(\nu), \text{Law}(\tilde{W}) = \text{Law}(W) \) and \( \tilde{\nu}_n \rightarrow \tilde{\nu} \) a.s. in \( \mathcal{P}_1(E \times Q) \). Let \( q > 1 \). If \( \nu_n \) is issued from a sequence of random variables bounded in \( L^q(Q; E) \) (as this will be the case in our context), say

\[ \nu_n = \delta_u, \quad u_n \in L^q(Q; E), \quad (4.4) \]

then \( \tilde{\nu}_n \) is itself a Dirac mass.

The proof of the above statement is as follows: using the Jensen’s Formula, \( (4.4) \) can be characterized by

\[ \mathbb{E} \int_{Q \times E} \psi(p) d\nu_{n,z}(p) d\lambda(z) = \mathbb{E} \int_Q \psi \left( \int_E pd\nu_{n,z}(p) \right) d\lambda(z), \]

where \( \psi : \mathbb{R}^m \rightarrow \mathbb{R} \) is a strictly convex function satisfying the growth condition \( |\psi(p)| \leq C(1 + |p|^q) \). In other words,

\[ \mathbb{E}\varphi(\nu_n) = \mathbb{E}\theta(\nu_n), \quad (4.5) \]

where

\[ \varphi : \mu \mapsto \int_{Q \times E} \psi(p) d\mu_z(p) d\lambda(z), \quad \theta : \mu \mapsto \int_Q \psi \left( \int_E pd\mu_z(p) \right) d\lambda(z). \]

Of course, \( \varphi \) is continuous on \( \mathcal{Y} \) and, by the Lebesgue dominated convergence theorem, \( \theta \) is continuous on the subset

\[ \left\{ \mu \in \mathcal{Y}; \int_{Q \times E} |p|^q d\mu_z(p) d\lambda(z) \leq R \right\}. \]

Consequently, by identity of the laws, \( (4.5) \) is satisfied by \( \tilde{\nu}_n \), i.e.

\[ \tilde{\nu}_n = \delta_{\tilde{u}_n}, \quad \tilde{u}_n(z) := \int_E pd\tilde{\nu}_{n,z} \]

almost surely.
4.3 Convergence to a random Young measure

We apply the results of paragraphs 4.1-4.2 to the case $\nu^\varepsilon := \delta_{U^\varepsilon}$, $Q = Q_T$, $E = \mathbb{R}_+ \times \mathbb{R}$. Strictly speaking we will consider a sequence $(\varepsilon_n)$ decreasing to 0 and will obtain some limits along some subsequences of $(\varepsilon_n)$. However, for the simplicity of notations we will keep the notation $\nu^\varepsilon$ instead of $\nu^\varepsilon_n$ in what follows.

Proposition 4.6. Let $U^\varepsilon_0 \in W^{1,\infty}(T)$ be bounded in $L^{\infty}(T)$. Let $U^\varepsilon$ be a martingale solution to (3.1) satisfying (??) and let $\nu^\varepsilon := \delta_{U^\varepsilon}$ be the associated random Young measure. Then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, some random variables $(\tilde{\nu}^\varepsilon, \tilde{W})$ and $(\tilde{\nu}, \tilde{W})$ with values in $Y \times X$ such that

1. the law of $(\tilde{\nu}^\varepsilon, \tilde{W})$ under $\tilde{\mathbb{P}}$ coincide with the law of $(\nu^\varepsilon, W)$,
2. $(\tilde{\nu}^\varepsilon, \tilde{W})$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{\nu}, \tilde{W})$ in the topology of $Y \times X$.

Proof. We apply the Proposition 4.4 to $(\nu^\varepsilon, W)$ taking for $\eta$ the energy, i.e. the entropy given by (2.7) with $g(\xi) = |\xi|^2$ (which is infinite at infinity by (3.22)). We use the estimate (??) for such an $\eta$. ■

5 Reduction of the Young measure

Proposition 4.6 above gives the existence of a random young measure $\nu^\varepsilon$ such that $\nu^\varepsilon := \delta_{U^\varepsilon}$ converges in law in the sense of Young measures to $\nu$. We will now apply the compensated compactness method to prove that a.s., for a.e. $(x, t) \in Q_T$, either $\nu^\varepsilon_{x,t}$ is a Dirac mass or $\nu^\varepsilon_{x,t}$ is concentrated on the vacuum region $\{\rho = 0\}$. To do this, we will use the probabilistic compensated compactness method of [FN08] to obtain a set of functional equations satisfied by $\nu$. Then we conclude by adapting the arguments of [LPS96].

5.1 Compensated compactness

In this section, we fix an entropy - entropy flux couple $(\eta, H)$ of the form (2.7)-(2.9) where $g \in C^2(\mathbb{R})$ is convex, with $g$ sub-quadratic and $g'$ sub-linear:

$$|g(\xi)| \leq C(1 + |\xi|^2), \quad |g'(\xi)| \leq C(1 + |\xi|), \quad (5.1)$$

for all $\xi \in \mathbb{R}$, for a given $C \geq 0$.

5.1.1 Murat’s Lemma

For $p \in [1, +\infty]$, we denote by $W^{1,p}_0(Q_T)$ the set of function $u$ in the Sobolev space $W^{1,p}(Q_T)$ such that $u = 0$ on $\mathbb{T} \times \{0\}$ and $\mathbb{T} \times \{T\}$. We denote by $W^{-1,p}(Q_T)$ the dual of $W^{1,p}_0(Q_T)$, where $p'$ is the conjugate exponent to $p$. 

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Proposition 5.1. Let $q \in (1, 2)$. Let $\eta$ be an entropy of the form (2.7) with $g$ convex satisfying (5.1). Let $U^\varepsilon_0 \in L^\infty(\mathbb{T})$. Then the sequence of random variables $(\varepsilon \eta(U^\varepsilon))_x > 0$ is tight in $W^{-1,q}(Q_T)$.

Proof. Let $\tilde{q} \in (q, 2)$. Let $m > 1$, 

\[ m = \max \left( 1, \frac{2(2 - \gamma)(2 - q)}{(\gamma - 1)q}, \frac{2\tilde{q} - q}{\tilde{q} - 1} \right). \]

Let $R > 0$. Assume that $\omega$ is fixed such that 

\[ X_1^\varepsilon := \|\rho^\varepsilon\|^{\frac{2}{\gamma - 2}}_{L^\infty(Q_T)} \leq R, \quad X_2^\varepsilon := \|u^\varepsilon\|_{L^m(Q_T)} \leq R, \quad (5.2) \]

and 

\[ X_3^\varepsilon := \varepsilon \int_{Q_T} \left\{ |\rho^\varepsilon|^{\gamma - 2} \left[ |\rho^\varepsilon|^\gamma + |u^\varepsilon|^4 \right] |\partial_x \rho^\varepsilon|^2 + \rho^\varepsilon \left[ 1 + |\rho^\varepsilon|^{\max(2, \gamma)} + |u^\varepsilon|^4 \right] |\partial_x u^\varepsilon|^2 \right\} dx dt \leq R. \quad (5.3) \]

Then we have the convergence 

\[ \lim_{\varepsilon \to 0} \varepsilon \eta(U^\varepsilon)_x = 0 \text{ in } L^q(Q_T), \quad (5.4) \]

which we admit for the moment. This convergence (5.4) implies in particular that $(\varepsilon \eta(U^\varepsilon))_x$ is in a compact $K_R$ of $W^{-1,q}(Q_T)$. Consequently, we have 

\[ \mathbb{P}(\varepsilon \eta(U^\varepsilon)_x \notin K_R) \leq \sum_{i=1}^3 \mathbb{P}(X_i^\varepsilon > R). \]

