Diffusion-approximation in stochastically forced kinetic equations

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Abstract

We derive the hydrodynamic limit of a kinetic equation where the interactions in velocity are modelled by a linear operator (Fokker-Planck or Linear Boltzmann) and the force in the Vlasov term is a stochastic process with high amplitude and short-range correlation. In the scales and the regime we consider, the hydrodynamic equation is a scalar second-order stochastic partial differential equation. Compared to the deterministic case, we also observe a phenomenon of enhanced diffusion.

Keywords: diffusion-approximation, kinetic equation, hydrodynamic limit

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Contents

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1 Introduction

1.1 Kinetic equations

Let $N \in \mathbb{N}^*$. We denote by $\mathbb{T}^N$ the N-dimensional torus. Let $\varepsilon > 0$. We consider the following kinetic equation

$$\partial_t f + \varepsilon v \cdot \nabla_x f + \bar{E}(t,x) \cdot \nabla_v f = Qf, \quad t > 0, x \in \mathbb{T}^N, v \in \mathbb{R}^N, \tag{1.1}$$

which is a perturbation of the equation

$$\partial_t f + \bar{E}(t,x) \cdot \nabla_v f = Qf \quad t > 0, x \in \mathbb{T}^N, v \in \mathbb{R}^N. \tag{1.2}$$

The operator Q is either the linear Boltzmann (LB) operator

$$Q_{\rm LB}f=\rho(f)M-f,\quad \rho(f)=\int_{\mathbb{R}^N}f(v)dv,\quad M(v)=\frac{1}{(2\pi)^{N/2}}\exp\left(-\frac{|v|^2}{2}\right),\tag{1.3}$$

or the Fokker-Planck (FP) operator

$$Q_{\rm FP}f=\mathrm{div}_v(\nabla_v f+vf).\tag{1.4}$$

The force field $\bar{E}(t,x)$ in (1.2) is a Markov, stationary mixing process $t\mapsto\bar{E}(t)$ with state space $F=C^1(\mathbb{T}^N;\mathbb{R}^N)$ (see Section 2 for more details). We show in Section 3 that there is a unique, ergodic, invariant measure for (1.2) and that this invariant measure is the law of an invariant solution $(x,v)\mapsto\rho(x)\bar{M}_t(x,v)$ parametrized by $\rho(x)$. See (3.6)-(3.7) for the definition of $\bar{M}_t$. Consider the solution f to (1.1) starting from a state

$$f_{\rm in}(x,v)\approx\rho_{\rm in}(x)\overline{M}_0(x,v).\tag{1.5}$$

Rescale over time intervals of order ε^{-2}:

$$f^\varepsilon(t,x,v)=f(\varepsilon^{-2}t,x,v).\tag{1.6}$$

Then f^ε is solution to the equation

$$\partial_t f^\varepsilon+\frac{v}{\varepsilon}\cdot\nabla_x f^\varepsilon+\frac{1}{\varepsilon^2}\bar{E}(\varepsilon^{-2}t,x)\cdot\nabla_v f^\varepsilon=\frac{1}{\varepsilon^2}Qf^\varepsilon,\quad t>0,x\in\mathbb{T}^N,v\in\mathbb{R}^N.\tag{1.7}$$

On bounded time intervals $[0,T]$, we expect

$$f^\varepsilon(t,x,v)\approx\rho(x,t)\overline{M}_{\varepsilon^{-2}t}(x,v),\tag{1.8}$$

where ρ is solution to a given equation (the *hydrodynamic* equation) which we would like to identify. We do not prove (1.8), but find the limit equation satisfied by $\rho=\lim_{\varepsilon\to 0}\rho^\varepsilon$, where $\rho^\varepsilon=\rho(f^\varepsilon)$. We show in Theorem 1.2 that ρ satisfies a diffusion equation, where the drift term is a second order differential operator in divergence form with respect to the space-variable x. Showing that ρ^ε is close to ρ with ρ a diffusion (in infinite dimension) is therefore a result of diffusion-approximation (in infinite dimension). See Theorem 1.2 for the precise statement. Theorem 1.1 is concerned with the limit behaviour of the average $\mathbb{E}\rho^\varepsilon$, a deterministic issue. The proof of this result is easier and related to characteristic equations associated to (1.1), which we discuss in the next section.

1.2 Trajectories

The phase space associated to (1.1) is $\mathbb{T}^N\times\mathbb{R}^N$. Consider the following systems of stochastic differential equations:

$$\begin{aligned}dX_t&=\varepsilon dV_t,\\ dV_t&=\bar{E}(t,X_t)dt+\text{jumps},\end{aligned}\tag{1.9}$$

and

$$
\begin{aligned}
dX_t &= \varepsilon dV_t,\\
dV_t &= (\bar{E}(t,X_t)-V_t)dt+\sqrt{2}dB_t.
\end{aligned}
\tag{1.10}
$$

In (1.9) the second equation describes the following piecewise deterministic Markov process (PDMP). Consider the Poisson process associated to the times (T_n) and to the probability measure Mdv: the increments $T_{n+1}-T_n$ are i.i.d. with exponential law of parameter 1. At each time $t=T_n$, V_t is jumping to a new value V_{T_n+} chosen at random, according to the probability law Mdv. Between each jump, (V_t) is evolving by the differential equation

$$
\frac{dV_t}{dt}=E(t,X_t),\quad T_n<t<T_{n+1},
\tag{1.11}
$$

which is coupled with the first equation of (1.9). In (1.10), B_t is an N-dimensional Wiener process. In both the LB case and the FP case, the extra stochastic processes which we introduce are independent of $(\bar{E}(t))$. In this context, the equation (1.1) gives the evolution of the density, with respect to the Lebesgue measure on $\mathbb{T}^N_x\times\mathbb{R}^N_v$, of the conditional law of (X_t,V_t): let $\mathcal{F}^E_t=\sigma((\bar{E}_s)_{0\leq s\leq t})$. If the law of (X_0,V_0) has density f_{in} with respect to the Lebesgue measure on $\mathbb{T}^N_x\times\mathbb{R}^N_v$, then

$$
\mathbb{E}\left[\varphi(X_t,V_t)|\mathcal{F}^E_t\right]=\iint_{\mathbb{T}^N\times\mathbb{R}^N}\varphi(x,v)f_t(x,v)dxdv,
\tag{1.12}
$$

for all $\varphi\in C_b(\mathbb{T}^N\times\mathbb{R}^N)$. From (1.12), it follows that

$$
\mathbb{E}\left[\varphi(X_t)\right]=\int_{\mathbb{T}^N}\varphi(x)\mathbb{E}\rho_t(x)dx,\quad \rho_t=\rho(f_t),
\tag{1.13}
$$

for all $\varphi\in C_b(\mathbb{T}^N)$.

We are interested in equation (1.7), the associated process is $(X_{\varepsilon^{-2}t},V_{\varepsilon^{-2}t})$ and the quantity of interest is $\rho_{\varepsilon^{-2}t}$.

We have two main results. The first one, Theorem 1.1, gives the limit behaviour of $\mathbb{E}\rho_{\varepsilon^{-2}t}$. The second one, Theorem 1.2, describes the limit behaviour of $\rho_{\varepsilon^{-2}t}$. The first result should follow from the second one. However, there are various reasons for giving two separate statements:

1. we obtain the limit behaviour of $\mathbb{E}\rho_{\varepsilon^{-2}t}$, by proving the convergence in law of $X_{\varepsilon^{-2}t}$ (hence focusing on the left-hand side of (1.13)),

2. on the contrary, the limit behaviour of $\rho_{\varepsilon^{-2}t}$ is obtained by working at the level of the PDE (1.7),

3. the proof of Theorem 1.1 uses the central limit theorem for martingales. This approach to the limiting behaviour of (1.9) or (1.10) is very classical in a certain mathematical community (see, *e.g.*, the second paragraph of the introduction to [14], and also Chapter 13 of the same reference), but is certainly not familiar to a large group of analysts, and we wanted to emphasize these probabilistic aspects here,

4. in the proof of Theorem 1.1, we introduce some tools and some results that are used later on in the proof of Theorem 1.2; with this progression, the proof of Theorem 1.2, which is quite long, is more gradual.

Note, however, that we establish Theorem 1.1 in the restrictive case of a field $\bar{E}_t$ independent on x.

1.3 Main results

Notations. The three first moments of a function $f \in L^1(\mathbb{R}^N, |v|^2 dv)$ are

$$\rho(f) = \int_{\mathbb{R}^N} f(v)dv, \quad J(f) = \int_{\mathbb{R}^N} vf(v)dv, \quad K(f) = \int_{\mathbb{R}^N} v \otimes vf(v)dv, \tag{1.14}$$

where $a \otimes b$ is the $N \times N$ rank-one matrix built on $a, b \in \mathbb{R}^N$ with ij-th elements $a_i b_j$. We denote by K the second moment of M (due to the particular fact that M is a Maxwellian, this is simply the identity matrix of size $N \times N$):

$$K = K(M) = \int_{\mathbb{R}^N} v \otimes vM(v)dv = \mathrm{Id}_N. \tag{1.15}$$

For $m \in \mathbb{N}$, we denote by $\bar{J}_m(f)$ the total m-th moment of f:

$$\bar{J}_m(f) = \iint_{\mathbb{T}^N \times \mathbb{R}^N} |v|^m f(x, v)dxdv. \tag{1.16}$$

Let us also introduce the Banach space

$$G_m = \left\{ f \in L^1(\mathbb{T}^N \times \mathbb{R}^N); \bar{J}_0(f) + \bar{J}_m(f) < +\infty \right\}, \tag{1.17}$$

with norm $\|f\|_{G_m} = \bar{J}_0(f) + \bar{J}_m(f)$. Eventually, we define the diffusion matrix $K_\sharp$ and the vector field Ψ of our limit equations by the formula

$$K_\sharp = K + \mathbb{E}\left[\bar{E}(0) \otimes [R_0(\bar{E}(0)) + (b-1)R_1(\bar{E}(0))]\right], \tag{1.18}$$

and

$$\Psi = \mathbb{E}\left[b\bar{E}(0) \cdot \nabla_x R_1(\bar{E}(0)) + [R_0(\bar{E}(0)) + (b-1)R_1(\bar{E}(0))]\mathrm{div}_x(\bar{E}(0))\right], \tag{1.19}$$

where $b^{\mathrm{LB}} = 2$ in the case $Q = Q_{\mathrm{LB}}$ and $b^{\mathrm{FP}} = 1$ in the case $Q = Q_{\mathrm{FP}}$, and where the resolvent R_λ is defined by (2.14).

Deterministic convergence Our first result gives the convergence of the average $\mathbb{E}\rho^\varepsilon$ in the case of a spatially constant $\bar{E}$.

Theorem 1.1. *Let $K_\sharp$ be defined by (1.18). Let $f_{\text{in}} \in G_3$ be non-negative. Let $(\bar{E}_t)$ be a mixing force process according to Definition 2.2. Let $f^\varepsilon \in C([0,T]; L^1(\mathbb{T}^N \times \mathbb{R}^N))$ be the mild solution to (1.7) with initial condition f_{in}, in the sense of Definition 4.1 or 4.2, depending on the nature of the collision operator Q. Let $r^\varepsilon = \mathbb{E}\rho(f^\varepsilon)$. Assume that f_{in} has the following structure:*

$$f_{\text{in}}(x,v) = \rho_{\text{in}}(x) g(v), \tag{1.20}$$

where $g \in L^1(\mathbb{R}^N)$, $\rho(g) = 1$. Then, $r^\varepsilon \to r$ in $C([0,T]; L^2(\mathbb{T}^N) - \text{weak})$, where r is the solution to the diffusion equation

$$\partial_t r - \text{div}_x(K_\sharp \nabla_x r) = 0, \tag{1.21}$$

with initial condition

$$r(0) = r_{\text{in}}. \tag{1.22}$$

We show in (5.31) that $K_\sharp \geq K$. It is a remarkable fact that the stochastic forcing term $\bar{E}_t$ has an influence on the diffusion matrix at the limit, and that it increases the diffusion effects. Note that the influence of stochastic mixing forcing terms in kinetic equations has also been investigated in [17, 10]. The context and the results in these two papers are different from the present one however. Indeed,

1. the starting kinetic equations in [17, 10] are not collisional,
2. In [17, 10], in the scaling that is considered, a collisional kinetic equation is obtained at the limit. The collision operator (an operator acting on functions of the variable v thus) is a diffusion operator. At the level of trajectories, this operator appears due to the convergence of the velocity V_t of particles to a diffusion like equation (1.10) with $E = 0$.

Diffusion-approximation Our main result of diffusion-approximation for ρ^ε is the following one.

Theorem 1.2. *Let $K_\sharp$ and Ψ be defined by (1.18) and (1.19) respectively.. Let $f^\varepsilon_{\text{in}} \in G_3$ be non-negative. Let $(\bar{E}_t)$ be a mixing force field on $H^{\bar{\sigma}}(\mathbb{T}^N; \mathbb{R}^N)$ according to Definition 2.1. Let $f^\varepsilon \in C([0,T]; L^1(\mathbb{T}^N \times \mathbb{R}^N))$ be the mild solution to (1.7) with initial condition $f^\varepsilon_{\text{in}}$, in the sense of Definition 4.1 or 4.2, depending on the nature of the collision operator Q. Let $\rho^\varepsilon = \rho(f^\varepsilon)$. Assume the convergence*

$$\rho(f^\varepsilon_{\text{in}}) \to \rho_{\text{in}} \text{ in } L^1(\mathbb{T}^N), \text{ with } \rho_{\text{in}} \in L^2(\mathbb{T}^N). \tag{1.23}$$

Let $K_\sharp$ and Ψ be defined by (1.18) and (1.19) respectively. Then (ρ^ε) converges in law on $C([0,T]; H^{-1}(\mathbb{T}^N))$ to ρ, the weak solution in the sense of Definition 6.1 of the stochastic equation

$$d\rho = \text{div}_x(K_\sharp \nabla_x \rho + \Psi\rho)dt + \sqrt{2}\text{div}_x(\rho S^{1/2} dW(t)), \tag{1.24}$$

with initial condition

$$\rho(0)=\rho_{\rm in}. \tag{1.25}$$

In (1.24), $W(t)$ is a cylindrical Wiener process on $\left[L^2(\mathbb{T}^N)\right]^N$, and $S^{1/2}$ is the Hilbert-Schmidt operator on $\left[L^2(\mathbb{T}^N)\right]^N$ defined in Section 6.5.1.

Remark 1.1 (Stratonovitch Formulation). The Stratonovitch formulation of (1.24) is

$$d\rho=\mathrm{div}_x(\tilde{K}_\sharp\nabla_x\rho+\tilde{\Psi}\rho)dt+\tilde{F}\cdot\nabla\rho dt+\sqrt{2}\mathrm{div}_x(\rho\circ S^{1/2}dW(t)), \tag{1.26}$$

where

$$\begin{aligned}\tilde{K}_\sharp&=K+(b-1)\mathbb{E}\left[R_1(\bar{E}(0))\right],\\ \Psi&=\mathbb{E}\left[b\bar{E}(0)\cdot\nabla_x R_1(\bar{E}(0))+(b-1)R_1(\bar{E}(0))\mathrm{div}_x(\bar{E}(0))\right],\end{aligned}$$

and $\tilde{F}=-\mathrm{div}(\mathbb{E}[R_0(\bar{E}(0))\otimes\bar{E}(0)])$, with $b^{\rm LB}=2$, $b^{\rm FP}=1$.