By the estimates (3.3), (3.4), (3.5) and the Markov Inequality, we deduce that, given $\eta > 0$, we may choose $R > 0$ such that 

\[ \mathbb{P}(\varepsilon \eta(U^\varepsilon)_x \notin K_R) < \eta, \]

uniformly in $\varepsilon$. This shows that $(\varepsilon \eta(U^\varepsilon)_x)$ is tight in $W^{-1,q}(Q_T)$. The proof of (5.4) is similar to the analysis in [LPS96], pp 627-629, with the minor difference that we have here only $L^r$ estimates with $r$ arbitrary large on the unknown, instead of the $L^\infty$ estimates of the deterministic case. We note first that, by (5.1), we have 

\[ |\partial_\rho \eta(U)| \leq C \left( 1 + |u|^2 + \rho^{(\gamma - 1)} \right), \]

and 

\[ |\partial_u \eta(U)| \leq C \rho \left( 1 + |u| + \rho^{\frac{\gamma - 1}{2}} \right), \]

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for a given non-negative constant that we still denote by $C$. By the Young Inequality, it follows that

$$
|\eta(U^s)_x|^q \leq C \left( 1 + |u^s|^{2q} + |\rho^s|^q(\gamma - 1) \right) |\partial_x \rho^s|^q \\
+ C|\rho^s|^q \left( 1 + |u^s|^q + |\rho^s|^q(\gamma - 1) \right) |\partial_u u^s|^q \\
\leq C + C \left( 1 + |u^s|^{2q} \right) |\partial_x \rho^s|^q + C|\rho^s|^{2(\gamma - 1)}|\partial_x \rho^s|^2 \\
C + C \rho^s \left( 1 + |\rho^s|^{\max(2, \gamma)} \right) + |u^s|^4 |\partial_x u^s|^2,
$$

where $C$ depends on $q$. By (5.2), (5.3), we obtain

$$
\varepsilon^q \int_Q |\eta(U^s)_x|^q \, dx \, dt \leq C_R \varepsilon^{q-1} + C \varepsilon^q \int_Q \left( 1 + |u^s|^{2q} \right) |\partial_x \rho^s|^q \, dx \, dt,
$$

where the constant $C_R$ depends on $R$. If $\gamma \leq 2$, we have furthermore

$$
(1 + |u^s|^{2q}) |\partial_x \rho^s|^q = |\rho^s|^{\frac{q}{2}(2 - \gamma)} \left( 1 + |u^s|^{2q} \right) |\rho^s|^{\frac{q}{2}(\gamma - 2)}|\partial_x \rho^s|^q \\
\leq C|\rho^s|^{\frac{2q}{q-2}(2 - \gamma)} + C \left( 1 + |u^s|^q \right) |\rho^s|^{(\gamma - 2)}|\partial_x \rho^s|^2.
$$

By (5.2), (5.3), we obtain

$$
\varepsilon^q \int_Q |\eta(U^s)_x|^q \, dx \, dt \leq C_R \varepsilon^{q-1}.
$$

If $\gamma > 2$, we decompose

$$
(1 + |u^s|^{2q}) |\partial_x \rho^s|^q = (1 + |u^s|^{2q}) |\partial_x \rho^s|^q 1_{\rho^s > \delta} + (1 + |u^s|^{2q}) |\partial_x \rho^s|^q 1_{\rho^s \leq \delta}
$$

The first term is bounded as follows:

$$
(1 + |u^s|^{2q}) |\partial_x \rho^s|^q 1_{\rho^s > \delta} \leq C \delta^{-\frac{q}{2}(\gamma - 2)} \left( 1 + (1 + |u^s|^{q}) |\rho^s|^{(\gamma - 2)}|\partial_x \rho^s|^2 \right).
$$

In particular, by (5.2) and (5.3), we have

$$
\varepsilon^q \int_Q \left( 1 + |u^s|^{2q} \right) |\partial_x \rho^s|^q 1_{\rho^s > \delta} \leq C_R \delta^{-\frac{q}{2}(\gamma - 2)} \varepsilon^{q-1}.
$$

To estimate the part $\{\rho^s \leq \delta\}$, we first use the Hölder Inequality and the estimate (5.2) to obtain

$$
\varepsilon^q \int_Q \left( 1 + |u^s|^{2q} \right) |\partial_x \rho^s|^q 1_{\rho^s \leq \delta} \\
\leq \left( \varepsilon^q \int_Q \left( 1 + |u^s|^{2q} \right)^{\frac{q}{q'-\gamma}} \right)^{\frac{q'-\gamma}{q'}} \left( \varepsilon^q \int_Q |\partial_x \rho^s|^q 1_{\rho^s \leq \delta} \right)^{\frac{q}{q'}} \\
\leq C_R \left( \varepsilon^q \int_Q |\partial_x \rho^s|^q 1_{\rho^s \leq \delta} \right)^{\frac{q}{q'}}.
$$

(5.8)
Then, we multiply the first Equation of the system (3.1a), the equation
\[ \partial_t \rho^\varepsilon + \partial_x (\rho^\varepsilon u^\varepsilon) = \varepsilon \partial_x^2 \rho^\varepsilon, \]
by \( \min(\rho^\varepsilon, \delta) \), and then sum the result over \( Q_T \). This gives
\[ \varepsilon \int_{Q_T} |\partial_x \rho^\varepsilon|^2 1_{\rho^\varepsilon \leq \delta} \leq C \delta^2 + C \left( \int_{Q_T} \rho^\varepsilon ||\partial_x \rho^\varepsilon|| 1_{\rho^\varepsilon \leq \delta} \right) \]
\[ \leq C \delta^2 + C \delta \|u^\varepsilon\|_{L^\frac{4}{3}(Q_T)} \left( \int_{Q_T} |\partial_x \rho^\varepsilon|^2 1_{\rho^\varepsilon \leq \delta} \right)^{\frac{1}{2}} \]
\[ \leq C \delta^2 + C R \delta \left( \int_{Q_T} |\partial_x \rho^\varepsilon|^2 1_{\rho^\varepsilon \leq \delta} \right)^{\frac{1}{2}}, \]
from which we deduce
\[ \left( \varepsilon^{\frac{3}{2}} \int_{Q_T} |\partial_x \rho^\varepsilon|^2 1_{\rho^\varepsilon \leq \delta} \right)^{\frac{3}{2}} \leq C R \delta^q. \]
Reporting this result in (5.8) and using (5.7) and (5.5), and also recalling the estimate (5.6), we obtain
\[ \varepsilon^{q} \int_{Q_T} |\eta(U^\varepsilon)|_x^q dx dt \leq C R \left( \varepsilon^{q-1} + \delta^{-\frac{2}{\gamma}} (\gamma-2)^{\varepsilon^{q-1}} + \delta^q \right), \]
whatever the value of \( \gamma > 1 \). This concludes the proof of (5.4) and of Proposition 5.1. ⊓⊔

The next Proposition is similar to Lemma 4.20 in [FN08].

Proposition 5.2. Let \( \eta \) be an entropy of the form (2.7) with \( g \) convex satisfying (5.1). Let \( U_0^\varepsilon \in L^\infty(T) \). Let
\[ M^\varepsilon(t) = \int_0^t \partial_q \eta(U^\varepsilon)(s) \Phi(U^\varepsilon)(s) dW(s). \]
Then \( \partial_t M^\varepsilon \) is tight in \( H^{-1}(Q_T) \).

Proof. The proof is in essential the proof of Lemma 4.19 in [FN08]. However, we will proceed slightly differently (instead of using Marchaud fractional derivative we work directly with fractional Sobolev spaces and an Aubin-Simon compactness lemma). Let \( m \geq 2 \). Let \( 0 \leq s \leq t \leq T \). In what follows we denote by \( C \) any constant, that may vary from line to line, which depends on the data only and independent on \( \varepsilon \). By the Burholder-Davis-Gundy Inequality, we have
\[ \mathbb{E} \|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^m(T)}^m \leq C \int_T \mathbb{E} \left| \int_s^t |\partial_q \eta(U^\varepsilon)|^2 G^2(U^\varepsilon) d\sigma \right|^{m/2} dx, \]
46
and, using the Hölder Inequality,
\[ \mathbb{E}\|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^m(T)} \leq C|t-s|^{\frac{m}{2}} \int_s^t \mathbb{E}\int_T \left| \partial_q \eta(U^\varepsilon) \right|^2 G^2(U^\varepsilon) \frac{d\sigma dx}{|t-s|^{1+\nu m}}. \]

By (2.2) and (5.1), we have \( |\partial_q \eta(U^\varepsilon)|^2 G^2(U^\varepsilon) \leq R(\rho^\varepsilon, |u^\varepsilon|) \), where \( R \) is a polynomial. In particular, by (3.3), we obtain
\[ \mathbb{E}\|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^m(T)} \leq C|t-s|^{\frac{m}{2}}. \]

and, by integration with respect to \( t \) and \( s \),
\[ \mathbb{E}\int_0^T \int_0^T \frac{\|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^m(T)}^m}{|t-s|^{1+\nu m}} \, dt \, ds \leq C, \quad (5.9) \]
as soon as \( \nu < 1/2 \). The left-hand side in this inequality (5.9) is the norm of \( M^\varepsilon \) in \( L^m(T) \). Since \( L^m(T) \hookrightarrow H^{-1}(T) \), it follows that
\[ \mathbb{E}\|M^\varepsilon\|_{W^{\nu,m}(0,T;H^{-1}(T))} \leq C. \quad (5.10) \]

Besides, by the growth inequalities (2.2) and the \( L^p \) estimate (3.3), \( M^\varepsilon \) also satisfies the bound
\[ \mathbb{E}\|M^\varepsilon\|_{L^2(Q_T)} \leq C. \quad (5.11) \]

Let us assume \( m > 2 \) now. We then have the continuous injection
\[ W^{\nu,m}(0,T;H^{-1}(T)) \hookrightarrow C^{0,\mu}([0,T];H^{-1}(T)) \]
for every \( 0 < \mu < \nu - \frac{1}{m} \). By the Aubin-Simon compactness Lemma (Simon, [Sim87]), the set
\[ A_R := \{ M \in L^2(Q_T); \|M^\varepsilon\|_{W^{\nu,m}(0,T;H^{-1}(T))} \leq R, \|M\|_{L^2(Q_T)} \leq R \} \]
is compact in \( C([0,T];H^{-1}(T)) \), hence compact in \( L^2(0,T;H^{-1}(T)) \). Consequently (5.10) and (5.11) show that \( (M^\varepsilon) \) is tight as a \( L^2(0,T;H^{-1}(T)) \)-random variable, and we conclude that \( (\partial_t M^\varepsilon) \) is tight as a \( H^{-1}(Q_T) \)-random variable. \( \blacksquare \)

5.1.2 Functional equation

Let us recall that a sequence of random variables \( (a_n) \) with values in a normed space \( X \) is said to be stochastically bounded if, for all \( \eta > 0 \), there exists \( M > 0 \) such that \( \mathbb{P}(\|a_n\|_X \geq M) \leq \eta \) for all \( n \). By the Markov inequality, \( \mathbb{E}\|a_n\|_X \leq C \) implies \( (a_n) \) stochastically bounded.

We consider the Itô Formula (3.6). Let \( p > 2 \). By the \( L^p \) estimates (3.3), which gives bounds on \( \eta(U^\varepsilon), H(U^\varepsilon) \) in \( L^p(Q_T) \), the left-hand side of (3.6) is stochastically bounded in \( W^{-1,p}(Q_T) \). By Proposition 5.2 above, the term
∂tMε in the right-hand side of (3.6) is tight in $H^{-1}(Q_T)$. The two remaining terms

$$
\varepsilon \eta''(U^\varepsilon) \cdot (U^\varepsilon_x, U^\varepsilon_x) \quad \text{and} \quad \frac{1}{2}|G(U^\varepsilon)|^2 \partial^2_{g_\varepsilon} \eta(U^\varepsilon)
$$

are stochastically bounded in measure on $Q_T$ by (3.4)-(3.5) and (2.2)-(3.3) respectively. Eventually, by Proposition 5.1, the term $\varepsilon(\eta(U^\varepsilon))_{xx}$ is tight in $H^{-1}(Q_T)$ for any $q \in (1, 2)$. We want now to apply the stochastic version of the Murat’s Lemma, Lemma A.3 in [FN08]. If we refer strictly to the statement of Lemma A.3 in [FN08], there is an obstacle here, due to the fact that $\varepsilon(\eta(U^\varepsilon))_{xx}$ is neither tight in $H^{-1}(Q_T)$, neither stochastically bounded in measure on $Q_T$. However, in the proof of Lemma A.3 in [FN08], the property which is used regarding the term that is stochastically bounded in measure on $Q_T$ is only the fact that it is tight in $W^{-1,q}(Q_T)$ for $1 < q < 2$. The argument about interpolation theory which combines this compactness result with the stochastic bound in $W^{-1,p}(Q_T)$ can therefore be directly applied here: we deduce that the sequence of $H^{-1}(Q_T)$ random variables

$$
\text{div}_{t,x}(\eta(U^\varepsilon), H(U^\varepsilon)) = \eta(U^\varepsilon)_t + H(U^\varepsilon)_x
$$

is tight. Let now $(\tilde{\eta}, \tilde{H})$ be a second entropy - entropy flux couple associated by (2.7)-(2.9) to a convex $C^2$ function $\tilde{g}$ satisfying (5.1). Similarly, the sequence of random variables

$$
\text{curl}_{t,x}(-\tilde{H}(U^\varepsilon), \tilde{\eta}(U^\varepsilon)) = \tilde{\eta}(U^\varepsilon)_t + \tilde{H}(U^\varepsilon)_x
$$

is tight. By Theorem A.2 in [FN08] and Proposition 4.6, we obtain, for a.e. $(x,t) \in Q_T$,

$$
\langle \tilde{\eta} \rangle \langle H \rangle - \langle \eta \rangle \langle \tilde{H} \rangle = \langle \tilde{\eta} H - \eta \tilde{H} \rangle,
$$

(5.12)

where $\langle \phi \rangle := \langle \phi, \nu_{x,t} \rangle$ and where the identity in (5.12) is the identity between the laws of the processes involved. To obtain an almost sure equality, we apply Skorokhod Theorem: by Remark 4.5, there exists a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a new $C([0,T];W^{1,2}(\mathbb{T}))$ random variable $\tilde{U}^\varepsilon$, $\tilde{\nu}$ a random Young measure, such that $\text{Law}(\delta_{\tilde{U}^\varepsilon}) = \text{Law}(\delta_{U^\varepsilon})$, $\text{Law}(\nu) = \text{Law}(\tilde{\nu})$ and, a.s.,

$$
\delta_{\tilde{U}^\varepsilon} \rightarrow \tilde{\nu}.
$$

Then, by a slight adaptation of the probabilistic div-curl Lemma, Theorem 2 in [FN08], we obtain (5.12) almost everywhere in $\tilde{\Omega}$, and with $\langle \phi \rangle := \langle \phi, \tilde{\nu}_{x,t} \rangle$.

### 5.2 Reduction of the Young measure

We now follow [LPS96] to conclude. We switch from the variables $(\rho, u)$ or $(\rho, q)$ to $(w, z)$, where