Note the weak mode of convergence of ρ^ε in Theorem 1.2. It is weak in the probabilistic sense (convergence in law). This is inherent to the limit theorems (like the Donsker theorem) which lay the bases of diffusion-approximation results. The convergence is weak with respect to the space-variable also. We obtain below a bound in G_3 on f^ε thus by interpolation a better convergence than convergence in $C([0,T];H^{-1}(\mathbb{T}^N))$ holds. But this is still in a space with negative regularity in space. We intend to improve this point, and to consider non-linear equations in a similar regime, in a future work. A final remark in this direction is that if $\bar{E}$ is spatially independent, then the spatial derivatives of f^ε satisfy the same equation as f^ε so that if the spatial derivatives of $f^\varepsilon_{\rm in}$ are in G_3 we have a bound in $W^{1,1}(\mathbb{T}^N)$ and obtain strong convergence.

The plan of the paper is the following one. In Section 2 we describe the type of forcing field $\bar{E}(t)$ which we consider. In Section 3, we prove some mixing properties and compute the invariant measures for the unperturbed equation (1.2). In Section 4, we solve the Cauchy Problem for the kinetic equation (1.1). In Section 5, we prove Theorem 1.1 (deterministic limit). In Section 6, we establish our main result of diffusion-approximation, Theorem 1.2.

Note that the present paper is quite long. There are various reasons for these, first the fact that the whole proof of Theorem 1.2 requires many step. However, the heart of our diffusion-approximation result is the computations done by the perturbed test-function method in Section 6.1. An other reason for the length of the paper is that we have taken the care to present all the details of some intuitive facts, like the statements of Theorem 4.3 or Theorem B.3 for example. Indeed, semi-groups, generator and Markov processes in infinite dimension require some circumspection. With that regard, we have used in particular the references [9] and [18].

2 Mixing force field

Let $F = C^1(\mathbb{T}^N; \mathbb{R}^N)$. This is the state space for the mixing force field $\bar{E}$. Let $(\bar{E}_t)_{t\geq 0}$ be a stationary, homogeneous Markov process of generator A over F (the generator is defined according to the theory developed in Appendix B). Let $P(t, \mathbf{e}, B)$ be a transition function for $(\bar{E}_t)$ associated to the filtration generated by $(\bar{E}_t)$ (see, e.g., [9, p. 156] for the definition), satisfying the Chapman-Kolmogorov relation

$$P(t+s, \mathbf{e}, B) = \int_F P(s, \mathbf{e}_1, B) dP(t, \mathbf{e}, d\mathbf{e}_1), \tag{2.1}$$

for all $s, t \geq 0$, $\mathbf{e} \in F$, B Borel subset of F. Let $\mathcal{P}(F)$ be the set of Borel probability measures on F. By [9, p. 157], up to a modification of the probability space $(\Omega, \mathcal{F})$, say into a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}})$, there exists a collection $\{\mathbb{P}_\mu; \mu \in \mathcal{P}(F)\}$ of probability measures and some Markov processes $(E(t,s))_{t\geq s}$ with transition function P such that, $\mathbb{P}_\mu(E(s,s) \in D_0) = \mu(D_0)$ for all Borel subset D_0 of F. When μ is the Dirac mass $\mu = \delta_{\mathbf{e}}$, we use the shorter notation $\mathbb{P}_{\mathbf{e}}$ instead of $\mathbb{P}_{\delta_{\mathbf{e}}}$. By [9, p. 157] additionally, for all $D \in \mathcal{F}$, $\mathbf{e} \mapsto \mathbb{P}_{\mathbf{e}}(D)$ is Borel measurable. Let $\mathbf{e}_0$ be a random variable on F of law μ. We do a slight abuse of notation and denote by $(E(t, s; \mathbf{e}_0), \mathbb{P})$ the couple $(E(t,s), \mathbb{P}_\mu)$. This means that the finite-dimensional distribution of both processes are the same, *i.e.*

$$\mathbb{P}(E(t_1, s; \mathbf{e}_0) \in D_1, \dots, E(t_n, s; \mathbf{e}_0) \in D_n) = \mathbb{P}_\mu(E(t_1, s) \in D_1, \dots, E(t_n, s) \in D_n), \tag{2.2}$$

for all $s \leq t_1 \leq \cdots \leq t_n$, and $D_1, \dots, D_n$ Borel subsets of F. For simplicity, we use the notation $E(t; \mathbf{e})$, or $E_t(\mathbf{e})$, instead of $E(t, 0; \mathbf{e})$. Note that, by iteration of (2.1), we have

$$\begin{aligned}
&\mathbb{P}(\bar{E}(0) \in D_0, \bar{E}(t_1) \in D_1, \dots, \bar{E}(t_n) \in D_n) \\
&= \int_{D_0} \cdots \int_{D_{n-1}} P(t_n - t_{n-1}, \mathbf{e}_{n-1}, D_n) P(t_{n-1} - t_{n-2}, \mathbf{e}_{n-2}, d\mathbf{e}_{n-1}) \cdots P(t_1, \mathbf{e}_0, d\mathbf{e}_1) d\nu(\mathbf{e}_0) \\
&= \mathbb{P}_\nu(E(t_1, 0) \in D_1, \dots, E(t_n, 0) \in D_n),
\end{aligned} \tag{2.3}$$

where ν is the law of $\bar{E}(0)$. Therefore $\bar{E}_t$ and $E_t(\bar{E}_0)$ have the same finite-dimensional distributions: $\bar{E}_t$ is a version $E_t(\bar{E}_0)$. The probability space $\tilde{\Omega}$ used in [9, p. 157] to define the probability measures $\mathbb{P}_{\mathbf{e}}$ is the path-space $F^{[0,+\infty)}$ (the σ-algebra $\tilde{\mathcal{F}}$ is the product σ-algebra). Assume in addition that $(\bar{E}_t)$ is càdlàg. Then it is clear that we can take the Skorohod space $D([0, +\infty); F)$ as a path space to define $\mathbb{P}_{\mathbf{e}}$. The σ-algebra $\tilde{\mathcal{F}}$ is then the trace of the product σ-algebra, which coincide with the Borel σ-algebra when the Skorohod topology is considered on $D([0, +\infty); F)$. In this context, it holds true that $\mathbf{e} \mapsto \mathbb{P}_{\mathbf{e}}(D)$ is Borel measurable for all $D \in \tilde{\mathcal{F}}$ (see the proof of Proposition 1.2 p. 158 in [9]). To sum up (see [20, Section I-3]), if $(\bar{E}_t)$ is càdlàg, we can assume that $t \mapsto E(t, s; \mathbf{e})$ is càdlàg, for all $s \in \mathbb{R}$ and $\mathbf{e} \in F$. As a last remark, note that it is always possible, using the Kolmogorov extension theorem, to build a càdlàg stationary process $(\check{E}(t))_{t\in\mathbb{R}}$ indexed by $t \in \mathbb{R}$ with the finite-dimensional distributions

$$\begin{aligned}
&\mathbb{P}(\check{E}(s) \in D_0, \check{E}(s+t_1) \in D_1, \dots, \check{E}(s+t_n) \in D_n) \\
&= \mathbb{P}(\bar{E}(0) \in D_0, \bar{E}(t_1) \in D_1, \dots, \bar{E}(t_n) \in D_n),
\end{aligned} \tag{2.4}$$

for all $s \in \mathbb{R}$, $0 \leq t_1, \ldots, t_n$. Instead of adding a new notation $(\check{E}(t))_{t\in\mathbb{R}}$, we simply denote this process by $(\bar{E}(t))_{t\in\mathbb{R}}$. We also denote by $(\mathcal{G}_t^E)$ the usual augmentation (*cf.* [20, Definition (4.13), Section I-4]) of the canonical filtration $(\mathcal{F}_t)$ on $D([0,+\infty);F)$ with respect to the family $(\mathbb{P}_{\mathbf{e}})_{\mathbf{e}\in F}$. In successive order, $(\mathcal{F}_t)$ is the filtration generated by the evaluation maps (π_t), $\pi_t(\omega)=\omega(t)$; $\mathcal{F}_t^*$ is the intersection over $\mathbf{e}\in F$ of the σ-algebras $\mathcal{F}_t^{\mathbb{P}_{\mathbf{e}}}$ obtained by completing $\mathcal{F}_t$ with $\mathbb{P}_{\mathbf{e}}$-negligible sets; and $\mathcal{G}_t$ is $\mathcal{F}_{t+}^*$:

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s^*. \tag{2.5}$$

Definition 2.1 (Mixing force field)**.** Let $(\bar{E}_t)_{t\geq 0}$ be a càdlàg, stationary, homogeneous Markov process of generator A, in the sense of Appendix B, over F. We say that $(\bar{E}_t)_{t\geq 0}$ is a mixing force field if the conditions (2.6), (2.7), (2.9), (2.13), (2.16) below are satisfied.

Our first hypothesis is that there exists a stable ball: there exists $\mathtt{R}\geq 0$ such that: almost-surely, for all $\mathbf{e}$ with $\|\mathbf{e}\|_F\leq \mathtt{R}$, for all $t\geq 0$,

$$\|E(t;\mathbf{e})\|_F \leq \mathtt{R}. \tag{2.6}$$

Our second hypothesis is about the law ν of $\bar{E}_t$. We assume that it is supported in the ball $\bar{B}_{\mathtt{R}}$ of F (therefore, it has moments of all orders) and that it is centred:

$$\int_F \mathbf{e}\, d\nu(\mathbf{e}) = \mathbb{E}\left[\bar{E}_t\right] = 0, \tag{2.7}$$

for all $t\geq 0$. Note that a consequence of this hypothesis is that: almost-surely, for all $t\geq 0$,

$$\|\bar{E}_t\|_F \leq \mathtt{R}. \tag{2.8}$$

Our third hypothesis is a mixing hypothesis: we assume that there exists a continuous, non-increasing, positive and integrable function $\gamma_{\mathrm{mix}}\in L^1(\mathbb{R}_+)$ such that, for all probability measures μ, μ' on F, for all random variables $\mathbf{e}_0$, $\mathbf{e}_0'$ on F of law μ and μ' respectively, there is a coupling $((E_t^*(\mathbf{e}_0))_{t\geq 0}, (E_t^*(\mathbf{e}_0'))_{t\geq 0})$ of $((E_t(\mathbf{e}_0))_{t\geq 0}, (E_t(\mathbf{e}_0'))_{t\geq 0})$ such that

$$\mathbb{E}\|E_t^*(\mathbf{e}_0) - E_t^*(\mathbf{e}_0')\|_F \leq \mathtt{R}\gamma_{\mathrm{mix}}(t), \tag{2.9}$$

for all $t\geq 0$. Typically, we expect γ_{mix} to be of the form $\gamma_{\mathrm{mix}}(t) = C_{\mathrm{mix}}e^{-\beta_{\mathrm{mix}}t}$, $\beta_{\mathrm{mix}}>0$ (see the example treated in Section 2.3 for instance).

2.1 Some consequences of the mixing hypothesis

Let φ be a Lipschitz continuous function on F. We have

$$\mathbb{E}\varphi(E_t^*(\mathbf{e}_0)) = \langle e^{tA}\varphi, \mu\rangle$$

(where e^{tA} denote the semi-group associated to A: $\mathbb{E}_{\mathbf{e}}\varphi(E_t) = e^{tA}\varphi(\mathbf{e})$). From (2.9), it follows that

$$\left|\langle e^{tA}\varphi, \mu\rangle - \langle e^{tA}\varphi, \mu'\rangle\right| \leq \|\varphi\|_{\mathrm{Lip}}\mathtt{R}\gamma_{\mathrm{mix}}(t), \tag{2.10}$$

for all $t \geq 0$. Let ν denote the law of $(\bar{E}(t))$ and let $\mathbf{e} \in \bar{B}_{\mathtt{R}}$. We will use (2.10) in particular when $\mathbf{e}_0 = \mathbf{e}$ a.s. and $\mathbf{e}_0'$ has law ν. Then (2.10) gives the following mixing estimate:

$$\|e^{tA}\varphi(\mathbf{e}) - \langle\varphi,\nu\rangle\|_F \leq \mathtt{R}\|\varphi\|_{\mathrm{Lip}}\gamma_{\mathrm{mix}}(t), \tag{2.11}$$

for all $t \geq 0$, for all $\mathbf{e} \in \bar{B}_{\mathtt{R}}$. The estimate (2.11) has an extension to quadratic functionals: for all linear and continuous $\Lambda\colon F \to \mathbb{R}$, for all bi-linear and continuous $q\colon F \times F \to \mathbb{R}$, we have, for all $\mathbf{e} \in \bar{B}_{\mathtt{R}}$,

$$\|e^{tA}[\Lambda + q](\mathbf{e}) - \langle\Lambda + q,\nu\rangle\|_F \leq \mathtt{R}\left(\|\Lambda\|_{B(F)} + 2\mathtt{R}\|q\|_{B(F\times F)}\right)\gamma_{\mathrm{mix}}(t), \tag{2.12}$$

where $\|\Lambda\|_{B(F)}$ is the norm of the linear form of Λ and $\|q\|_{B(F\times F)}$ is the norm of the bi-linear form of q. Note that, actually, $\langle\Lambda,\nu\rangle = 0$ by (2.7). The factor $\mathtt{R}$ in front of $\|q\|_{B(F\times F)}$ in (2.12) is due to the decomposition (recall that $\mathbf{e}_0 = \mathbf{e}$ a.s. and $\mathbf{e}_0'$ has law ν)

$$\begin{aligned} e^{tA}q(\mathbf{e}) - \langle q,\nu\rangle = \mathbb{E}\left[q(E_t^*(\mathbf{e}_0), E_t^*(\mathbf{e}_0)) - q(E_t^*(\mathbf{e}_0), \mathbb{E}_t^*(\mathbf{e}_0'))\right] \\ + \mathbb{E}\left[q(E_t^*(\mathbf{e}_0), E_t^*(\mathbf{e}_0')) - q(E_t^*(\mathbf{e}_0'), E_t^*(\mathbf{e}_0'))\right]. \end{aligned}$$

We have indeed

$$\begin{aligned} |e^{tA}q(\mathbf{e}) - \langle q,\nu\rangle| &\leq \|q\|_{B(F\times F)}\mathbb{E}\left[(\|E_t^*(\mathbf{e}_0)\|_F + \|E_t^*(\mathbf{e}_0')\|_F)\|E_t^*(\mathbf{e}_0) - E_t^*(\mathbf{e}_0')\|_F\right] \\ &\leq 2\mathtt{R}\|q\|_{B(F\times F)}\mathbb{E}\|E_t^*(\mathbf{e}_0) - E_t^*(\mathbf{e}_0')\|_F \quad \text{by (2.6)}, \\ &\leq 2\mathtt{R}^2\|q\|_{B(F\times F)}\gamma_{\mathrm{mix}}(t) \quad \text{by (2.9)}. \end{aligned}$$