$$
z = u - \rho^\theta, \quad w = u + \rho^\theta.
$$
By (5.12), we have, a.s., for a.e. \((x, t) \in Q_T,\)
\[
2\lambda \left( \langle \chi(v)w \rangle \langle \chi(v') \rangle - \langle \chi(v) \rangle \langle \chi(v')w \rangle \right) = (v - v') \left( \langle \chi(v) \chi(v') \rangle - \langle \chi(v) \rangle \langle \chi(v') \rangle \right),
\]
where
\[
\langle \chi(v) \rangle = \int \chi(w, z, v) d\tilde{\nu}_{x,t}(w, z), \quad \chi(w, z, v) := (v - z)^{\lambda^+}_+(w - v)^{\lambda^-}.
\]
Let us fix \((\omega, x, t)\) such that (5.13) is satisfied. Let \(C\) denote the set \(\{v \in \mathbb{R}; \langle \chi(v) \rangle > 0\} = \bigcup_{(w, z) \in \text{supp} \tilde{\nu}_{x,t}, (w, z) \in C} \{v; z < v < w\} \cup \{v, \rho = 0\}\). Denote \(\nu \) the vacuum region. If \(C\) is empty, then \(\tilde{\nu}_{x,t}\) is concentrated on \(V\). Assume \(C\) not empty. By Lemma I.2 in [LPS96] then, \(C\) is an open interval in \(\mathbb{R}\), say \(C = ]a, b[\), where \(-\infty \leq a < b \leq +\infty\) (we use here the french notation for open intervals to avoid the confusion with the point \((a, b)\) of \(\mathbb{R}^2\)). Furthermore all the computations of [LPS96] apply here, and thus, as in Section I.6 of [LPS96], we obtain
\[
\langle \rho^{2\lambda}\langle \chi \circ \pi_i \rangle \phi \circ \pi_i \rangle = 0,
\]
for any continuous function \(\phi\) with compact support in \(C\), where \(\pi_i : \mathbb{R}^2 \to \mathbb{R}\) denote the projection on the first coordinate \(w\) if \(i = 1\), and the projection on the second coordinate \(z\) if \(i = 2\). If we assume that there exists \(Q \in \mathbb{R}^2\) satisfying
\[
Q \in \text{supp}(\tilde{\nu}_{x,t}) \setminus V, \quad \pi_i(Q) \in C, \tag{5.15}
\]
for \(i \in \{1, 2\}\), then there exists a neighbourhood \(K\) of \(Q\) such that \(K \cap V = \emptyset, \nu_{x,t}(K) > 0, \pi_i(K) \subset C\). But then \(\langle \chi \circ \pi_i \rangle > 0\) on \(K, \rho > 0\) on \(K\) and, choosing a continuous function \(\phi\) compactly supported in \(C\) such that \(\phi > 0\) on \(K\) we obtain a contradiction to (5.14). Consequently (5.15) cannot be satisfied. This implies that there cannot exists two distinct points \(P, Q\) in \(\text{supp}(\tilde{\nu}_{x,t}) \setminus V\). Indeed, if two such points exists, then either \(\pi_1(Q) < \pi_1(P)\), and then \(Q\) satisfies (5.15) with \(i = 1\), or \(\pi_1(Q) = \pi_1(P)\) and, say, \(\pi_2(P) < \pi_2(Q)\) and then \(Q\) also satisfies (5.15). The other cases are similar by symmetry of \(P\) and \(Q\).
Therefore if \(C \neq \emptyset\), then the support of the restriction of \(\tilde{\nu}_{x,t}\) to \(C\) is reduced to a point. In particular, \(a\) and \(b\) are finite. Then, by Lemma I.2 in [LPS96], \(P := (a, b) \in \text{supp}(\nu_{x,t})\) and \(\tilde{\nu}_{x,t} = \tilde{\mu}_{x,t} + \alpha \delta_{U(t,x)}\), where \(\tilde{\mu}_{x,t} = \tilde{\nu}_{x,t}|V\). Using (5.13), we obtain
\[
0 = (v - v')\chi(b, a, v)\chi(b, a, v')(\alpha - \alpha^2),
\]
and thus \(\alpha = 0\) or \(1\). We have therefore proved the following result.
Proposition 5.3 (Reduction of the Young measure). Either \( \tilde{\nu}_{x,t} \) is concentrated on the vacuum region \( V \), or \( \tilde{\nu}_{x,t} \) is reduced to a Dirac mass \( \delta_{\tilde{U}(t,x)} \).

5.3 Martingale solution

By Proposition 5.3, we have

\[
\int_Q \int_S \langle \tilde{U}_\varepsilon(t,x) \rangle \varphi(t,x) dx dt \to \int_Q \int_S \langle \tilde{U}(t,x) \rangle \varphi(t,x) dx dt \quad (5.16)
\]

almost surely for every continuous and bounded function \( S \) on \( \mathbb{R}_+ \times \mathbb{R} \) which vanishes on the vacuum region \( \{0\} \times \mathbb{R} \) and every \( \varphi \in L^1(Q_T) \). There is also strong convergence of \( S(\tilde{U}_\varepsilon) \) to \( S(\tilde{U}) \) in \( L^2(Q_T) \), almost-surely. This can be deduced from (5.16), which gives directly the weak convergence in \( L^2(Q_T) \) by taking \( \varphi \in L^2(Q_T) \), but also the convergence of the norms by taking \( S^2 \) instead of \( S \) and \( \varphi \equiv 1 \).

For the moment we have only supposed that \( U_\varepsilon^0 \in W^{1,\infty}(T) \) and that \( (U_\varepsilon^0) \) is bounded in \( L^\infty(T) \). Assume furthermore

\[
U_0^\varepsilon \to U_0 \text{ in } L^1(T). \quad (5.17)
\]

The entropy and entropy flux as defined in (2.10), (2.11) are examples of functions \( S \) as above: we can therefore pass to the limit in (3.6). We will proceed as follows (along the lines of the proof of Theorem 3.4.12 in [Hof13b]): let \( \varphi \in C^1(T) \) be fixed. Since \( \eta(\tilde{U}_\varepsilon) \to \eta(\tilde{U}) \) in \( L^1(\Omega \times Q_T) \) we have, for possibly a further subsequence,

\[
\langle \eta(\tilde{U}_\varepsilon), \varphi \rangle(t) \to \langle \eta(\tilde{U}), \varphi \rangle(t), \quad \forall t \in D, \quad \text{almost surely,} \quad (5.18)
\]

where \( D \) is a set of full measure in \([0,T]\) which contains the point \( t = 0 \). Set

\[
e_\varepsilon = \varepsilon \eta''(\tilde{U}_\varepsilon) \cdot (U^\varepsilon_x, U^\varepsilon_x) \quad \tilde{e}_\varepsilon = \varepsilon \eta''(\tilde{U}_\varepsilon) \cdot (\tilde{U}^\varepsilon_x, \tilde{U}^\varepsilon_x).
\]

Let \( \mathcal{M}_b(Q_T) \) denote the set of bounded Borel measures on \( Q_T \) and \( \mathcal{M}_b^+(Q_T) \) denote the subset of nonnegative measures. By Remark ??, there exists a random variable

\[
\tilde{e} \in L^2_m(\tilde{\Omega}; \mathcal{M}_b(Q_T)), \quad \tilde{e} \in \mathcal{M}_b^+(Q_T) \quad \tilde{P} \text{-almost surely,}
\]

such that, for all \( \psi \in C(Q_T) \), for all \( Y \in L^2(\tilde{\Omega}) \), and up to a subsequence,

\[
\tilde{E}(\langle \tilde{e}_\varepsilon, \psi \rangle Y) \to \tilde{E}(\langle \tilde{e}, \psi \rangle Y). \quad (5.19)
\]

In (5.19), \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( \mathcal{M}_b(Q_T) \) and \( C(\tilde{Q}_T) \) (which we extend below in the right-hand side of (5.20) to the pairing between \( \mathcal{M}_b(Q_T) \) and \( \mathcal{B}_b(Q_T) \), the set of Borel bounded functions over \( Q_T \).
The subscript \( w \) in \( L^2_w(\tilde{\Omega}; \mathcal{M}_b(Q_T)) \) indicates that we consider weak-star measurable mappings \( e \) from \( \tilde{\Omega} \) into \( \mathcal{M}_b(Q_T) \), i.e. maps \( e \) such that \( \langle e, \psi \rangle \) is \( \tilde{\mathcal{F}} \)-measurable for every \( \psi \in C(\tilde{Q}_T) \). Then \( L^2_w(\tilde{\Omega}; \mathcal{M}_b(Q_T)) \) is the dual of the space \( L^2(\tilde{\Omega}; C(\tilde{Q}_T)) \) (Edwards [Edw65, Theorem 8.20.3]), hence (5.19) follows from the Banach-Alaoglu Theorem and from the estimate (??) with \( I = (0, T) \). Besides (??) with arbitrary \( I \) implies that, almost surely, \( \tilde{e} \) has no atom in time, i.e.

\[ \mathbb{E} \tilde{e}(\mathcal{T} \times \{ t \}) = 0, \quad \forall t \in [0, T]. \]

This shows in particular that (5.19) holds true when \( \psi = 1_{[0,t]} \varphi \) where \( \varphi \in C(\mathcal{T}) \): for all \( Y \in L^2(\tilde{\Omega}) \),

\[ \mathbb{E} \left( \int_0^t \langle \tilde{e}^\varepsilon, \varphi \rangle ds Y \right) \to \mathbb{E} \left( \langle \tilde{e}, 1_{[0,t]} \varphi \rangle Y \right). \]  
(5.20)

Let now \((\tilde{\mathcal{F}}_t)\) be the \( \tilde{\mathbb{P}} \)-augmented canonical filtration of the process \((\tilde{U}, \tilde{W})\), i.e.

\[ \tilde{\mathcal{F}}_t = \sigma(\sigma(\varrho_t \tilde{U}, \varrho_t \tilde{W}) \cup \{ N \in \tilde{\mathcal{F}}; \tilde{\mathbb{P}}(N) = 0 \}), \quad t \in [0, T], \]

where the restriction operator \( \varrho_t \) is defined in (3.77). Then the sextuplet

\[ (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}, \tilde{U}) \]

is a weak martingale solution to (1.1). To show this, we use a reasoning analogous to the one followed in Section 3.1.8. Let \( \varphi \in C^1(\mathcal{T}) \). Define

\[ M^\varepsilon(t) = \langle \eta(U^\varepsilon)(t), \varphi \rangle - \langle \eta(U_0^\varepsilon), \varphi \rangle + \int_0^t \langle \partial_x H(U^\varepsilon) - \varepsilon \partial^2_x \eta(U^\varepsilon), \varphi \rangle ds 
- \int_0^t \langle e^\varepsilon(s), \varphi \rangle ds, \]

\[ \tilde{M}^\varepsilon(t) = \langle \eta(\tilde{U}^\varepsilon)(t), \varphi \rangle - \langle \eta(U_0^\varepsilon), \varphi \rangle + \int_0^t \langle \partial_x H(\tilde{U}^\varepsilon) - \varepsilon \partial^2_x \eta(\tilde{U}^\varepsilon), \varphi \rangle ds 
- \int_0^t \langle \tilde{e}^\varepsilon(s), \varphi \rangle ds, \]

\[ \tilde{M}(t) = \langle \eta(\tilde{U})(t), \varphi \rangle - \langle \eta(U_0), \varphi \rangle + \int_0^t \langle \partial_x H(\tilde{U}), \varphi \rangle ds - \langle \tilde{e}, 1_{[0,t]} \varphi \rangle. \]

Then \( M^\varepsilon \) and \( \tilde{M}^\varepsilon \) have same laws. By the convergence (5.16), (5.18) and (5.20) we show, as in Section 3.1.8 that the processes

\[ \tilde{M}, \tilde{M}^2 - \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{U}) \partial_q \eta(\tilde{U}), \varphi \rangle^2 dr, \tilde{M} \tilde{\beta}_k - \int_0^t \langle \sigma_k(\tilde{U}) \partial_q \eta(\tilde{U}), \varphi \rangle dr \]
(5.21)
are \((\mathcal{F}_t)\)-martingales. There is however a notable difference between the result of Lemma 3.24 and the result (5.21) here, in the fact that the martingales in (5.21) are indexed by \(D \subset [0, T]\) since we have used the convergence (5.18). If all the processes in (5.21) were continuous martingales indexed by \([0, T]\), we would infer, as in the proof of Proposition 3.22, that

\[
\langle \eta(\tilde{U}(t)), \varphi \rangle - \langle \eta(U_0), \varphi \rangle - \int_0^t \langle H(\tilde{U}), \partial_x \varphi \rangle \, ds
\]

\[
= -\langle \tilde{e}, 1_{[0,t]} \varphi \rangle + \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{U}) \partial_q \eta(\tilde{U}), \varphi \rangle \, d\tilde{\beta}_k(s),
\]

(5.22)

for all \(t \in [0, T]\), \(\tilde{\mathbb{P}}\)-almost surely. Nevertheless, \(D\) contains 0 and is dense in \([0, T]\) since it is of full measure, and it turns out, by the Proposition A.1 in [Hof13b] on densely defined martingales, that this is sufficient to obtain (5.22) for all \(t \in D\), \(\tilde{\mathbb{P}}\)-almost surely. Then we conclude as in the proof of Theorem 4.13 of [Hof13b]: let \(N(t)\) denote the continuous semi-martingale defined by

\[
N(t) = \int_0^t \langle H(\tilde{U}), \partial_x \varphi \rangle \, ds + \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{U}) \partial_q \eta(\tilde{U}), \varphi \rangle \, d\tilde{\beta}_k(s).
\]

Let \(t \in (0, T]\) be fixed and let \(\alpha \in C^1_c([0, t])\). By the Itô Formula we compute the stochastic differential of \(N(s)\alpha(s)\) to get

\[
0 = \int_0^t N(s)\alpha'(s) \, ds + \int_0^t \langle H(\tilde{U}), \partial_x \varphi \rangle \alpha(s) \, ds
\]

\[
+ \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{U}) \partial_q \eta(\tilde{U}), \varphi \rangle \alpha(s) \, d\tilde{\beta}_k(s).
\]

(5.23)

By (5.22), we have

\[
N(t) = \langle \eta(\tilde{U})(t), \varphi \rangle - \langle \eta(U_0), \varphi \rangle + \langle \tilde{e}, 1_{[0,t]} \varphi \rangle,
\]

for all \(t \in D\), \(\tilde{\mathbb{P}}\)-almost surely. In particular, by the Fubini Theorem,

\[
\int_0^t N(s)\alpha'(s) \, ds = \int_0^t \langle \eta(\tilde{U})(s), \varphi \rangle \alpha'(s) \, ds
\]

\[
+ \langle \eta(U_0), \varphi \rangle \alpha(0) - \int_0^t \alpha(\sigma) d\tilde{\rho}(\sigma),
\]

(5.24)

\(\tilde{\mathbb{P}}\)-almost surely, where we have defined the measure \(\tilde{\rho}\) by \(\tilde{\rho}(B) = \langle \tilde{e}, 1_B \varphi \rangle\), for \(B\) a Borel subset of \([0, T]\). If \(\alpha, \varphi \geq 0\), then

\[
\int_0^t \alpha(\sigma) d\tilde{\rho}(\sigma) \geq 0, \quad \tilde{\mathbb{P}} - \text{almost surely}
\]

and (2.16) follows from (5.23), (5.24). This concludes the proof of Theorem 2.7.
6 Conclusion

We want to discuss in this concluding section some open questions related to the qualitative behaviour of solutions to (1.1). The first one is the question of uniqueness. Uniqueness of weak entropy solutions is an unsolved question in the deterministic setting. For the stochastic problem (1.1), one may consider the question of uniqueness of martingale solutions, i.e. uniqueness in law. Is it a promising question? Would it be easier to answer than the question of uniqueness in the deterministic setting, as, for example, uniqueness in law manifests itself in stochastic differential equations with continuous coefficients, which represent therefore stochastic perturbations of o.d.e’s displaying some indeterminacy [SV79]? It may well be the case that this issue of uniqueness is simpler for a noise of the form \((\rho u \Phi \dot{W})_x\), but for such a noise occurring in the flux term, we do not know for the moment how to prove existence (see [LPS13] for the equivalent problem in the scalar case).

An other problem concerns the long-time behaviour of solutions to (1.1). It is known that for scalar stochastic conservation laws with additive noise, and for non-degenerate fluxes, there is a unique ergodic invariant measure, cf. [EKMS00, DV13]. Since both fields of (1.1) are genuinely non-linear, a form of non-degeneracy condition is clearly satisfied in (1.1). Actually, in the deterministic case \(\Phi \equiv 0\), the solution converges to the constant state determined by the conservation of mass [CF99, Theorem 5.4], which indicates that some kind of dissipation effects (via interaction of waves, cf. also [GL70]) occur in the Euler system for isentropic gas dynamics. However, in a system there is in a way more room for waves to evolve than in a scalar conservation law, and the long-time behaviour in (1.1) may be different from the one described in [EKMS00, DV13].