Without loss of generality (as we can rescale γ_{mix} if we rescale $\mathtt{R}$), we assume

$$\|\gamma_{\mathrm{mix}}\|_{L^1(\mathbb{R}_+)} = 1. \tag{2.13}$$

Using (2.11), the resolvent

$$R_\lambda\varphi(\mathbf{e}) := \int_0^\infty e^{-\lambda t}\left(e^{tA}\varphi\right)(\mathbf{e})dt, \tag{2.14}$$

is well defined for all $\lambda \geq 0$, $\mathbf{e} \in \bar{B}_{\mathtt{R}}$ and all $\varphi\colon F \to \mathbb{R}$ which is Lipschitz continuous and satisfies the cancellation condition $\langle\varphi,\nu\rangle = 0$. Using (2.7), we can therefore define $R_\lambda\varphi_h(\mathbf{e})$ for $\lambda \geq 0$, where $\varphi_h(e) = \langle e, h\rangle_{L^2(\mathbb{T}^N)}$. Moreover by (2.11), there exists $T_\lambda\colon F \to F$ such that $R_\lambda\varphi_h(\mathbf{e}) = \langle T_\lambda(e), h\rangle_{L^2(\mathbb{T}^N)}$. By a slight abuse of notation, we write $R_\lambda(\mathbf{e}) = T_\lambda(e)$. By (2.9) (with $\mathbf{e}_0 = \mathbf{e}$ a.s. and $\mathbf{e}_0' \sim \nu$) and (2.13), we have

$$\|R_0(\mathbf{e})\|_F \leq \mathtt{R}, \tag{2.15}$$

for all $\mathbf{e}$ with $\|\mathbf{e}\|_F \leq \mathtt{R}$. Eventually, let $\Lambda\colon F \to \mathbb{R}$ be a linear functional. Then, with the notations above, $\varphi_\Lambda := \Lambda \circ R_0$ is a map $F \to \mathbb{R}$. The generator A acts on φ_Λ and on the square of φ_Λ and we will assume that there exists a constant $C^0_{\mathtt{R}} \geq 0$ such that the following bounds are satisfied:

$$|[A|\varphi_\Lambda|^2](\mathbf{e})| \leq C^0_{\mathtt{R}}\|\Lambda\|^2_{B(F)}, \quad |[A\varphi_\Lambda](\mathbf{e})| \leq C^0_{\mathtt{R}}\|\Lambda\|_{B(F)}, \tag{2.16}$$

for all $\mathbf{e}$ with $\|\mathbf{e}\|_F \leq \mathtt{R}$.

2.2 Covariance

Our mixing hypothesis has the following consequence on the covariances of (E_t) and $(\bar{E}_t)$: let

$$\Gamma_{\mathbf{e}}(s,t) = \mathbb{E}\left[E_s(\mathbf{e}) \otimes E_t(\mathbf{e})\right], \quad \bar{\Gamma}(t) = \mathbb{E}\left[\bar{E}(t) \otimes \bar{E}(0)\right]. \tag{2.17}$$

Let $t \geq s \geq r \geq 0$. Conditioning on $\mathcal{G}^E_{t-s}$, we have

$$\Gamma_{\mathbf{e}}(t-r, t-s) = e^{(t-s)A}(e^{(s-r)A}\theta \otimes \theta)(\mathbf{e}), \quad \theta(\mathbf{e}) = \mathbf{e}$$

It follows from (2.12) that, for all $\mathbf{e}$ with $\|\mathbf{e}\|_F \leq \mathtt{R}$,

$$\|\Gamma_{\mathbf{e}}(t-r, t-s) - \bar{\Gamma}(s-r)\|_F \leq 2\mathtt{R}^2 \gamma_{\mathrm{mix}}(t-s). \tag{2.18}$$

2.3 Some simple examples

Let $(E_n(\mathbf{e}))_{n\geq 0}$ be a Markov chain on F with $E_0(\mathbf{e}) = \mathbf{e}$, and let $(N_t)_{t\geq 0}$ be a Poisson process of rate 1 ($N_0 = 0$) independent on (E_n). We assume that the ball $\bar{B}_{\mathtt{R}}$ of F is stable by (E_n), that $(E_n(\mathbf{e}))_{n\geq 0}$ has the invariant measure ν and the mixing property

$$\mathbb{E}\|E^*_n(\mathbf{e}_0) - E^*_n(\mathbf{e}'_0)\| \leq C\mathtt{R}\gamma^n, \tag{2.19}$$

where $\gamma < 1$ for a coupling $(E^*_n(\mathbf{e}_0), E^*_n(\mathbf{e}'_0))$ of $(E_n(\mathbf{e}_0), E_n(\mathbf{e}'_0))$. Let

$$E(t,s;\mathbf{e}_0) = E_{N_{t-s}}(\mathbf{e}_0) \tag{2.20}$$

and let $\bar{E}_t = E(t,0;\bar{\mathbf{e}}_0)$, where $\bar{\mathbf{e}}_0$ is a random variable of law ν independent on $(E_n)_{n\geq 0}$ and $(N_t)_{t\geq 0}$. Then $(\bar{E}_t)$ is a stationary process (it is a time-homogeneous Markov process and is initially at equilibrium). It is càdlàg, it satisfies (2.6), (2.7) if ν is centred, and also (2.9) since

$$\begin{aligned}\mathbb{E}\|E^*_t(\mathbf{e}_0) - E^*_t(\mathbf{e}'_0)\|_F &= \sum_{n=0}^{\infty} \mathbb{P}(N_t = n)\mathbb{E}\|E^*_n(\mathbf{e}_0) - E^*_n(\mathbf{e}'_0)\|_F \\ &\leq C\mathtt{R}\sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}\gamma^n = C\mathtt{R}e^{-(1-\gamma)t} =: \mathtt{R}\gamma_{\mathrm{mix}}(t).\end{aligned}$$

Let us simplify still by considering the situation where $E_{n+1}(\mathbf{e})$ is drawn *independently* on $E_n(\mathbf{e})$, with law ν. We can then consider the synchronous coupling $(E^*_n(\mathbf{e}_0), E^*_n(\mathbf{e}'_0))$ of $(E_n(\mathbf{e}_0), E_n(\mathbf{e}'_0))$ which is such that $E^*_n(\mathbf{e}_0) = E^*_n(\mathbf{e}'_0)$ for all $n \geq 1$. It gives us

$$\mathbb{E}\|E^*_t(\mathbf{e}_0) - E^*_t(\mathbf{e}'_0)\|_F \leq 2\mathtt{R}\mathbb{P}(N_t = 0) = 2\mathtt{R}e^{-t}.$$

In addition, the semi-group, generator and resolvent R_0 have the explicit forms

$$e^{tA}\varphi(e) = e^{-t}\varphi(\mathbf{e}) + (1 - e^{-t})\langle\varphi, \nu\rangle,$$

and

$$A\varphi(\mathbf{e}) = \langle\varphi, \nu\rangle - \varphi(\mathbf{e}), \quad R_0\varphi(\mathbf{e}) = \mathbf{e}.$$

From these formula, we deduce the second inequality in (2.16) with $C^0_{\mathtt{R}} \geq \mathtt{R}$. The first inequality in (2.16) is obtained with any $C^0_{\mathtt{R}} \geq 2\mathtt{R}^2$.

An other instance of mixing force field is a function $\eta(X_t)$, $\eta\colon \mathbb{R}^m \to F$, of an Ornstein-Uhlenbeck process (X_t) on $\mathbb{R}^m$:

$$dX_t = -X_t dt + \sqrt{2} dB_t, \tag{2.21}$$

where (B_t) is a Wiener process on $\mathbb{R}^m$. We choose η Lipschitz and taking values in the ball $\bar{B}_{\mathtt{R}}$ of F. We will not develop that example much, but simply check that the mixing condition (2.9) is also satisfied here. Since η is Lipschitz, it is sufficient to check it directly on (X_t). We use once again a synchronous coupling: let

$$X^*(t; X_0) = e^{-t} X_0 + \sqrt{2} \int_0^t e^{-(t-s)} dB_s, \quad X^*(t; X_0') = e^{-t} X_0' + \sqrt{2} \int_0^t e^{-(t-s)} dB_s$$

and let $\gamma_{\text{mix}}(t) = C_{\text{mix}} e^{-t}$, where C_{mix} is a constant. Then

$$e^t |X^*(t; X_0) - X^*(t; X_0')| \leq |X_0| + |X_0'|,$$

hence (2.9) is satisfied provided we limit ourselves to initial laws μ and μ' with first moment below a given threshold. This is not a limitation since the invariant measure associated to (2.21), which is Gaussian, has a finite first moment.

2.4 Mixing force process

In Theorem 1.1 and Section 5, we consider the case where $\bar{E}_t$ is independent on x. This means that the state space is $\mathbb{R}^N$, and $(\bar{E}_t)$ is simply a process on $\mathbb{R}^N$. In this simpler framework, the notion of mixing force field is reduced to the following notion of mixing force process.

Definition 2.2 (Mixing force process)**.** Let $(\bar{E}_t)$ be a càdlàg, stationary, homogeneous Markov process of generator A over $\mathbb{R}^N$. We say that $(\bar{E}_t)$ is a mixing force process if the conditions (2.22), (2.23), (2.24) below are satisfied.

Condition (2.22) is the condition of localization

$$|E(t; \mathbf{e})| \leq \mathtt{R}, \tag{2.22}$$

almost-surely, for all $t \geq 0$, for all $\mathbf{e} \in \bar{B}_{\mathtt{R}}$, where $\bar{B}_{\mathtt{R}}$ is the closed ball of center 0 and radius $\mathtt{R}$ in $\mathbb{R}^N$. We require then that the invariant measure ν of (E_t) is supported in $\bar{B}_{\mathtt{R}}$ and that

$$\int_{\mathbb{R}^N} \mathbf{e}\, d\nu(\mathbf{e}) = \mathbb{E}[\bar{E}_t] = 0. \tag{2.23}$$

The mixing hypothesis is

$$\mathbb{E}\|E^*_t(\mathbf{e}_0) - E^*_t(\mathbf{e}_0')\| \leq \mathtt{R}\gamma_{\text{mix}}(t), \tag{2.24}$$

like in (2.9), except that now, $\mathbf{e}_0$ and $\mathbf{e}_0'$ are random variables on $\mathbb{R}^N$. Note in particular that $F \subset C_b(\mathbb{T}^N; \mathbb{R}^N)$. Therefore, if $(\bar{E}_t)$ is a mixing force field, then, for each $x \in \mathbb{T}^N$, $(\bar{E}_t(x))$ is a mixing force process.

3 Unperturbed equation: ergodic properties

We consider first the equation

$$\partial_t f_t + \bar{E}(t) \cdot \nabla_v f_t = Q f_t \quad t > 0, v \in \mathbb{R}^N, \tag{3.1}$$

where $Q = Q_{\mathrm{LB}}$ or $Q = Q_{\mathrm{FP}}$. In (3.1), $\bar{E}(t)$ should stand for $\bar{E}(x,t)$, where $(\bar{E}(t))$ is a mixing force field, since (3.1) is the instance of Equation 1.1 obtained for $\varepsilon = 0$. However, x is just a parameter and we may as well consider that $(\bar{E}(t))$ is a mixing force process. Thus, *in all this section*, $(\bar{E}(t))$ is a mixing force process in the sense of Definition 2.2.

To find the invariant measure for (3.1), we solve the equation starting from a given time $s \in \mathbb{R}$, and then let $s \to -\infty$. More precisely, given $\mathbf{e} \in \mathbb{R}^N$, we consider the following evolution equation:

$$\partial_t f_t + E(t,s;\mathbf{e}) \cdot \nabla_v f_t = Q f_t \quad t > s, v \in \mathbb{R}^N. \tag{3.2}$$

Let $f \in L^1(\mathbb{R}^N)$ and $s \in \mathbb{R}$. The solution to (3.2) with initial condition $f_{t=s} = f$ is

$$\begin{aligned} f^{\mathrm{LB}}_{s,t}(v) = e^{-(t-s)} f\left(v - \int_s^t E(r,s;\mathbf{e}) dr\right) \\ + \rho(f) \int_s^t e^{-(t-\sigma)} \left[M\left(v - \int_\sigma^t E(r,s;\mathbf{e})dr\right)\right] d\sigma, \end{aligned} \tag{3.3}$$

when $Q = Q_{\mathrm{LB}}$, and

$$f^{\mathrm{FP}}_{s,t}(v) = e^{N(t-s)} \int_{\mathbb{R}^N} f\left(e^{(t-s)} v - \int_s^t e^{-(s-\sigma)} E(\sigma,s,\mathbf{e}) d\sigma + \sqrt{e^{2(t-s)} - 1}\, w\right) M(w) dw, \tag{3.4}$$

when $Q = Q_{\mathrm{FP}}$. A brief explanation to (3.3) and (3.4) is given in Appendix A. By the term "solution to (3.2)", we mean weak solutions, *i.e.* functions $f \in C([s,+\infty); L^1(\mathbb{R}^N))$ satisfying the identity

$$\langle f_t, \varphi \rangle = \langle f, \varphi \rangle + \int_s^t \langle f_\sigma, E(\sigma,t;\mathbf{e}) \cdot \nabla_v \varphi \rangle + \langle f_\sigma, Q^* \varphi \rangle d\sigma,$$

almost-surely, for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, for all $t \geq s$. We may also consider mild solutions (this is equivalent, actually), as we do in Section 4. We do not need to be very specific on that point here. All that matter to us is to understand the limit behaviour of $f_{s,t}$ defined by (3.3)-(3.4) when $s \to -\infty$. This is the content of the following result, Theorem 3.1.

Theorem 3.1 (Invariant solutions)**.** *Let $(\bar{E}(t))$ be a mixing force process in the sense of Definition 2.2. Let $f^{\mathrm{LB}}_{s,t}$ and $f^{\mathrm{FP}}_{s,t}$ be defined by (3.3) and (3.4) respectively, with $\mathbf{e} \in \bar{B}_{\mathrm{R}}$. Then*

$$(f^{\mathrm{LB}}_{s,t}, E(t,s;\mathbf{e})) \to (\rho(f)\bar{M}^{\mathrm{LB}}_t, \bar{E}_t) \text{ and } (f^{\mathrm{FP}}_{s,t}, E(t,s;\mathbf{e})) \to (\rho(f)\bar{M}^{\mathrm{FP}}_t, \bar{E}_t) \tag{3.5}$$

in law on $L^1(\mathbb{R}^N)\times\mathbb{R}^N$ when $s\to-\infty$, where $\bar{M}_t^{\mathrm{LB}}$ and $\bar{M}_t^{\mathrm{FP}}$ are defined by

$$\bar{M}_t^{\mathrm{LB}} = \int_{-\infty}^{t} e^{-(t-\sigma)}\left[M\left(v-\int_\sigma^t \bar{E}(r)dr\right)\right]d\sigma, \tag{3.6}$$

and

$$\bar{M}_t^{\mathrm{FP}} = M\left(v-\int_{-\infty}^{t} e^{-(t-r)}\bar{E}(r)dr\right), \tag{3.7}$$

respectively.

We denote by μ_ρ the invariant measure (parametrized by ρ) defined by

$$\langle\varphi,\mu_\rho\rangle = \mathbb{E}\varphi(\rho\bar{M}_t,\bar{E}_t), \tag{3.8}$$

for all continuous and bounded function φ on $L^1(\mathbb{R}^N)\times\mathbb{R}^N$.

Remark 3.1. We call $\bar{M}_t^{\mathrm{LB}}$ and $\bar{M}_t^{\mathrm{FP}}$ the invariant solutions, since their laws are the invariant measure for (3.1). Note that $(\bar{E}(r))$ in (3.6) and (3.7) is defined for all $r\in\mathbb{R}$ (see the discussion and convention of notations around (2.4)).

Remark 3.2. Let φ be a bounded continuous function on $\mathbb{R}^N\times\mathbb{R}^N$. Similarly to (1.12), we have, by conditioning on the natural filtration $(\mathcal{F}_t^E)$ of (E_t):

$$\mathbb{E}\left[\varphi(V_{s,t},E(t,s;\mathbf{e}))\right] = \mathbb{E}\int_{\mathbb{R}^N} f_{s,t}(v)\varphi(v,E(t,s;\mathbf{e}))dv, \tag{3.9}$$

where $V_{s,t}$ is the solution to (1.9) or (1.10) (with $\bar{E}(t)$ instead of $\bar{E}(t,X_t)$) starting from V_s at time $t=s$, where V_s follows the law of density f with respect to the Lebesgue measure on $\mathbb{R}^N$. Since

$$\Phi\colon (f,\mathbf{e})\mapsto \int_{\mathbb{R}^N} f(v)\varphi(v,\mathbf{e})dv$$

is continuous and bounded on $L^1(\mathbb{R}^N)\times\mathbb{R}^N$, we deduce from Theorem 3.1 that

$$\lim_{s\to-\infty}\mathbb{E}\left[\varphi(V_{s,t},E(t,s;\mathbf{e}))\right] = \langle\lambda_\rho,\varphi\rangle := \rho\mathbb{E}\int_{\mathbb{R}^N}\bar{M}_t(v)\varphi(v,\bar{E}_t)dv, \tag{3.10}$$

where $\rho=\rho(f)$.

The proof of Theorem 3.1 uses the estimates (3.13) and (3.14) in the following lemma.

Lemma 3.2. *For $w,z\in\mathbb{R}^N$, we have the estimates and identities*

$$\|M(\cdot-w)\|^2_{L^2(M^{-1})} = e^{|w|^2}, \tag{3.11}$$

$$\|M(\cdot-w)-M(\cdot-z)\|^2_{L^2(M^{-1})} = e^{|w|^2}+e^{|z|^2}-2e^{w\cdot z}, \tag{3.12}$$

in $L^2(M^{-1})$, and

$$\|M(\cdot-w)\|_{L^1(\mathbb{R}^N)} = 1, \tag{3.13}$$

$$\|M(\cdot-w)-M(\cdot-z)\|_{L^1(\mathbb{R}^N)} \le 2\wedge\left[\frac{|w-z|}{(1-|w-z|)^+}\right]^{1/2} \tag{3.14}$$

in $L^1(\mathbb{R}^N)$.

Proof of Lemma 3.2. Standard manipulations and identities for Gaussian densities give (3.11), (3.12) and (3.13) (one can also use (3.15) below to prove (3.11) and (3.12)). By (3.13) and the triangular inequality, we have the bound by 2 in (3.14). To obtain the second estimate, we use the identity

$$\|M(\cdot - w) - M(\cdot - z)\|_{L^1(\mathbb{R}^N)} = \|M(\cdot - w + z) - M\|_{L^1(\mathbb{R}^N)},$$

and the expansion

$$M(v - w) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{|v-w|^2}{2}} = M(w) \sum_{n \in \mathbb{N}^N} H_n(v) w^n, \tag{3.15}$$

where H_n is the n-th Hermite polynomial (see [15, Section 1.1.1]). This yields the inequality

$$\|M(\cdot - w) - M\|_{L^1(\mathbb{R}^N)} \leq M(w) \sum_{n \in \mathbb{N}^N \setminus \{0\}} \|H_n\|_{L^1(\mathbb{R}^N)} |w|^n.$$

Since $\|H_n\|_{L^1(\mathbb{R}^N)} \leq \|H_n\|_{L^2(M^{-1})} = \frac{1}{\sqrt{n!}}$ (*cf.* [15, Lemma 1.1.1]), the Cauchy-Schwarz inequality yields, for $|w| < 1$,

$$\|M(\cdot - w) - M\|_{L^1(\mathbb{R}^N)} \leq M(w) \left[\frac{e^{|w|}|w|}{1 - |w|}\right]^{1/2} \leq \left[\frac{|w|}{1 - |w|}\right]^{1/2}.$$

Indeed, setting $a = |w|$, we have $a \in [0, 1]$ and

$$M(w) e^{|w|/2} = \left[\frac{1}{(2\pi)^N} e^{a - a^2}\right]^{1/2} \leq \left[\frac{1}{(2\pi)^N} e^{1/4}\right]^{1/2} \leq 1$$

since $e^{1/4} \leq 2\pi$. □

Proof of Theorem 3.1. Let $\mathbf{e} \in \bar{B}_{\mathrm{R}}$ $t \in \mathbb{R}$, let $\Phi \colon L^1(\mathbb{R}^N) \times F \to \mathbb{R}$ be a bounded and uniformly continuous function and let $\varepsilon > 0$. Our aim is to show that

$$|\mathbb{E}\Phi(f_{s,t}(v), E(t, s; \mathbf{e})) - \mathbb{E}\Phi(\rho \bar{M}_t, \bar{E}_t)| < K\varepsilon, \tag{3.16}$$

for $s < \min(0, t)$, $|s|$ large enough, where K is a finite constant (it will turn out that $K = 5$, but this does not matter). Note that it is sufficient to consider uniformly continuous functions in (3.16), *cf.* Proposition I-2.4 in [11]. We denote by η a modulus of uniform continuity of Φ associated to ε.

Step 1. Reduction to the case $f \in L^2(M^{-1})$. The maps $f \mapsto f_{s,t}$, $f \mapsto \rho(f)\bar{M}_t$ are continuous on L^1, uniformly in $s \leq t$:

$$\|f_{s,t}\|_{L^1(\mathbb{R}^N)}, \|\rho(f)\bar{M}_t^{\mathrm{LB}}\|_{L^1(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)}.$$

Using the uniform continuity of Φ on K, we have

$$|\mathbb{E}\Phi(f_{s,t}(v),E(t,s;\mathbf{e}))-\mathbb{E}\Phi(\rho\bar{M}_t,\bar{E}_t)|<2\varepsilon+|\mathbb{E}\Phi((\tilde{f})_{s,t},E(t,s;\mathbf{e}))-\mathbb{E}\Phi(\rho(\tilde{f})\bar{M}_t,\bar{E}_t)|$$

if $\|f-\tilde{f}\|_{L^1(\mathbb{R}^N)}<\eta$. Therefore, to prove (3.16), we turn to the case $f\in L^2(M^{-1})$.

Step 2. Cut-off after time s. For $s\leq t$, introduce

$$\bar{M}^{\mathrm{LB}}_{s,t}=\int_s^t e^{-(t-\sigma)}\left[M\left(v-\int_\sigma^t\bar{E}(r)dr\right)\right]d\sigma, \tag{3.17}$$

and

$$\bar{M}^{\mathrm{FP}}_{s,t}=M\left(v-\int_s^t e^{-(t-r)}\bar{E}(r)dr\right). \tag{3.18}$$

We have $\|\bar{M}^{\mathrm{LB}}_{s,t}-\bar{M}^{\mathrm{LB}}_t\|_{L^1(\mathbb{R}^N)}\leq e^{-(t-s)}$ by a direct computation and

$$\|\bar{M}^{\mathrm{FP}}_{s,t}-\bar{M}^{\mathrm{FP}}_t\|_{L^1(\mathbb{R}^N)}\leq b\left(\int_{-\infty}^s e^{-(t-r)}\bar{E}(r)dr\right)$$

where $b(|w-z|)$ is the right-hand side of (3.14). We use the bound

$$b(r)\leq\frac{\sqrt{5}}{2}r^{1/2} \tag{3.19}$$

and (2.6) to obtain, almost-surely, $\|\bar{M}^{\mathrm{FP}}_{s,t}-\bar{M}^{\mathrm{FP}}_t\|_{L^1(\mathbb{R}^N)}\leq\frac{\sqrt{5}}{2}\mathtt{R}^{1/2}e^{-\frac{1}{2}(t-s)}$. To sum up, in both the LB and FP case, we have a bound almost-sure on $\|\bar{M}_{s,t}-\bar{M}_t\|_{L^1(\mathbb{R}^N)}$ by a deterministic quantity which tends to 0 when $t-s\to+\infty$. It follows that, for $t-s$ large enough,

$$|\mathbb{E}\Phi(\rho(f)\bar{M}_t)-\mathbb{E}\Phi(\rho(f)\bar{M}_{s,t},\bar{E}_t)|<\varepsilon.$$

In the next step we prove that

$$|\mathbb{E}\Phi(f_{s,t},E(t,s;\mathbf{e}))-\mathbb{E}\Phi(\rho(f)\bar{M}_{s,t})|<2\varepsilon, \tag{3.20}$$

for $t-s$ large enough.

Step 3. Convergence in law. Let $\mathbf{e}\in\bar{B}_{\mathtt{R}}$. Let $\mathbf{e}_0=\mathbf{e}$ a.s. and $\mathbf{e}'_0=\bar{E}_s$. Since $E(s,t;\mathbf{e})$ has the same law as $E_{t-s}(\mathbf{e}_0)$ and $\bar{E}(t)$ has the same law as $E_{t-s}(\mathbf{e}'_0)$, (2.24) gives a coupling

$$(E(s,t;\mathbf{e}),\bar{E}(t))_{t\geq s}\to(E^*(s,t;\mathbf{e}),\bar{E}^*_t)_{t\geq s}$$

such that

$$\mathbb{E}\|E^*(t,s;\mathbf{e})-\bar{E}^*_t\|_F\leq\mathtt{R}\gamma_{\mathrm{mix}}(t-s), \tag{3.21}$$

for all $t\geq s$. We have

$$\mathbb{E}\Phi(f_{s,t},E(t,s;\mathbf{e}))-\mathbb{E}\Phi(\rho(f)\bar{M}_{s,t},\bar{E}_t)=\mathbb{E}\Phi(f^*_{s,t},E^*(s,t;\mathbf{e}))-\mathbb{E}\Phi(\rho(f)\bar{M}^*_{s,t},\bar{E}^*_t), \tag{3.22}$$

where the superscript star in $f_{s,t}$ and $\bar{M}_{s,t}$ indicates that $E(s,t;\mathbf{e})$ has been replaced by $E^*(s,t;\mathbf{e})$ and $\bar{E}(t)$ by $\bar{E}^*_t$. Since

$$
\begin{aligned}
&|\mathbb{E}\Phi(f^*_{s,t}, E^*(s,t;\mathbf{e})) - \mathbb{E}\Phi(\rho(f)\bar{M}^*_{s,t}, \bar{E}^*_t)| \\
&\qquad \leq \varepsilon + \|\Phi\|_{BC}\left[\mathbb{P}(\|f^*_{s,t} - \rho(f)\bar{M}^*_{s,t}\|_{L^1(\mathbb{R}^N)} > \eta) + \mathbb{P}(\|E^*(s,t;\mathbf{e}) - \bar{E}^*_t\|_F > \eta)\right],
\end{aligned}
$$

it is sufficient to prove that $f^*_{s,t} - \rho(f)\bar{M}^*_{s,t} \to 0$ and $E^*(s,t;\mathbf{e}) - \bar{E}^*_t \to 0$ in probability on $L^1(\mathbb{R}^N)$ and F respectively. We show the strongest (strongest, as is proved classically by means of the Markov inequality) property

$$
\lim_{s\to-\infty} \mathbb{E}\|f^*_{s,t} - \rho(f)\bar{M}^*_{s,t}\|_{L^1(\mathbb{R}^N)} = 0, \qquad \lim_{s\to-\infty} \mathbb{E}\|E^*(s,t;\mathbf{e}) - \bar{E}^*_t\|_F = 0. \tag{3.23}
$$

The second limit in (3.23) is a consequence of (3.21). Let us prove the first limit. Consider first the LB case. Using (3.13) and the estimate $|\rho(f)| \leq \|f\|_{L^1(\mathbb{R}^N)}$, we have

$$
\begin{aligned}
\mathbb{E}\|f^{\mathrm{LB},*}_{s,t} - \rho(f)\bar{M}^{\mathrm{LB},*}_{s,t}\|_{L^1(\mathbb{R}^N)} &\leq \|f\|_{L^1(\mathbb{R}^N)} e^{-(t-s)} \\
&\quad + \|f\|_{L^1(\mathbb{R}^N)}\mathbb{E}\int_s^t e^{-(t-\sigma)} b\left(\int_\sigma^t |E^*(r,s,\mathbf{e}) - \bar{E}^*(r)|dr\right) d\sigma,
\end{aligned}
$$

where, as in (3.19), we denote by $b(|w-z|)$ the right-hand side of (3.14). From (3.19) follows

$$
2b(r) \leq \varepsilon + \frac{5}{4\varepsilon} r.
$$

We deduce the estimate

$$
\begin{aligned}
\mathbb{E}\|f^{\mathrm{LB},*}_{s,t} - \rho(f)\bar{M}^{\mathrm{LB},*}_{s,t}\|_{L^1(\mathbb{R}^N)} &\leq \|f\|_{L^1(\mathbb{R}^N)}(e^{-(t-s)} + \varepsilon) \\
&\quad + \frac{5}{4\varepsilon}\|f\|_{L^1(\mathbb{R}^N)} \int_s^t e^{-(t-r)}\mathbb{E}|E^*(r,s,\mathbf{e}) - \bar{E}^*(r)|dr.
\end{aligned}
$$

By (3.21), this yields the following estimate:

$$
\begin{aligned}
\mathbb{E}\|f^{\mathrm{LB},*}_{s,t} - \rho(f)\bar{M}^{\mathrm{LB},*}_{s,t}\|_{L^1(\mathbb{R}^N)} &\leq \|f\|_{L^1(\mathbb{R}^N)}\left(e^{-(t-s)} + \varepsilon + \frac{5\mathtt{R}}{4\varepsilon}\int_s^t e^{-(t-r)}\gamma_{\mathrm{mix}}(t-r)dr\right) \\
&= \|f\|_{L^1(\mathbb{R}^N)}\left(e^{-(t-s)} + \varepsilon + \frac{5\mathtt{R}}{4\varepsilon}\int_0^{t-s} e^{r-(t-s)}\gamma_{\mathrm{mix}}(r)dr\right).
\end{aligned} \tag{3.24}
$$

We fix r_1 such that $\frac{5}{4}\mathtt{R}\int_{r_1}^\infty \gamma_{\mathrm{mix}}(r)dr < \varepsilon^2$. Then

$$
\frac{5}{4}\mathtt{R}\int_0^{t-s} e^{r-(t-s)}\gamma_{\mathrm{mix}}(r)dr \leq \varepsilon^2 + \frac{5}{4}\mathtt{R}\int_0^{r_1}\gamma_{\mathrm{mix}}(r)dr\, e^{r_1-(t-s)} < 2\varepsilon^2
$$

for $t-s$ large enough and (3.23) follows from (3.24). In the FP case, we start first from the exponential estimate

$$
\|f^{\mathrm{FP}}_{s,t}|_{E\equiv 0} - \rho(f)M\|_{L^2(M^{-1})} \leq e^{s-t}\|f\|_{L^2(M^{-1})}. \tag{3.25}
$$