Specifically, consider the case \(\gamma = 2\). For such a value the system of Euler equations for isentropic gas dynamics is equivalent to the following Saint-Venant system of equations for shallow water:

\[
\begin{align*}
    h_t + (hu)_x & dt = 0, & \text{in } Q_T, & \quad (6.1a) \\
    (hu)_t + (hu^2 + g\frac{h^2}{2})_x + ghZ_x &= 0, & \text{in } Q_T, & \quad (6.1b)
\end{align*}
\]

with \(Z(t, x) = \Phi^\ast(x)\frac{dW}{dt}\) and \(Q_T = \mathbb{T} \times (0, T)\). When \(Z = Z(x)\), (6.1) is a model for the one-dimensional flow of a fluid of height \(h\) and speed \(u\) over a ground described by the curve \(z = Z(x)\) (\(u(x)\) is the speed of the column of water over the abscissa \(x\)\(^4\)). For a random \(Z\) as in (6.1b), the system (6.1) describes the evolution of the fluid in terms of \((h, u)\) when its behaviour is

\(^4\)the fact that \(u\) is independent on the altitude \(z\) is admissible as long as \(h\) is small compared to the longitudinal length \(L\) of the channel, \(L = 1\) here, cf. [GP01]
forced by the moving topography, the question being thus to determine if
an equilibrium in law (and which kind of equilibrium) for such a random
process can be reached when time goes to $+\infty$. An hint for the existence of
a unique, ergodic, invariant measure is the “loss of memory in the system”
given by the ergodic theorem: if $f$ is a bounded, continuous functional of
the solution $U(t)$, then

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(U(t))dt \to \langle f, \mu \rangle \text{ a.s.} \quad (6.2)$$

where $\mu$ is the invariant measure. Before testing the ergodic convergence
(6.2), one has first to restrict the evolution to the right manifold. Indeed,
in the scalar case [EKMS00, DV13], say for the equation

$$dv + (A(v))_x = \partial_x \phi(x) dW(t), \quad x \in \mathbb{T}, t > 0,$$

there is a unique invariant measure $\mu_\lambda$ indexed by the constant parameter

$$\lambda = \int_\mathbb{T} v(x)dx \in \mathbb{R}.$$ 

For (6.1), the entropy solution is evolving on the manifold

$$\int_\mathbb{T} h(x)dx = \text{cst}.$$ 

Is there or is there not an other quantity preserved in the evolution? This
is currently not clear. A way to enforce preservation of a second quantity,
is to use symmetry, by choosing $Z$ to be an odd function of $x$, as well as $u$,
while $h$ is an even function of $x$. Then $q = hu$ is an odd function of $x$ and
the identity

$$\int_\mathbb{T} q(t,x)dx = 0$$

is maintained in the evolution. In that case the convergence (6.2) is clearly
observed on our numerical test. Actually, it is even not necessary to start
with an $h$ and a $u$ having a particular symmetry: as long as $Z$ is symmetric,
it will enforce the symmetry on $h$ and $u$. This is illustrated by Figure ?? on
page ??, where are depicted, respectively, the evolution of

$$t \mapsto \frac{1}{t} \int_0^t \mathcal{E}(U(s))ds \text{ and } t \mapsto \int_\mathbb{T} q(s,x)ds,$$

where

$$\mathcal{E}(U) := \int_\mathbb{T} \frac{1}{2} hu^2 + \frac{1}{2} gh^2 \, dx$$

denotes the energy of the system. For the four test cases considered, the
quantity $\int_\mathbb{T} hdx$ is the same of course. Note also that the numerical simulations for each test case are done simultaneously, therefore the same noise is
drawn for each test. In the non-symmetric case, we are for the moment not able to conclude, cf. Figure ?? on page ??, where no significant convergence of the energy is observed. The question of the large-time behaviour of solutions will be addressed in a future work.

A  A parabolic uniformization effect

The following result, a phenomenon of uniformization from below for a non-negative solution for a parabolic equation with a drift-term with quite low regularity, will be proved with techniques similar to those used in [MV09].

Theorem A.1 (Positivity). Let $\tau > 0$. Let $1_{\text{det}}$ be the step function defined by (3.12). Let $\rho, u \in L^2(0,T;H^1(T))$ satisfy

$$\rho|\partial_x u|^2 \in L^1(\mathbb{T} \times (0,T)), \quad \partial_t \rho \in L^2(0,T;H^{-1}(\mathbb{T})).$$

(A.1)

Assume that $\rho$ is a non-negative solution in which sense? to the equation

$$\frac{1}{2} \partial_t \rho + 1_{\text{det}} [\partial_x (\rho u) - \partial_x^2 \rho] = 0$$

(A.2)

and that $\rho_0 := \rho(0)$ satisfy $\frac{1}{\rho_0} \in L^q(\mathbb{T})$ where $q > 3$. Then, for all $\tau \in [0,T]$ and $m > 3$, there exists a constant $c > 0$ depending on $\|\rho_0^{-1}\|_{L^q(\mathbb{T})}$, $q$, $T$, $\tau$, $m$ and

$$\int_{Q_T} \rho |\partial_x u|^2 \, dx \, dt \quad \text{and} \quad \|u\|_{L^m(Q_T)}$$

(A.3)

only, such that

$$\rho \geq c$$

(a.e. in $T \times [\tau,T]$).

Proof. Note first that the $L^2_t H^1_x$ regularity of $\rho$, together with the $L^2_t H^{-1}_x$ regularity of $\partial_t \rho$ implies the existence of a representative still denoted $\rho$ which is continuous in time with values in $L^2$. This is the value of this representative at time $t = 0$ which is denoted by $\rho_0$. Now, to prove the result, we will show a bound from above on $w = \frac{1}{\rho}$. Actually, $w$ is not yet well-defined and we shall more rigorously prove a bound from above on $w_\varepsilon = \frac{1}{\rho_0 + \varepsilon}$ that is uniform with respect to the parameter $\varepsilon > 0$. For simplicity we will work directly on $w$, the main lines of the proof are easily adapted for $w_\varepsilon$. By a chain-rule formula (cf. Lemma 1.4 in Carrillo, Wittbold [CW99] for example) we derive the following equation for $w$:

$$\frac{1}{2} \partial_t w - 1_{\text{det}} \partial_x^2 w = 1_{\text{det}}(-8|\partial_x w^{1/2}|^2 + w \partial_x u - u \partial_x w).$$

(A.5)
Similarly, we have, for $p \geq 2$,

$$
\frac{1}{2} \frac{\partial_t}{p} z^2 - \frac{1}{2} \det \left( \frac{\partial_x^2 z^2}{p} \right) = \frac{1}{2} \det \left( -\frac{4(p+1)}{p^2} |\partial_x z|^2 + z^2 \partial_x u - \frac{u}{p} \partial_x z^2 \right),
$$

(A.6)

where $z := \frac{w^{p/2}}{2}$. We will use (A.6) to obtain an estimate on $\|w\|_{L^p(T)}$ in function of $\|u\|_{L^r(Q_T)}$ and $\|w(0)\|_{L^p(T)}$ only (see Equation A.9 below). Let us sum (A.6) on $T$: we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_T z^2 dx + \frac{4(p+1)}{p} \frac{1}{2} \det \int_T |\partial_x z|^2 dx = -(2p+1) \frac{1}{2} \det \int_T u z \partial_x z dx.
$$

Consequently, we have

$$
\frac{1}{2} \frac{d}{dt} \int_T z^2 dx + \frac{2(p+1)}{p} \frac{1}{2} \det \int_T |\partial_x z|^2 dx = \frac{p(p+1)}{2} \frac{1}{2} \det \int_T u^2 dx.
$$

Integrating then over $t \in [0, \sigma]$ where $\sigma \leq T$, we obtain

$$
U_\sigma \leq \frac{p(p+1)}{2} \int_{Q_\sigma} \det u^2 z^2 dx + \frac{1}{2} \|z(0)\|_{L^2(T)}^2,
$$

where

$$
U_\sigma := \frac{1}{2} \sup_{t \in [0, \sigma]} \|z(t)\|_{L^2(T)}^2 + \frac{2(p+1)}{p} \frac{1}{2} \det \partial_x z \|L^2(Q_\sigma)\|.
$$