In (3.25), $f^{\rm FP}_{s,t}|_{E\equiv 0}$ denote the function (3.4) obtained when $E\equiv 0$. The estimate (3.25) is a consequence of the dual estimate in $L^2(M)$ for functions h such that $\langle h, M\rangle_{L^2(\mathbb{R}^N)}=0$, *cf.* [1, p. 179]. It implies

$$\|f^{\rm FP}_{s,t}|_{E\equiv 0}-\rho(f)M\|_{L^1(\mathbb{R}^N)}\leq e^{s-t}\|f\|_{L^2(M^{-1})}. \tag{3.26}$$

The translations

$$v\mapsto v-\int_s^t e^{-(t-\sigma)}\tilde{E}(\sigma,s,\mathbf{e})d\sigma,\quad v\mapsto v-\int_s^t e^{-(t-\sigma)}\tilde{E}^*_s(\sigma)d\sigma,$$

leave invariant the L^1-norm. Therefore (3.26) yields

$$\begin{aligned}&\mathbb{E}\|f^{\rm FP,*}_{s,t}-\rho(f)\bar{M}^{\rm FP,*}_t\|_{L^1(\mathbb{R}^N)}\leq e^{s-t}\|f\|_{L^2(M^{-1})}\\ &+|\rho(f)|\mathbb{E}\left\|M\left(\cdot-\int_s^t e^{-(t-\sigma)}\bar{E}^*(\sigma)d\sigma\right)-M\left(\cdot-\int_s^t e^{-(t-\sigma)}E^*(\sigma,s,\mathbf{e})d\sigma\right)\right\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$

We conclude as in the case $Q=Q_{\rm LB}$ by means of (3.14). □

4 Resolution of the kinetic equation

We consider the resolution of the Cauchy problem of (1.1) or (1.7) at fixed $\varepsilon>0$. We set $\varepsilon=1$ for simplicity. Then (1.1) and (1.7) are the same equation

$$\partial_t f+v\cdot\nabla_x f+\bar{E}(t,x)\cdot\nabla_v f=Qf. \tag{4.1}$$

More generally, what matters to us is the dynamics given by $(f,\mathbf{e})\mapsto(f_t,E_t(\mathbf{e}))$, where f_t is the solution to the equation

$$\partial_t f+v\cdot\nabla_x f+E(t,x)\cdot\nabla_v f=Qf, \tag{4.2}$$

with $E(t,x)=E_t(\mathbf{e}(x))$. Therefore, this is (4.2) which we solve. We simply assume that $t\mapsto E(t,\cdot)$ is a càdlàg function with values in F (see Section 2 for the definition of the state space F). In the particular case $E(t,x)=E_t(\mathbf{e}(x))$, we define in this way pathwise solutions. We solve the Cauchy Problem for (4.2) in the LB-case and in the FP-case in Section 4.1 and Section 4.2 respectively. Then, in Section 4.3, we establish the Markov property of the process $(f_t,E_t(\mathbf{e}))$, where the first component f_t is the solution to (4.2) with the forcing $E(t,x)=E_t(\mathbf{e}(x))$.

4.1 Cauchy Problem in the LB case

Let $t\mapsto E(t,\cdot)$ be a càdlàg function with values in F. Let $\Phi_t(x,v)=(X_t(x,v),V_t(x,v))$ denote the flow associated to the field $(v,E(t,x))$:

$$\begin{aligned}\dot{X}_t&=V_t,\quad X_0=x,\\ \dot{V}_t&=E(t,X_t),\quad V_0=v.\end{aligned}$$

The partial map $(x,v) \mapsto \Phi_t(x,v)$ is a C^1-diffeomorphism of $\mathbb{T}^N \times \mathbb{R}^N$. We denote by Φ^t the inverse application: $\Phi^t \circ \Phi_t = \mathrm{Id}$. Note that Φ^t and Φ_t preserve the Lebesgue measure on $\mathbb{T}^N \times \mathbb{R}^N$.

Definition 4.1 (Mild solution, LB case)**.** Let $f_{\mathrm{in}} \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$. Assume $Q = Q_{\mathrm{LB}}$. A continuous function from $[0,T]$ to $L^1(\mathbb{T}^N \times \mathbb{R}^N)$ is said to be a mild solution to (4.2) with initial datum f_{in} if

$$f(t) = e^{-t} f_{\mathrm{in}} \circ \Phi^t + \int_0^t e^{-(t-s)} [\rho(f(s))M] \circ \Phi^{t-s} ds, \tag{4.3}$$

for all $t \in [0,T]$.

Proposition 4.1 (The Cauchy Problem, LB case)**.** *Let $f_{\mathrm{in}} \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$. Assume (2.6). There exists a unique mild solution to (4.2) in $C([0,T]; L^1(\mathbb{T}^N \times \mathbb{R}^N))$ with initial datum f_{in}. It satisfies*

$$\|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \leq \|f_{\mathrm{in}}\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \quad \text{for all } t \in [0,T]. \tag{4.4}$$

If $f_{\mathrm{in}} \geq 0$, then $f(t) \geq 0$ for all $t \in [0,T]$ and (4.4) is an identity. In addition, if $f_{\mathrm{in}} \in W^{k,1}(\mathbb{T}^N \times \mathbb{R}^N)$ with $k \leq 2$, then

$$\|f\|_{L^\infty(0,T; W^{k,1}(\mathbb{T}^N \times \mathbb{R}^N))} \leq C(k, T, f_{\mathrm{in}}), \tag{4.5}$$

where the constant $C(k,T,f_{\mathrm{in}})$ depends on k, T, N, and on the norms

$$\sup_{t \in [0,T]} \|E(t,\cdot)\|_F \text{ and } \|f_{\mathrm{in}}\|_{W^{k,1}(\mathbb{T}^N \times \mathbb{R}^N)}$$

only. Eventually, if $f_{\mathrm{in}} \in G_m$, then $f(t) \in G_m$ for all $t \in [0,T]$.

Proof of Proposition 4.1. Let E_T denote the space of continuous functions from $[0,T]$ to $L^1(\mathbb{T}^N \times \mathbb{R}^N)$. We use the norm

$$\|f\|_{E_T} = \sup_{t \in [0,T]} \|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)}$$

on E_T. Note that

$$\|\rho(f)\|_{L^1(\mathbb{T}^N)} \leq \|f\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)}. \tag{4.6}$$

Let $f \in E_T$. Assume that (4.3) is satisfied. Then, by (4.6), we have

$$\|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \leq e^{-t} \|f_{\mathrm{in}}\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} + \int_0^t e^{-(t-s)} \|f(s)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} ds.$$

By Gronwall's Lemma applied to $t \mapsto e^t \|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)}$, we obtain (4.4) as an a priori estimate. Besides, the L^1-norm of the integral term in (4.3) can be estimated by $(1 - e^{-T})\|f\|_{E_T}$. Therefore existence and uniqueness of a solution to (4.3) in $L^1(\Omega; E_T)$ follow from the Banach fixed point Theorem. To obtain the additional regularity (4.5), we do the same kind of estimates on the system satisfied by the derivatives and incorporate these estimates in the fixed-point space. To conclude the proof, let us assume $f_{\text{in}} \geq 0$. Since $s \mapsto s^-$ (negative part) is convex and satisfies $(a+b)^- \leq a^- + b^-$, we deduce from (4.3) and the Jensen inequality that

$$f^-(t) \leq \int_0^t e^{-(t-s)}[\rho(f(s))M]^- \circ \Phi^{t-s} ds.$$

Since $M \geq 0$ and $\rho(f)^- \leq \rho(f^-)$, (4.6) yields the estimate

$$\|f^-(t)\|_{L^1(\mathbb{T}^N\times\mathbb{R}^N)} \leq \int_0^t e^{-(t-s)}\|f^-(s)\|_{L^1(\mathbb{T}^N\times\mathbb{R}^N)} ds.$$

We conclude to $f^- = 0$ by the Gronwall Lemma. Eventually, that $f_{\text{in}} \in G_m$ implies $f(t) \in G_m$ for all $t \in [0,T]$ (propagation of moments) is proved in Proposition 6.3. $\square$

4.2 Cauchy Problem in the FP case

Let $K_t(x,v;y,w)$ denote the kernel associated to the kinetic Fokker-Planck equation

$$\partial_t f = Q_{\text{FP}} f - v \cdot \nabla_x f. \tag{4.7}$$

Let us recall some elementary facts about K_t (see [4] for more results about the analytical properties of K_t, and [19] for the probabilistic interpretation of K_t). The function $K_t(\cdot;y,w)$ is the density with respect to the Lebesgue measure on $\mathbb{T}^N \times \mathbb{R}^N$ of the law $\mu_t^{(y,w)}$ of the solution (X_t, V_t) to the SDE

$$dX_t = V_t dt, \qquad X_0 = y, \tag{4.8}$$

$$dV_t = -V_t dt + \sqrt{2} dB_t, \qquad V_0 = w. \tag{4.9}$$

where B_t is a Wiener process over $\mathbb{R}^N$. Therefore

$$K_t f(x,v) := \iint_{\mathbb{T}^N\times\mathbb{R}^N} K_t(x,y;y,w) f(y,w) dy dw$$

satisfies the identity

$$\langle K_t f, \varphi\rangle = \iint_{\mathbb{T}^N\times\mathbb{R}^N} \mathbb{E}\varphi(X_t,V_t) f(y,w) dy dw, \tag{4.10}$$

for $f \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$ and $\varphi\colon \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ continuous and bounded. The solution to (4.8)-(4.9) is given explicitly by

$$\begin{aligned} X_t &= y + (1-e^{-t})w + \int_0^t (1-e^{-(t-s)}) dB_s, \\ V_t &= e^{-t} w + \int_0^t e^{-(t-s)} dB_s. \end{aligned} \tag{4.11}$$

The process (X^0_t, V^0_t) given by (4.11) when $y = 0$, $w = 0$ is a Gaussian process with covariance matrix

$$Q_t := \begin{pmatrix} \displaystyle\int_0^t |1-e^{-s}|^2 ds & \int_0^t e^{-s}(1-e^{-s})ds \\ \displaystyle\int_0^t e^{-s}(1-e^{-s})ds & \int_0^t e^{-2s}ds \end{pmatrix} \otimes \mathrm{I}_N. \tag{4.12}$$

Using (4.12) and (4.10)-(4.11), one can show that $K_t\colon L^p(\mathbb{T}^N \times \mathbb{R}^N) \to L^p(\mathbb{T}^N \times \mathbb{R}^N)$ with norm bounded by $e^{\frac{N}{p'}t}$. We have also the estimate

$$\iint_{\mathbb{T}^N \times \mathbb{R}^N} |\nabla_w K_t(x, v; y, w)| dx dv \le C t^{-1/2}, \tag{4.13}$$

for all $(y, w) \in \mathbb{T}^N \times \mathbb{R}^N$, $t \in [0, T]$, with a constant C independent on (y, w) and T. The estimate (4.13) also follows from the estimate between (26) and (27) that can be found in [4].

Definition 4.2 (Mild solution, FP case)**.** Let $t \mapsto E(t, \cdot)$ be a càdlàg function with values in F. Let $p \in [1, +\infty[$. Let $f_{\text{in}} \in L^p(\mathbb{T}^N \times \mathbb{R}^N)$. Assume $Q = Q_{\text{FP}}$. A continuous function from $[0, T]$ to $L^p(\mathbb{T}^N \times \mathbb{R}^N))$ is said to be a mild solution to (4.2) in L^p with initial datum f_{in} if

$$f(t) = K_t f_{\text{in}} + \int_0^t \nabla_w K_{t-s}[E(s)f(s)] ds, \tag{4.14}$$

for all $t \in [0, T]$.

Proposition 4.2 (The Cauchy Problem, FP case)**.** *Let $t \mapsto E(t, \cdot)$ be a càdlàg function with values in F. Let $p \in [1, +\infty[$. Let $f_{\text{in}} \in L^p(\mathbb{T}^N \times \mathbb{R}^N)$. Then (4.2) has a unique mild solution f in L^p with initial datum f_{in}. If $f_{\text{in}} \ge 0$, then $f(t) \ge 0$, for all $t \in [0, T]$. In addition, for every $k \le 2$, the regularity $W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N))$ is propagated:*

$$\sup_{t \in [0,T]} \|f(t)\|_{W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N)} \le C(k, T) \|f_{\text{in}}\|_{W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N)}, \tag{4.15}$$

where the constant $C(k, T)$ depends on k, T, N and $\sup_{t \in [0,T]} \|E(t, \cdot)\|_F$. If $p = 1$ and $f_{\text{in}} \ge 0$, then $\|f(t)\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} = \|f_{\text{in}}\|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)}$. If, more generally, there is no sign condition on $f_{\text{in}} \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$, then (4.4) is satisfied. Eventually, if $f_{\text{in}} \in G_m$, then $f(t) \in G_m$ for all $t \in [0, T]$.

Proof of Proposition 4.2. The existence-uniqueness follows from the Banach fixed point Theorem using (4.13), in a manner similar to the proof of Proposition 4.1. To obtain (4.15) for $k = 1$, we assume first that $f(t)$ is in $W^{k,p}(\mathbb{T}^N \times \mathbb{R}^N)$ for all t and we use the relations

$$\begin{aligned} \nabla_x K_t(x, v; y, w) &= -\nabla_y K_t(x, v; y, w), \\ \nabla_v K_t(x, v; y, w) &= -(1 - e^{-t}) \nabla_y K_t(x, v; y, w) - e^{-t} \nabla_w K_t(x, v; y, w), \end{aligned}$$

and Gronwall's Lemma, to obtain (4.15). We can drop the a priori requirement that $f(t)$ is in $W^{k,p}(\mathbb{T}^N\times\mathbb{R}^N)$ for all t either by incorporating this in the fixed-point space, or by working with differential quotients. The case $k=2$ is obtained similarly. To prove that $f_{\rm in}\geq 0$ implies $f(t)\geq 0$, we use a duality argument: it is sufficient to prove the propagation of the sign for L^∞ solutions to the dual equation

$$\varphi(T)=\psi, \tag{4.16}$$
$$\partial_t\varphi=-v\cdot\nabla_x\varphi-\bar{E}_t\cdot\nabla_v\varphi-Q_{\rm FP}^*\varphi,\quad 0<t<T. \tag{4.17}$$

This follows from the maximum principle, since $Q_{\rm FP}^*\varphi=\Delta_v\varphi-v\cdot\nabla_v\varphi$. The maximum principle for the solutions to (4.16)-(4.17) also yields the L^1-estimate (4.4). The propagation of moments is proved in Proposition 6.3. □

4.3 Markov property

We prove the following result.

Theorem 4.3 (Markov property)**.** *Let $(\bar{E}(t))$ be a mixing force field in the sense of Definition 2.1. We denote by A the generator of $(\bar{E}_t)$. Let $\mathcal{X}$ denote the state space*

$$\mathcal{X}=L^1(\mathbb{T}^N\times\mathbb{R}^N)\times F. \tag{4.18}$$

For $(f,\mathbf{e})\in\mathcal{X}$, let f_t denote the mild solution to (4.2) with initial datum f and forcing $E_t(\mathbf{e})$. Then $(f_t,E_t(\mathbf{e}))_{t\geq 0}$ is a time-homogeneous Markov process over $\mathcal{X}$.

Proof of Theorem 4.3. For a synthetic treatment of the proof, we use the following notations:

$$G=L^1(\mathbb{T}^N\times\mathbb{R}^N),\quad H=W^{2,1}(\mathbb{T}^N\times\mathbb{R}^N).$$

For Φ bounded and measurable on $\mathcal{X}$, $\mathbf{e}\in F$, define

$$P_t\Phi(f,\mathbf{e})=\mathbb{E}_{(f,\mathbf{e})}\Phi(f_t,E_t(\mathbf{e})). \tag{4.19}$$

Let $(\mathcal{G}_t)$ be the filtration defined in (2.5). The process $(f_t,E_t(\mathbf{e}))$ is $(\mathcal{G}_t)$-adapted. To prove this assertion, note that f_t is obtained as the solution of a fixed-point equation (see the proof of Proposition 4.1 and Proposition 4.2 respectively). Consequently (f_t) is the limit of the sequence obtained by iterating the fixed-point map, starting from f. It is simple to check that each element in this sequence is $(\mathcal{G}_t)$-adapted. Since $(\mathcal{G}_t)$ is complete, this implies that (f_t) also is $(\mathcal{G}_t)$-adapted. Assume that Φ is continuous and bounded. Our aim, first, is to prove the following identity: for $0\leq s,t$,

$$\mathbb{E}_{(f,\mathbf{e})}(\Phi(f_{t+s},E_{t+s}(\mathbf{e}))|\mathcal{G}_s)=(P_t\Phi)(f_s,E_s(\mathbf{e})). \tag{4.20}$$

We can use the propagation of the $W^{2,1}$-regularity stated in Proposition 4.1 and Proposition 4.2 and an argument of density to reduce the proof of (4.20) to the case where $f\in H$. In that case, (f_t,E_t) is seen as a process with state space $\mathcal{Y}=H\times F$. We apply this reduction because, when f_t has the regularity $W^{2,1}(\mathbb{T}^N \times \mathbb{R}^N)$, it is simple to prove that

$$f_t = \Psi_{0,t}(f, (E(\sigma))_{0\leq\sigma\leq t}), \tag{4.21}$$

where $\Psi_{0,t}(f, \cdot)$ is a *continuous* map from $L^1([0,t]; F)$ to $L^1(\mathbb{T}^N \times \mathbb{R}^N)$. Indeed, if f_t^i, $i \in \{1,2\}$ are two solutions to (4.2) corresponding to two different forcing terms $E^i(t,x)$, $i \in \{1,2\}$, we just need to write

$$\left[\partial_t + E^1 \cdot \nabla_v - Q\right](f_t^1 - f_t^2) = (E^2 - E^1) \cdot \nabla_v f_t^2,$$

multiply the equation by $\mathrm{sgn}(f_1 - f_2)$ and integrate, to obtain

$$\|f_t^1 - f_t^2\|_{L^1(\mathbb{T}^N\times\mathbb{R}^N)} \leq C \int_0^t \|E^2(s) - E^1(s)\|_F ds, \tag{4.22}$$

where the constant C depends on the $L_t^\infty L_{x,v}^1$-norm of $\nabla_v f_t^2$. Similarly, we may introduce the solution map $\Psi_{s,t}$, which associates to the path $(E(\sigma))_{s\leq\sigma\leq t}$ and to the datum f_σ at $\sigma = s$, the solution to (4.2) at time t. We have then the semi-group property

$$\Psi_{0,t+s}(f, (E(\sigma))_{0\leq\sigma\leq t+s}) = \Psi_{s,t+s}(\Psi_{0,s}(f, (E(\sigma))_{0\leq\sigma\leq s}), (E(\sigma))_{s\leq\sigma\leq t+s}). \tag{4.23}$$

Using (4.23), the relation (4.20) is equivalent to

$$\mathbb{E}\Big(\Phi\left[\Psi_{s,t+s}(\xi, (E_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}), E_{t+s}(\mathbf{e})\right] \Big| \mathcal{G}_s\Big) = (P_t\Phi)(\xi, E_s(\mathbf{e})). \tag{4.24}$$

where $\xi = f_s$ is $\mathcal{G}_s$-measurable. The property (4.24) is true for every random variable ξ which is $\mathcal{G}_s$-measurable. The reason for this is that $\Psi_{s,t}(\xi, (E_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s})$ is a functional of $(E_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}$. The proof of (4.24) is clear if we replace $(E_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}$ by the piecewise constant path $(\tilde{E}_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}$, where $\tilde{E}_\sigma(\mathbf{e}) = E_{s+t_i}(\mathbf{e})$ for all $\sigma \in [s + t_i, s + t_{i+1})$, where

$$s = s + t_0 < s + t_1 < \cdots < s + t_N = s + t$$

is a subdivision of the interval $[s, s+t]$. Indeed, we have then

$$\Phi\left[\Psi_{s,t+s}(\xi, (\tilde{E}_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}), E_{t+s}(\mathbf{e})\right] = \bar{\Phi}_N(\xi; E_{s+t_0}(\mathbf{e}), \ldots, E_{s+t_N}(\mathbf{e})),$$

where $\bar{\Phi}_N$ is continuous in its arguments. By considering the finite-dimensional distributions of (E_t), we obtain

$$\mathbb{E}\left[\bar{\Phi}_N(\xi; E_{s+t_0}(\mathbf{e}), \ldots, E_{s+t_N}(\mathbf{e}))|\mathcal{G}_s\right] = P_{N,t}\Phi(\xi, E_s(\mathbf{e})), \tag{4.25}$$

where

$$\begin{aligned} P_{N,t}\Phi(f, \mathbf{e}) &= \mathbb{E}_{\mathbf{e}}\bar{\Phi}_N(f; E_{t_0}(\mathbf{e}), \ldots, E_{t_N}(\mathbf{e}))) \\ &= \mathbb{E}_e\Phi\left[\Psi_{0,t}(f, (\hat{E}_\sigma(\mathbf{e}))_{0\leq\sigma\leq t}), E_t(\mathbf{e})\right], \end{aligned}$$

where $(\hat{E}_\sigma(\mathbf{e}))_{0\leq\sigma\leq t}$ is the piecewise-constant path defined by $\hat{E}_\sigma(\mathbf{e}) = E_{t_i}(\mathbf{e})$ for all $\sigma \in [t_i, t_{i+1})$, $i = 0, \dots, N-1$. Since $(E_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}$ is càdlàg, we can approximate $(E_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}$ by $(\tilde{E}_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}$, due to the continuous dependence of $\Psi_{s,t+s}$ on $(E_\sigma(\mathbf{e}))_{s\leq\sigma\leq t+s}$ with respect to the $L^1([s, t+s]; F)$-norm. When the size $\max(t_{i+1} - t_i)$ of the subdivision tends to 0, we have $P_{N,t}\Phi(f, \mathbf{e}) \to P_t\Phi(f, \mathbf{e})$. This yields (4.20). Using (4.21) and approaching $f \in G$ by a sequence of H, using also the fact that $e \mapsto \mathbb{P}_e(D)$ is measurable for all Borel subset D of $D([0, +\infty); F)$ (see Section 2), we see that $(f, \mathbf{e}) \mapsto \mathbb{P}_{(f,\mathbf{e})}(A)$ is measurable for all Borel set $A \subset \mathcal{X}$. We conclude then by Proposition B.2 and Remark B.1 in the appendix. □

Let us introduce the operators

$$\mathscr{L}_\sharp\varphi(f, \mathbf{e}) = A\varphi(f, \mathbf{e}) + (Qf - \mathbf{e}\cdot\nabla_v f, D_f\varphi(f, \mathbf{e})), \tag{4.26}$$

$$\mathscr{L}_\flat\varphi(f, \mathbf{e}) = -(v\cdot\nabla_x f, D_f\varphi(f, \mathbf{e})), \tag{4.27}$$

and $\mathscr{L} = \mathscr{L}_\sharp + \mathscr{L}_\flat$. Formally, $\mathscr{L}$ is the generator associated to the Markov process (f_t, E_t). We do not need to be much specific on that point here. Indeed, what is relevant to apply the perturbed test-function method in Section 6 (see (6.2)) are sufficient conditions for a test function to be both in the domain of $\mathscr{L}_\sharp$ and in the domain of $\mathscr{L}_\flat$. We prove the following result.

Proposition 4.4. *Let $(\bar{E}(t))$ be a mixing force field in the sense of Definition 2.1. Let A be the generator of (E_t), let $\mathcal{X}$ be the state space defined by (4.18), and let $\mathscr{L}_\sharp$ and $\mathscr{L}_\flat$ be defined by (4.26)-(4.27). Let $\psi\colon \mathbb{R}^m \times F \to \mathbb{R}$ be a continuous function which is bounded on bounded sets of $\mathbb{R}^m \times F$ and satisfies the following properties:*

1. *for all $u \in \mathbb{R}^m$, $\mathbf{e} \mapsto \psi(u; \mathbf{e})$ is in the domain of A and $(u, \mathbf{e}) \mapsto A\psi(u; \mathbf{e})$ is bounded on bounded sets of $\mathbb{R}^m \times F$,*
2. *for all $\mathbf{e} \in F$, $u \mapsto \psi(u; \mathbf{e})$ is differentiable, $(u, \mathbf{e}) \mapsto \nabla_u\psi(u; \mathbf{e})$ is bounded on bounded sets of $\mathbb{R}^m \times F$ and continuous with respect to $\mathbf{e}$.*

Let $\xi_1, \dots, \xi_m \in C_c^\infty(\mathbb{T}^N \times \mathbb{R}^N)$. Then the test-function

$$\varphi\colon (f, \mathbf{e}) \mapsto \psi(\langle f, \xi_1\rangle, \dots, \langle f, \xi_m\rangle; \mathbf{e}) \tag{4.28}$$

satisfies $\mathscr{L}_\sharp\varphi(f, \mathbf{e}), \mathscr{L}_\flat\varphi(f, \mathbf{e}) < +\infty$ for all $(f, \mathbf{e}) \in \mathcal{X}$ and φ is in the domain of $\mathscr{L}$ in the sense that

$$P_t\varphi(f, \mathbf{e}) = \varphi(f, \mathbf{e}) + t\mathscr{L}\varphi(f, \mathbf{e}) + o(t), \tag{4.29}$$

for all $(f, \mathbf{e}) \in \mathcal{X}$.

Proof of Proposition 4.4. Let $\xi = (\xi_i)_{1,m}$. We have

$$\mathscr{L}_\sharp\varphi(f, \mathbf{e}) = \left\{A\psi(u; \mathbf{e}) + \langle f, Q^*\xi + \mathbf{e}\cdot\nabla_v\xi\rangle\nabla_u\psi(u; \mathbf{e})\right\}\big|_{u=\langle f,\xi\rangle},$$

$$\mathscr{L}_\flat\varphi(f, \mathbf{e}) = \langle f, v\cdot\nabla_x\xi\rangle\nabla_u\psi(u; \mathbf{e})\big|_{u=\langle f,\xi\rangle},$$

therefore $(f, \mathbf{e}) \mapsto (\mathscr{L}_\sharp\varphi(f, \mathbf{e}), \mathscr{L}_\flat\varphi(f, \mathbf{e}))$ is bounded on bounded sets of $\mathcal{X}$. To obtain (4.29), we use the decomposition of $P_t\varphi(f, \mathbf{e}) - \varphi(f, \mathbf{e})$ into the sum of the terms

$$\mathbb{E}_{(f,\mathbf{e})}\varphi(f, E_t) - \varphi(f, \mathbf{e}) \tag{4.30}$$

and

$$\mathbb{E}_{(f,\mathbf{e})}\left[\varphi(f_t, E_t) - \varphi(f, E_t)\right]. \tag{4.31}$$

By item 1, we have the asymptotic expansion (4.30) $= tA\psi(u; \mathbf{e})\big|_{u=\langle f,\xi\rangle} + o(t)$. In addition, by (4.2), we have

$$u_t = u + t\big(\langle f, Q^*\xi + \mathbf{e}\cdot\nabla_v\xi\rangle + \langle f, v\cdot\nabla_x\xi\rangle\big) + o(t),$$

where $u_t = \langle f_t, \xi\rangle$, $u = \langle f, \xi\rangle$. By item 2, we obtain the asymptotic expansion

$$(4.31) = t\big(\langle f, Q^*\xi + \mathbf{e}\cdot\nabla_v\xi\rangle + \langle f, v\cdot\nabla_x\xi\rangle\big)\nabla_u\psi(u; \mathbf{e})\big|_{u=\langle f,\xi\rangle} + o(t).$$

This concludes the proof. □

Remark 4.1. The result of Proposition 4.4 holds true if we consider some functions ξ_i not as smooth and localised as C_c^∞ functions, provided there is a sufficient balance with the regularity and integrability properties of f. For example, we apply Proposition 4.4 in Section 6.1.3 with $\xi_i(x, v) = \hat{\xi}_i(x)\zeta_i(v)$, where $\hat{\xi}_i$ is in some Sobolev space $H^s(\mathbb{T}^N)$ and $\zeta_i(v)$ is a polynomial in v of degree less than two. In that case, we view (f_t, E_t) as a Markov process on $\mathcal{X}_3 := G_3 \times F$ and the conclusion of Proposition 4.4 is valid for $f \in G_3$.

Remark 4.2. Note that, in the context of Proposition 4.4, the function $|\psi|^2$ has the same properties (item 1 and item 2) as ψ. Therefore $|\psi|^2$ is also in the domain of $\mathscr{L}$.

5 Deterministic convergence

In this section, we prove Theorem 1.1. We establish the convergence

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left|\int_{\mathbb{T}^N} r_t^\varepsilon(x)\varphi(x)dx - \int_{\mathbb{T}^N} r_t(x)\varphi(x)dx\right| = 0 \tag{5.1}$$

for all $\varphi \in C(\mathbb{T}^N)$. We use the two following results.