By the H"older Inequality, it follows that

$$
U_\sigma \leq \frac{p(p+1)}{2} \|u\|_{L^1(Q_\sigma)} \|1_{\text{det}} z\|_{L^2(Q_\sigma)}^2 + \frac{1}{2} \|z(0)\|_{L^2(T)}^2.
$$

(A.7)

To obtain an estimate on the right hand-side of (A.7), we apply the following inequality (proven below)

$$
\|1_{\text{det}} z\|_{L^6(Q_\sigma)} \leq C \left( \sup_{t \in [0, \sigma]} \|z(t)\|_{L^2(T)} \right)^{2/3} \|1_{\text{det}} \partial_x z\|_{L^2(Q_\sigma)}^{1/3},
$$

(A.8)

where $C$ is a numerical constant. This gives

$$
U_\sigma \leq C^2 \frac{p(p+1)}{2} \|u\|_{L^1(Q_\sigma)}^2 U_\sigma + \frac{1}{2} \|z(0)\|_{L^2(T)}^2,
$$

and then, since $m > 3$,

$$
U_\sigma \leq C^2 \frac{p(p+1)}{2} \sigma^e \|u\|_{L^m(Q_T)}^2 U_\sigma + \frac{1}{2} \|z(0)\|_{L^2(T)}^2, \quad e := \frac{2}{3} - \frac{2}{m}.
$$

Consequently, for $\sigma < \sigma_0$, we obtain $U_\sigma \leq C \|w^{p/2}(0)\|_{L^p(T)}^2$, where $\sigma_0 > 0$ and $C \geq 0$ are some constant depending only on $\|u\|_{L^m(Q_T)}$ and $p$. Since an
estimate on $U_\sigma$ gives in turn an estimate on $\|z(\sigma)\|_{L^2(T)}^2 = \|w^{p/2}(\sigma)\|_{L^2(T)}^2$, we can iterate our procedure to deduce the following bound:

$$\sup_{t \in [0,T]} \|w(t)\|_{L^p(T)}^p + \|1_{\text{det}} \partial_x w^{p/2}\|_{L^2(Q_T)}^2 \leq C\|w(0)\|_{L^p(T)}^p,$$

(A.9)

where $C$ is a constant depending on $p$, $m$, $T$, $\|u\|_{L^m(Q_T)}$. This estimate is the first step in the proof of Theorem A.1. To achieve the proof of (A.9), we still have to show (A.8). We use the injection $H^\delta(T) \subset L^r(T)$, $\delta \in [0,1/2)$, $\frac{1}{r} := \frac{1}{2} - \delta$, an interpolation inequality and the Poincaré Inequality to obtain

$$\|z(t)\|_{L^r(T)} \leq C_F \|z(t)\|_{L^2(T)}^{(1-\delta)} \|\partial_x z(t)\|_{L^2(T)}^\delta,$$

(A.10)

for a given numerical constant $C_F$. Then we multiply the result by $1_{\text{det}}(t)$ and we sum over $t \in [0,\sigma]$. For $r\delta = 2$ (an equality which sets the value of $(\delta, r)$ to $(1/3, 6)$), we obtain (A.8).

In the second step of the proof, we will derive the following $L^\infty$ estimate on $w$:

$$\|w\|_{L^\infty(Q_{\tau}, T)} \leq C\left(r, T; \|u\|_{L^m(Q_T)}, \|\rho^{1/2} \partial_x u\|_{L^2(Q_T)}, \|w\|_{L^{p_0}(Q_T)}\right),$$

(A.11)

for a $p_0 > 3$ that will be specified later. To prove (A.11), we use the equation (A.5), which we rewrite in the mild form justify

$$w(t) = S(t_2)w(0) + \int_0^{t_2} S(t_2 - s)f(s)ds,$$

where we have set

$$t_2 := \min(2t - t_{2n}, t_{2n+2}), \quad t_3 := \frac{t + t_{2n}}{2}, \quad t_{2n} \leq t \leq t_{2n+2},$$

and where $f$ is the right hand-side of (A.5). Since

$$f \leq 1_{\text{det}}(w\partial_x u - u\partial_x w),$$

we obtain

$$0 \leq w(t) \leq S(t_2)w(0) + W_1(t) + W_2(t),$$

with

$$W_1(t) = \int_0^{t_2} S(t_2 - s)(1_{\text{det}}w\partial_x u)(s)ds,$$

$$W_2(t) = -\int_0^{t_2} S(t_3 - s)(1_{\text{det}}u\partial_x w)(s)ds.$$

Let us set $g = \rho^{1/2} \partial_x u$ and rewrite $W_1(t)$ as

$$W_1(t) = \int_0^{t_2} S(t_2 - s)(1_{\text{det}}w^{3/2}g)(s)ds.$$
Fix $\sigma \in (0, T)$. Let $p_k \in [1, +\infty)$, $r_k \in [1, 2)$ be given. By (B.7) with $j = 0$, we have
\[
\|W_1\|_{L^p_{pk+1}(Q_{\sigma, T})} \leq C\|w^{3/2}g\|_{L^r_{pk}(Q_{\sigma, T})}, \quad \frac{1}{p_{k+1}} < \frac{1}{r_k} < \frac{1}{p_{k+1}} + \frac{2}{3}, \quad \text{(A.12)}
\]
\[
\leq C\|w\|_{L^r_{pk}(Q_{\sigma, T})}\|g\|_{L^2(Q_T)}, \quad \text{(A.13)}
\]
where $Q_{\sigma, T} = T \times (\sigma, T)$, and provided $p_k$ and $r_k$ satisfy the relation
\[
\frac{1}{2} + \frac{3}{2p_k} = \frac{1}{r_k}. \quad \text{(A.14)}
\]

Furthermore, by integration by parts,
\[
W_3(t) = -\int_0^t \partial_x S(t_t - s)(1_{\det uw})(s)ds.
\]

By (B.7) with $j = 1$, we have
\[
\|W_3\|_{L^p_{pk+1}(Q_{\sigma, T})} \leq C\|u\|_{L^q_{pk}(Q_{\sigma, T})}, \quad \frac{1}{p_{k+1}} < \frac{1}{q_k} < \frac{1}{p_{k+1}} + \frac{1}{3}, \quad \text{(A.15)}
\]
\[
\leq C\|w\|_{L^r_{pk}(Q_{\sigma, T})}\|u\|_{L^m(Q_T)}, \quad \text{(A.16)}
\]
provided $p_k$ and $q_k$ satisfy the relation
\[
\frac{1}{p_k} + \frac{1}{m} = \frac{1}{q_k}. \quad \text{(A.17)}
\]

Finally, the estimate (B.2) shows that
\[
\|S(t_t)w(0)\|_{L^\infty(Q_{\sigma, T})} \leq C\sigma^{-\frac{1}{p_0}}\|w(0)\|_{L^{p_0}(T)}, \quad 1 \leq p_0 \leq +\infty, \quad \text{(A.18)}
\]
for a constant $C$ depending on $T$ only. Let
\[
a = -\frac{1}{6} + \delta, \quad b = \frac{1}{m} - \frac{3}{4} + \delta,
\]
$\delta$ small, and let
\[
\frac{1}{p_{k+1}} = \min\left(\frac{3}{2p_k} + a, \frac{1}{p_k} + b\right). \quad \text{(A.19)}
\]

Then $r_k$ and $q_k$ defined by (A.14), (A.17) satisfy the constraints (A.12), (A.15) respectively. It follows then from (A.13), (A.16), (A.18) that $\|w\|_{L^p_{pk}(Q_{\sigma, T})}$ satisfies the recursion formula
\[
\|w\|_{L^p_{pk+1}(Q_{\sigma, T})} \leq C\sigma^{-\frac{1}{p_0}}\|w(0)\|_{L^{p_0}(T)} + C(1 + \|w\|_{L^r_{pk}(Q_{\sigma, T})}^{3/2}), \quad k \geq 0,
\]
where the constant $C$ depends on $T$, $\|u\|_{L^m(Q_T)}$, $\|\rho^{1/2}\partial_x u\|_{L^2(Q_T)}$ only. We choose $p_0 > \frac{3}{1 - 6\delta}$. Then there exists a finite $K > 0$ such that $\frac{1}{p_K} > 0$ while $\frac{1}{p_{K+1}}$ given by (A.19) is negative, which means that we may as well take $p_{K+1} = +\infty$. We deduce the estimate (A.11). Using then the estimate (A.9) for $p = p_0$, we obtain (A.4), which concludes the proof of the Theorem.

\[\blacksquare\]
Remark A.2. Note that, if $\rho_0$ is positive, say $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$, where $c_0 > 0$, then $w_0 \in L^\infty(\mathbb{T})$. We may therefore replace take $\sigma = 0$ in the estimates (A.13), (A.16) above and replace (A.18) by the estimate

$$\|S(t)w(0)\|_{L^\infty(Q_T)} \leq T\|w(0)\|_{L^\infty(\mathbb{T})}$$

to conclude that (A.4) holds on $[0, T]$.