Theorem 5.1 (Martingale characterization of Markov processes)**.** *Let (X_t) be a $\mathbb{R}^N$-valued time-homogeneous Markov process of generator L. Assume that (X_t) is càdlàg. Then, for all continuous bounded $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ in the domain of L,*

$$Z(t) := \varphi(X_t) - \varphi(X_0) - \int_0^t L\varphi(X_s)ds \tag{5.2}$$

is an $\mathbb{R}^N$*-valued* $(\mathcal{F}_t^X)$*-martingale. If* φ^2 *is in the domain of* L*, then the quadratic variation of* $(Z(t))$ *is*

$$[Z_i, Z_j]_t = \int_0^t (L\varphi_i\varphi_j - \varphi_i L\varphi_j - \varphi_j L\varphi_i)(X_s)ds. \tag{5.3}$$

Theorem 5.2 (CLT for martingales)**.** *Let* (Z_t) *be a* $\mathbb{R}^N$*-valued càdlàg martingale satisfying*

$$\lim_{\varepsilon\to 0} \varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T/\varepsilon^2} |Z_t - Z_{t-}|\right] = 0. \tag{5.4}$$

Assume

$$\frac{1}{t}[Z, Z]_t \to \sigma^*\sigma \text{ in probability,} \tag{5.5}$$

where σ *is an* $N \times N$ *matrix. Then*

$$\varepsilon Z_{t/\varepsilon^2} \to \sigma W_t \tag{5.6}$$

in law on $C([0, T]; \mathbb{R}^N)$*, where* (W_t) *is an* N*-dimensional Wiener process.*

Proof of Theorem 5.1. We apply [9, Proposition 1.7, p. 162]. Since (X_t) is càdlàg, it is progressive. We obtain the fact that $(Z(t))$ is an $(\mathcal{F}_t^X)$-martingale. The proof of (5.3) is a consequence of Theorem B.3 in the appendix B. □

Proof of Theorem 5.2. Let $(\varepsilon_n) \downarrow 0$. We apply [9, Theorem 1.4, p. 339] to

$$M_n(t) = \varepsilon_n Z_{t/\varepsilon_n^2}, \quad c_{ij}(t) = (\sigma^*\sigma)_{ij} t.$$

Indeed, Condition (1.14) in [9, p. 340] is fulfilled by (5.4). □

Remark 5.1. Assume, in the context of Theorem 5.1, that (X_t) is ergodic, with invariant measure λ and let Z be defined by (5.2). We have then

$$\begin{aligned}\frac{1}{t}[Z_i, Z_j]_t \to &\langle L\varphi_i\varphi_j - \varphi_i L\varphi_j - \varphi_j L\varphi_i, \lambda\rangle \quad \text{in probability}\\ = &- \langle \varphi_i L\varphi_j + \varphi_j L\varphi_i, \nu\rangle \quad \text{since } L^*\lambda = 0.\end{aligned}$$

Therefore, we obtain (5.6) with

$$(\sigma^*\sigma)_{ij} = -\langle \varphi_i L\varphi_j + \varphi_j L\varphi_i, \lambda\rangle. \tag{5.7}$$

5.1 Classical diffusion limit

Let us first illustrate the application of Theorem 5.1 and Theorem 5.2 in the case where $\bar{E} \equiv 0$ in (1.7) and $Q = Q_{\rm LB}$. The equation (1.7) is deterministic then. An argument using a PDE theory approach gives the convergence of $r^\varepsilon = \rho^\varepsilon$ to r solution to (1.21) in $L^2(0,T;L^2(\mathbb{T}^N))$, see [8, Theorem 1.1] for example. Using a probabilistic approach, we consider the jump process (V_t) associated to (1.2) (this is a pure jump process since we assume $\bar{E} \equiv 0$). We obtain the convergence of the process (X_t^ε) that is behind $r^\varepsilon = \rho^\varepsilon$, and a convergence to r in $C([0,T];L^2(\mathbb{T}^N) - \text{weak})$. Indeed, let us set

$$X_t^\varepsilon = X_0 + \left[\varepsilon \int_0^{t/\varepsilon^2} V_s ds\right], \tag{5.8}$$

where X_0 follows a law with density $\rho_{\rm in}$ with respect to the Lebesgue measure on $\mathbb{T}^N$. In (5.8), $[Y]$ denote the equivalence class in $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ of an element $Y \in \mathbb{R}^N$. By (1.20), X_0 and V_0 are independent, V_0 having the law of density g with respect to the Lebesgue measure on $\mathbb{R}^N$. Then r^ε is the density, with respect to the Lebesgue measure on $\mathbb{T}^N$, of the law of X_t^ε. For all $\psi\colon \mathbb{R}^N \to \mathbb{R}$ continuous and bounded, we have

$$\int_{\mathbb{T}^N} r^\varepsilon(x,t)\psi(x)dx = \mathbb{E}\psi(X_t^\varepsilon). \tag{5.9}$$

The generator L of (V_t) is $L\varphi(v) = \langle \varphi, M\rangle - \varphi$. We apply Theorem 5.1 to the Markov process (V_t) with φ such that

$$L\varphi = -\mathrm{Id}. \tag{5.10}$$

The solution to (5.10) is $\varphi = \mathrm{Id}$. We have then

$$X_t^\varepsilon = X_0 + \varepsilon Z_{t/\varepsilon^2} - \varepsilon(V_{t/\varepsilon^2} - V_0) = X_0 + \varepsilon Z_{t/\varepsilon^2} + \mathcal{O}(\varepsilon), \tag{5.11}$$

where the $\mathcal{O}(\varepsilon)$ is in $L^1(\Omega)$ (and thus, in probability) since V_t has either the distribution of V_0 or, if a jump has already occurred, the density M with respect to the Lebesgue measure on $\mathbb{R}^N$. We obtain, using (5.7), the convergence (5.5) with $\sigma^*\sigma = 2K$, where K is given by (1.15). By Theorem 5.2 thus, we have

$$X_t^\varepsilon \to X_t := X_0 - \sqrt{2K} W_t$$

in law on $C([0,T];\mathbb{R}^N)$. Using (5.9), we obtain

$$\int_{\mathbb{T}^N} r^\varepsilon(x,\cdot)\psi(x)dx \to \int_{\mathbb{T}^N} r(x,\cdot)\psi(x)dx \quad \text{in } C([0,T]), \tag{5.12}$$

for all $\psi\colon \mathbb{R}^N \to \mathbb{R}$ continuous and bounded, where r satisfies (1.21). Since (r^ε) is bounded uniformly in ε in $C([0,T];L^2(\mathbb{T}^N))$ (see [8]), the convergence (5.12) is satisfied for all $\psi \in L^2(\mathbb{T}^N)$. We conclude to the convergence of (r^ε) to r in $C([0,T];L^2(\mathbb{R}^N) - \text{weak})$.

Remark 5.2. In the FP case, the proof is direct and does not require Theorem 5.1 and Theorem 5.2. Indeed, we have (5.8) where V_t satisfies (4.9). We replace V_s in the integral in (5.8) by $-dV_s + \sqrt{2}dB_s$ to obtain

$$X_t^\varepsilon = X_0 + \varepsilon B_{t/\varepsilon^2} - \varepsilon(V_{t/\varepsilon^2} - V_0)$$

Since (4.11) gives a bound (in $L^2(\Omega)$ for example) on (V_t), we have (5.11) with $Z_t = B_t$ and we conclude using the scale invariance of the Wiener process.

5.2 Diffusion limit for the stochastically forced equation

We consider now the general case of Equation (1.7) with a non-trivial mixing force process $(\bar{E}_t)$ (*cf.* Definition 2.2). Recall that $\mathcal{F}_t^E$ is the σ-algebra generated by $(E_s)_{0\leq s\leq t}$. For $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ continuous and bounded, we have

$$\int_{\mathbb{T}^N} \rho^\varepsilon(x,t)\varphi(x)dx = \mathbb{E}\left[\varphi(X_t^\varepsilon)|\mathcal{F}^E_{t/\varepsilon^2}\right],$$

and (5.9), where X_t^ε is like in (5.8) and (V_t) is either the stochastic process described in (1.9) in the case $Q = Q_{\mathrm{LB}}$ (given E, this is a PDMP), or the Ornstein-Uhlenbeck process (1.10) in the case $Q = Q_{\mathrm{FP}}$. The process (V_t, E_t) is Markov and has the generator L given by

$$L\varphi(v,\mathbf{e}) = Q^*\varphi(v,\mathbf{e}) - \mathbf{e}\cdot\nabla_v\varphi(v,\mathbf{e}) + A\varphi(v,\mathbf{e}). \tag{5.13}$$

The proof of this result is analogous to (actually, simpler than, since the state space has finite dimension) the proof of Theorem 4.3. By Theorem 3.1 (see Remark 3.2), (V_t, E_t) is ergodic and has the invariant measure λ defined by

$$\langle\lambda,\varphi\rangle = \mathbb{E}\int_{\mathbb{R}^N} \bar{M}_t(v)\varphi(v,\bar{E}_t)dv.$$

This is a consequence of (3.10) and of the fact that (V_t, E_t) is time homogeneous. More precisely, denoting by (e^{tL}) the semi-group generated by (V_t, E_t), (3.10) gives, for all $f \in L^1(\mathbb{R}^N)$, and for all φ bounded and continuous on $\mathbb{R}^N \times \mathbb{R}^N$,

$$\lim_{t\to+\infty}\int_{\mathbb{R}^N} f(v)e^{tL}\varphi(v,\mathbf{e})dv = \langle\lambda,\varphi\rangle\int_{\mathbb{R}^N} f(v)dv. \tag{5.14}$$

To apply Theorem 5.1 and Theorem 5.2 like in Section 5.1, we need to solve the Poisson equation

$$L\varphi(v,\mathbf{e}) = -v. \tag{5.15}$$

Using (5.7) then, we need to compute

$$(\sigma^*\sigma)_{ij} = \langle\varphi_i v_j + \varphi_j v_i, \lambda\rangle. \tag{5.16}$$

5.3 The auxiliary test-function

Our aim is to find the solution φ to the Poisson equation $L\varphi = \psi$ (with $\psi(v) = -v$ in our case). An perturbation argument (of the case $E \equiv 0$) suggests the solution

$$\varphi(v, \mathbf{e}) = v + R_0(\mathbf{e}), \tag{5.17}$$

where the resolvent $R_0(\mathbf{e})$ is defined in (2.14) and (2.15). The test-function (5.17) indeed satisfies (5.15) since $Q^*v = -v$ and $AR_0(\mathbf{e}) = -\mathbf{e}$. However, there is a more systematic way to obtain (5.17) than just guessing, and we explain it below, since it involves some computations that are necessary later in Section 6. To solve $L\varphi = \psi$ (with $\psi(v) = -v$ in our case, thus), we use the resolvent formula

$$\varphi = -\int_0^\infty e^{tL}\psi dt. \tag{5.18}$$

This may work at least if $\langle \psi, \lambda \rangle$, which is equal to $\lim_{t\to+\infty} e^{tL}\psi$ by ergodicity, vanishes. In our case $\psi(v) = v$, this condition is satisfied as we will see. Note first that, for $f \in L^1(\mathbb{R}^N)$ and $f_{0,t}$ given either by (3.3) or (3.4) with $s = 0$, we have, by (3.9),

$$\int_{\mathbb{R}^N} f(v)e^{tL}\psi(v, \mathbf{e})dv = \mathbb{E}_{\mathbf{e}} \int_{\mathbb{R}^N} f_{0,t}(v)\psi(v, \mathbf{e})dv. \tag{5.19}$$

With $\psi(v, \mathbf{e}) = -v$, we obtain

$$\int_{\mathbb{R}^N} f(v)e^{tL}\psi(v, \mathbf{e})dv = -\mathbb{E}_{\mathbf{e}} J(f_{0,t}). \tag{5.20}$$

Lemma 5.3. *Let $f_{s,t}$ be equal either to (3.3), or to (3.4). The two first moments of $f_{s,t}$ (see (1.14) for the definition of the moments) are, respectively, $\rho(f_{s,t}) = \rho(f)$, and*

$$J(f_{s,t}) = e^{-(t-s)} J(f) + \rho(f) \int_s^t e^{-(t-\sigma)} E(\sigma, s; \mathbf{e}) d\sigma. \tag{5.21}$$

Proof of Lemma 5.3. We use the formula

$$\int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ v^{\otimes 2} \end{pmatrix} M(v - w)dv = \begin{pmatrix} 1 \\ w \\ K + w^{\otimes 2}, \end{pmatrix}, \tag{5.22}$$

where K is defined by (1.15). By (5.22) (and a change of variable in the FP-case), we obtain (5.21). ☐

Corollary 5.4. *Let $f_{s,t}$ be equal either to (3.3), or to (3.4) and let $\bar{M}_t^{\mathrm{LB}}$ and $\bar{M}_t^{\mathrm{FP}}$ be defined by (3.6) and (3.7) respectively. We have then, for all $\mathbf{e} \in \bar{B}_{\mathtt{R}}$,*

$$\int_0^\infty |\mathbb{E} J(f_{0,t})| dt < +\infty, \quad \int_0^\infty \mathbb{E} J(f_{0,t}) dt = J(f) + \rho(f) R_0(\mathbf{e}), \tag{5.23}$$

$$J(\bar{M}_0^{\mathrm{LB}}) = J(\bar{M}_0^{\mathrm{FP}}) = \int_{-\infty}^0 e^\sigma \bar{E}(\sigma) d\sigma. \tag{5.24}$$

Proof of Corollary 5.4. We take $t = 0$, $\rho = 1$ and the limit $s \to -\infty$ in (5.21) to obtain (5.24). To prove (5.23), we use the following formula:

$$\int_0^\infty \int_0^t e^{-(t-r)} \mathbb{E}\varphi(E(0,r;\mathbf{e}))drdt = \int_0^\infty \mathbb{E}\varphi(E(0,r;\mathbf{e}))dr.$$

□

Comparing (5.18)-(5.20) and (5.23), we obtain (5.17).

5.4 Diffusion matrix

Now that we have found φ solution to (5.15), let us come back to (5.16).

Lemma 5.5. *Let $\bar{M}_t^{\rm LB}$ and $\bar{M}_t^{\rm FP}$ be defined by (3.6) and (3.7) respectively. The expectation of the second moment of $\bar{M}_0$ is*

$$\mathbb{E}\left[K(\bar{M}_0)\right] = K + b\mathbb{E}\left[\bar{E}(0) \otimes R_1(\bar{E}(0))\right], \tag{5.25}$$

where $b^{\rm LB} = 2$ and $b^{\rm FP} = 1$.

Proof of Lemma 5.5. We compute, using (5.22),

$$K(\bar{M}_0^{\rm LB}) = \int_{-\infty}^0 e^\sigma \left(K + \left[\int_\sigma^0 \bar{E}(r)dr\right]^{\otimes 2}\right) d\sigma.$$

This gives

$$\mathbb{E}\left[K(\bar{M}_0^{\rm LB})\right] = K + \int_{-\infty}^0 e^\sigma \int_\sigma^0 \int_\sigma^0 \bar{\Gamma}(r-s)drdsd\sigma,$$

where $\bar{\Gamma}(t)$ is the covariance of $(\bar{E}(t))$ (see (2.17)). By symmetry, we have

$$\int_\sigma^0 \int_\sigma^0 \bar{\Gamma}(r-s)drds = 2\int_\sigma^0 (r-\sigma)\bar{\Gamma}((r)dr.$$

By two successive integration by parts, we have next

$$2\int_{-\infty}^0 e^\sigma \int_\sigma^0 (r-\sigma)\bar{\Gamma}((r)drd\sigma = 2\int_{-\infty}^0 e^\sigma \int_\sigma^0 \bar{\Gamma}((r)drd\sigma = 2\int_{-\infty}^0 e^\sigma \bar{\Gamma}((\sigma)d\sigma.$$

Coming back to the definition (2.17), we obtain

$$\begin{aligned}\int_{-\infty}^0 e^\sigma \bar{\Gamma}((\sigma)d\sigma &= \int_{-\infty}^0 e^\sigma \mathbb{E}\left[\bar{E}(\sigma) \otimes \bar{E}(0)\right] d\sigma \\ &= \int_0^{+\infty} e^{-\sigma} \mathbb{E}\left[\bar{E}(-\sigma) \otimes \bar{E}(0)\right] d\sigma \\ &= \int_0^{+\infty} e^{-\sigma} \mathbb{E}\left[\bar{E}(0) \otimes \bar{E}(\sigma)\right] d\sigma = \mathbb{E}\left[\bar{E}(0) \otimes R_1(\bar{E}(0))\right],\end{aligned} \tag{5.26}$$

and we conclude to (5.25). Similarly, we have by (3.7) and (5.22),

$$K(\bar{M}_0^{\mathrm{FP}}) = K + \left[\int_{-\infty}^{0} e^{\sigma}\bar{E}(\sigma)d\sigma\right]^{\otimes 2}.$$

To conclude to (5.25), we use the following Lemma 5.6. □

Lemma 5.6 (Symmetry and positivity)**.** *For $\delta > 0$, we have*

$$\mathbb{E}\left[R_\delta(\bar{E}(0)) \otimes \bar{E}(0)\right] = \delta\mathbb{E}\left[\int_{-\infty}^{0} e^{\delta\sigma}\bar{E}(\sigma)d\sigma\right]^{\otimes 2}. \tag{5.27}$$

In particular, if $\delta \geq 0$, we have $\mathbb{E}\left[R_\delta(\bar{E}(0)) \otimes \bar{E}(0)\right] = \mathbb{E}\left[\bar{E}(0) \otimes R_\delta(\bar{E}(0))\right]$ and this quantity is non-negative.