Remark A.3. Note also that it is possible to give some precisions on the bound from below (I.57) in [LPS96], regarding the positivity of the density $\rho$ in the deterministic parabolic approximation of the isentropic Euler system. Since, for such a system, the terms in (A.3) are bounded, respectively, by the initial entropy

$$\int_{\mathbb{T}} \eta_E(U_0(x))dx \leq C(\|\rho_0\|_{L^\infty(\mathbb{T})}, \|u_0\|_{L^\infty(\mathbb{T})})$$

and the $L^\infty$ norm

$$\|u\|_{L^\infty(Q_T)} \leq TC(\|\rho_0\|_{L^\infty(\mathbb{T})}, \|u_0\|_{L^\infty(\mathbb{T})}),$$

where here $C$ is a continuous function of its arguments, we obtain $\rho \geq c_1$ a.e. in $Q_T$, where $c_1$ depends continuously on $T$, $\|\rho_0\|_{L^\infty(\mathbb{T})}$, $\|u_0\|_{L^\infty(\mathbb{T})}$, $c_0$, where $c_0 = \inf_{x \in \mathbb{T}} \rho_0(x)$.

B Regularizing effects of the one-dimensional heat equation

Let $T > 0$, let $z \in C([0, T]; L^2(\mathbb{T}))$ satisfy

$$z(t) = S(t)z_0 + \int_0^t S(t - s)f(s)ds, \quad (B.1)$$

for some given data $z_0$ and $f$, where $S(t)$ is the semi-group associated to the heat operator $\partial_t - \partial_x^2$ on $\mathbb{T}$. The function $z$ is a mild solution to the heat equation

$$(\partial_t - \partial_x^2)z = f \quad \text{in} \quad Q_T,$$

with initial condition $z(0) = z_0$. If $f \in L^p(Q_T)$, $p > \frac{6}{5}$ and $z_0 \in L^2(\mathbb{T})$ then, by the regularizing properties of $S(t)$, (B.1) gives a $z$ which is indeed in $C([0, T]; L^2(\mathbb{T}))$. More precisely, we have

$$\|S(t)\|_{L^\infty \to L^q} \leq Ct^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2})}, \quad (B.2)$$

for $1 < p \leq q \leq +\infty$, for a given constant $C$ and therefore, for possibly a different constant $C$,

$$\|S(t - s)f(s)\|_{L^2(\mathbb{T})} \leq C(t - s)^{-\mu}\|f(s)\|_{L^p(\mathbb{T})}, \quad \mu := \frac{1}{2}\left(\frac{1}{p} - \frac{1}{2}\right).$$

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By the Young inequality for the convolutions of functions, $L^p \ast L^p$ embeds in the space of continuous functions (here $p'$ is the conjugate exponent to $p$), hence $z$ defined by (B.1) is indeed continuous in time with values in $L^2(T)$ if $t \mapsto t^{-\nu}$ is in $L^{p'}$, which is equivalent to the condition $p > \frac{6}{5}$. Using more generally the regularizing properties

$$
\|\partial_t^j S(t)\|_{L^p_x(T)} \leq C t^{- \frac{j}{2} \left(\frac{1}{p'} - \frac{1}{q}\right)}
$$

for $j \in \mathbb{N}$, we obtain that, for a given constant $C \geq 0$,

$$
\left\| \partial_x^j \int_0^t S(t-s)f(s) ds \right\|_{L^p(Q_T)} \leq C \|f\|_{L^p(Q_T)} \quad \text{if} \quad \frac{1}{q} < \frac{1}{p} < \frac{1}{q} + \frac{2 - j}{3},
$$

(B.3a)

$$
\left\| \partial_x^j S(t)z_0 \right\|_{L^p(Q_T)} \leq C \|z_0\|_{L^p(T)} \quad \text{if} \quad \frac{1}{q} < \frac{1}{p} < \frac{3}{q} - j
$$

(B.3b)

for $j \in \{0,1\}$.

**Remark B.1** (Time regularity). Let $\nu > 0$, $m \geq 1$. If $t \mapsto t^{-\nu}$ is in $L^m(0,T)$ then it is also in $W^{\sigma,m}(0,T)$ for a certain $\sigma > 0$. This shows that we have actually

$$
\left\| \partial_x \int_0^t S(t-s)f(s) ds \right\|_{W^{\sigma,m}(0,T;L^p(T))} \leq C \|f\|_{L^p(Q_T)} \quad \text{if} \quad \frac{1}{q} > \frac{1}{p} - \frac{1}{3},
$$

(B.4)

for a given $\sigma > 0$ depending on $p$ and $q$.

We end this section with a variation over (B.4a) for the solution of a split evolution equation (this is used in Section 3.1.6): let $\tau > 0$, set $t_n = n\tau$, $n \in \mathbb{N}$, set

$$
t_1 := \min(2t - t_2, t_2 + 2), \quad t_2 := \frac{t + t_2}{2}, \quad t_2 \leq t \leq t_2 + 2,
$$

and let

$$
z(t) = \int_0^{t_2} S(t_2 - s) f(s) ds.
$$

The function $z$ is the solution to

$$
\frac{1}{2} \partial_t z - \partial_x^2 z = f, \quad \text{in} \ T \times (t_2, t_2 + 1), \quad \partial_t z = 0, \quad \text{in} \ T \times (t_2, t_2 + 2),
$$

for $n = 0,1,\ldots$. If $T = t_2 K$, $K \in \mathbb{N}^*$, we have, for $h \in L^1(0,T)$,

$$
\int_0^T h(t) dt = \frac{1}{2} \int_0^T h(t) dt + \sum_{n=0}^{K-1} \tau h(t_{2n+2})
$$

$$
= \int_0^T h(t) dt + \frac{1}{2} \sum_{n=0}^{K-1} \int_{t_{2n+2}}^{t_{2n+2}} (h(t_{2n+2}) - h(t)) dt.
$$

(B.5)
If there exists a $\sigma \in (0, 1)$ such that $h \in W^{\sigma,1}(0, T)$, we can estimate the remainder in (B.6) and obtain

$$\left| \int_0^T h(t) dt - \int_0^T h(t) dt \right| \leq C\|h\|_{W^{\sigma,1}(0, T)} \tau^\sigma.$$  

By Remark B.1 and (B.4), we deduce in particular that, for $\tau \leq 1$ and $j = 0$ or 1,

$$\left\| \partial_j^2 \int_0^{t_2} S(t_2 - s) f(s) ds \right\|_{L^q(T)} \leq C\|\tilde{f}\|_{L^p(T)} \quad \text{if } \frac{1}{q} < \frac{1}{p} < \frac{1}{q} + \frac{2-j}{3},$$

(B.7)

where $\tilde{f}$ is the extension by 0 of $f$ to the strips $T \times (t_{2n+1}, t_{2n+2})$, $n = 0, \ldots, K - 1$, and $C \geq 0$ a numerical constant. Similarly, by (B.4b), we have

$$\left\| \partial_j^2 S(t_2) z_0 \right\|_{L^q(T)} \leq C\|z_0\|_{L^p(T)} \quad \text{if } \frac{1}{q} < \frac{1}{p} < \frac{3}{q} - j, \quad \text{(B.8)}$$

References