Proof of Lemma 5.6. We compute

$$\begin{aligned}
\mathbb{E}\left[\int_{-\infty}^{0} e^{\delta\sigma}\bar{E}(\sigma)d\sigma\right]^{\otimes 2} =& \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\delta(\sigma+s)}\mathbb{E}[\bar{E}(s)\otimes\bar{E}(\sigma)]d\sigma ds\\
=&2\int_{-\infty}^{0}\int_{\sigma=-\infty}^{s} e^{\delta(\sigma+s)}\mathbb{E}[\bar{E}(s)\otimes\bar{E}(\sigma)]d\sigma ds\\
=&2\int_{-\infty}^{0}\int_{\sigma=-\infty}^{s} e^{\delta(\sigma+s)}\mathbb{E}[\bar{E}(0)\otimes\bar{E}(\sigma-s)]d\sigma ds\\
=&2\int_{-\infty}^{0}\int_{\sigma=-\infty}^{0} e^{\delta(\sigma+2s)}\mathbb{E}[\bar{E}(0)\otimes\bar{E}(\sigma)]d\sigma ds\\
=&\frac{1}{\delta}\int_{-\infty}^{0} e^{\delta\sigma}\mathbb{E}[\bar{E}(0)\otimes\bar{E}(\sigma)]d\sigma\\
=&\frac{1}{\delta}\int_{0}^{\infty} e^{-\delta\sigma}\mathbb{E}[\bar{E}(\sigma)\otimes\bar{E}(0)]d\sigma = \frac{1}{\delta}\mathbb{E}\left[R_\delta(\bar{E}(0))\otimes\bar{E}(0)\right],
\end{aligned} \tag{5.28}$$

which gives the result. □

By (5.16) and (5.17), we have

$$\sigma^*\sigma = 2\mathbb{E}\left[K(\bar{M}_0)\right] + \mathbb{E}\left[R_0(\bar{E}(0)) \otimes J(\bar{M}_0) + J(\bar{M}_0) \otimes R_0(\bar{E}(0))\right].$$

We use (5.24) to derive the following expression

$$\mathbb{E}\left[R_0(\bar{E}_0) \otimes J(\bar{M}_0)\right] = \int_{-\infty}^{0} e^{\sigma}\mathbb{E}\left[R_0(\bar{E}(0)) \otimes \bar{E}(\sigma)\right] d\sigma. \tag{5.29}$$

As in (5.26), this is $\mathbb{E}\left[R_1 R_0(\bar{E}(0)) \otimes \bar{E}(0)\right]$ and thus

$$\sigma^*\sigma = 2\mathbb{E}\left[K(\bar{M}_0)\right] + \mathbb{E}\left[R_1 R_0(\bar{E}(0)) \otimes \bar{E}(0) + \bar{E}(0) \otimes R_1 R_0(\bar{E}(0))\right].$$

To obtain a more tractable expression of $\sigma^*\sigma$, we use the resolvent identity $R_1R_0 = R_0 - R_1$ which yields, by the symmetry property stated in Lemma 5.6,

$$\sigma^*\sigma = 2\mathbb{E}\left[K(\bar{M}_0)\right] + 2\mathbb{E}\left[\bar{E}(0)\otimes(R_0(\bar{E}(0)) - R_1(\bar{E}(0)))\right].$$

Using (5.25), we obtain

$$\sigma^*\sigma = 2K + 2\mathbb{E}\left[\bar{E}(0)\otimes[R_0(\bar{E}(0)) + (b-1)R_1(\bar{E}(0))]\right]. \tag{5.30}$$

We have therefore (5.1) for all $\varphi \in C(\mathbb{T}^N)$, where r is the solution to (1.21) starting from r_{in}. Note that, as a consequence of Lemma 5.6, we have

$$K_\sharp \geq K, \tag{5.31}$$

in the sense of symmetric matrices.

6 Diffusion-approximation

In this section, we consider a spatially dependent process $(\bar{E}_t)$ and establish the limit behaviour of ρ^ε as stated in Theorem 1.2. We forget now the probabilistic origin of f^ε, the solution to (1.7). This probabilistic aspect has been used in the previous Section 5. Our main probabilistic object of study now is the process (f^ε_t) solution to the SPDE (1.7). More precisely, we consider the Markov process $(f^\varepsilon_t, \bar{E}^\varepsilon_t)$ (see Theorem 4.3). The generator $\mathscr{L}^\varepsilon$ of this process can be decomposed as

$$\mathscr{L}^\varepsilon = \frac{1}{\varepsilon^2}\mathscr{L}_\sharp + \frac{1}{\varepsilon}\mathscr{L}_\flat,$$

where $\mathscr{L}_\sharp$ and $\mathscr{L}_\flat$ are defined by (4.26) and (4.27) respectively. For every φ in the domain of $\mathscr{L}^\varepsilon$, the process

$$M^\varepsilon_\varphi(t) := \varphi(f^\varepsilon_t, \bar{E}^\varepsilon_t) - \varphi(f_{\mathrm{in}}, \bar{E}_0) - \int_0^t \mathscr{L}^\varepsilon\varphi(f^\varepsilon_s, \bar{E}^\varepsilon_s)ds \tag{6.1}$$

is a $(\mathcal{G}_{t/\varepsilon^2})$-martingale (this is a consequence of Theorem 4.3 and Theorem B.3 in Appendix B). The equation associated to the principal generator $\mathscr{L}_\sharp$ is (1.2). It has been analysed in Section 3. Our approach to the proof of the convergence of (ρ^ε) uses the perturbed test-function method introduced by Papanicolaou, Stroock, Varadhan in [16] and adapted in the setting of hydrodynamic limits in [7]. Let us explain the main steps of the proof.

1. **Limit generator.** To find the limit generator $\mathscr{L}$ associated to the equation satisfied by the limit ρ of (ρ^ε), which acts on test functions $\varphi(\rho)$, we seek two correctors φ_1 and φ_2 such that, for the perturbed test function

$$\varphi^\varepsilon(f, \mathbf{e}) = \varphi(\rho) + \varepsilon\varphi_1(f, \mathbf{e}) + \varepsilon^2\varphi_2(f, \mathbf{e}), \tag{6.2}$$

we may write $\mathscr{L}^\varepsilon\varphi^\varepsilon = \mathscr{L}\varphi + o(1)$. See Section 6.1.

2. **Tightness.** We prove the tightness of the sequence (ρ^ε) in an adequate space. First, we obtain some bounds uniform with respect to ε by perturbation of the functional which we try to estimate. See Section 6.2. Then we establish some uniform estimates on the time increments of (ρ^ε). See Section 6.3.

3. **Convergence.** We use the characterization of (1.24)-(1.25) as a martingale problem to take the limit of the processes (ρ_ε). This is a very classical approach to the convergence of stochastic processes, see the introduction to [12, Chapter III]. The class Θ of test-functions $\varphi(\rho)$ of the form

$$\varphi(\rho) = \psi\left(\langle \rho, \xi\rangle_{L^2(\mathbb{T}^N)}\right), \tag{6.3}$$

for ξ in a dense subset of $L^2(\mathbb{T}^N)$ and ψ a Lipschitz function on $\mathbb{R}$ such that $\psi' \in C_b^\infty(\mathbb{R})$, is a separating class in $L^2(\mathbb{T}^N)$: if two random variables ρ_1 and ρ_2 satisfy $\mathbb{E}\varphi(\rho_1) = \mathbb{E}\varphi(\rho_2)$ for all φ as in (6.3), then ρ_1 and ρ_2 have the same laws (this follows from the fact that Θ separates points and from Theorem 4.5 p. 113 in [9]). This is why we put a special emphasis in Section 6.1.3 on the application of the perturbed test-function method to test-functions as in (6.3).

6.1 Perturbed test-function

Let $\varphi\colon L^2(\mathbb{T}^N) \to \mathbb{R}$ be a given test-function in the variable ρ. We specify its regularity later. Consider the perturbation (6.2). To obtain the approximation $\mathscr{L}^\varepsilon\varphi^\varepsilon = \mathscr{L}\varphi + o(1)$, we identify the powers in ε in each side of this equality. This gives, for the scale ε^{-2}, the first equation $\mathscr{L}_\sharp\varphi = 0$. This equation is satisfied since φ is independent on $\mathbf{e}$, hence $A\varphi = 0$, and

$$(Qf - \mathbf{e}\cdot\nabla_v f, D_f\varphi(\rho)) = (\rho(Qf - \mathbf{e}\cdot\nabla_v f), D_\rho\varphi(\rho)) = 0$$

since $\rho(Qf) = 0$ and $\rho(\mathbf{e}\cdot\nabla_v f) = 0$ separately. At the scale ε^{-1} and ε^0, we obtain the equation for the first corrector

$$\mathscr{L}_\sharp\varphi_1 + \mathscr{L}_\flat\varphi = 0 \tag{6.4}$$

and the equation for the second corrector

$$\mathscr{L}_\sharp\varphi_2 + \mathscr{L}_\flat\varphi_1 = \mathscr{L}\varphi, \tag{6.5}$$

respectively. If (6.4) and (6.5) are satisfied, then $\mathscr{L}^\varepsilon\varphi^\varepsilon = \mathscr{L}\varphi + \varepsilon\mathscr{L}_\flat\varphi_2$. We solve (6.4) and (6.5) by formal computations first, see Section 6.1.1 and Section 6.1.2. In Section 6.1.3 then, we discuss more rigorously the resolution of (6.4) and (6.5).

6.1.1 First corrector

We seek a solution to (6.4) by means of the resolvent formula

$$\varphi_1(f,\mathbf{e}) = \int_0^\infty \mathbb{E}_{(f,\mathbf{e})}\psi(f_t, E_t)dt, \quad \psi = \mathscr{L}_\flat\varphi,$$

where f_t is obtained either by (3.3) or (3.4) with $s = 0$. The right-hand side ψ is

$$\psi(f, \mathbf{e}) = \mathscr{L}_\flat \varphi(f, \mathbf{e}) = -(\mathrm{div}_x(vf)), D_f\varphi(\rho)) = -(\mathrm{div}_x(J(f)), D_\rho\varphi(\rho)),$$

since $\rho(vf) = J(f)$. By (5.23), we obtain the candidate

$$\varphi_1(f, \mathbf{e}) = -(\mathrm{div}_x(H(f)), D_\rho\varphi(\rho)), \quad H(f) := J(f) + \rho(f)R_0(\mathbf{e}). \tag{6.6}$$

6.1.2 Second corrector and limit generator

Let μ_ρ be the invariant measure parametrized by ρ associated to $\mathscr{L}_\sharp$, defined by (3.8). Since $\mathscr{L}_\sharp^* \mu_\rho = 0$ and $\langle \mathscr{L}\varphi, \mu_\rho\rangle = \mathscr{L}\varphi(\rho)$, a necessary condition to (6.5) is that

$$\mathscr{L}\varphi(\rho) = \langle \mathscr{L}_\flat \varphi_1, \mu_\rho\rangle. \tag{6.7}$$

If (6.7) is satisfied, then we set

$$\varphi_2(f, \mathbf{e}) = \int_0^\infty \left(\mathbb{E}_{(f,\mathbf{e})}\mathscr{L}_\flat\varphi_1(f_t, E_t) - \langle \mathscr{L}_\flat\varphi_1, \mu_\rho\rangle\right) dt. \tag{6.8}$$

The equation (6.7) gives the limit generator $\mathscr{L}$. Since $f \mapsto H(f)$ is linear, we have

$$\begin{aligned}\mathscr{L}_\flat\varphi_1(f, \mathbf{e}) &= -(\mathrm{div}_x(vf), D_f\varphi_1(f, \mathbf{e})) \\ &= (\mathrm{div}_x[H(\mathrm{div}_x(vf))], D_\rho\varphi(\rho)) + D^2_\rho\varphi(\rho)\cdot(\mathrm{div}_x(H(f)), \mathrm{div}_x(J(f))),\end{aligned} \tag{6.9}$$

and thus

$$\mathscr{L}\varphi(\rho) = (\langle\psi, \mu_\rho\rangle, D_\rho\varphi(\rho)) + \int_{E\times F} D^2_\rho\varphi(\rho)\cdot(\mathrm{div}_x(H(f)), \mathrm{div}_x(J(f)))d\mu_\rho(f, \mathbf{e}), \tag{6.10}$$

where $\psi(f, \mathbf{e}) = \mathrm{div}_x(H(\mathrm{div}_x(vf))$. Let us compute the first term in the right-hand side of (6.10). Using (6.6), we have

$$\psi(f, \mathbf{e}) = D^2_x{:}K(f) + \mathrm{div}_x\left[R_0(\mathbf{e})\mathrm{div}_x(J(f))\right]. \tag{6.11}$$

The part $\langle D^2_x{:}K(f), \mu_\rho\rangle = D^2_x{:}\left[\rho\mathbb{E}K(\bar{M}_0)\right]$ is given by (5.25). To identify the contribution of the second part in (6.11), we adapt (5.29) to obtain

$$\begin{aligned}\langle \mathrm{div}_x\left[R_0(\mathbf{e})\mathrm{div}_x(J(f))\right], \mu_\rho\rangle =&\partial_{x_j}\int_{-\infty}^0 e^\sigma\mathbb{E}\left[R_0(E^*_j(0))\partial_{x_i}\left(\rho E^*_i(\sigma)\right)\right] \\ =&\partial_{x_j}\mathbb{E}\left[R_1R_0(E^*_j(0))\partial_{x_i}\left(\rho E^*_i(0)\right)\right].\end{aligned}$$

The first-order part in (6.10) is therefore $(\langle\psi, \mu_\rho\rangle, D_\rho\varphi(\rho))$, with

$$\langle\psi, \mu_\rho\rangle = D^2_x{:}\left[\rho\left(K + b\mathbb{E}\left[\bar{E}(0)\otimes R_1(\bar{E}(0))\right]\right)\right] + \mathrm{div}_x\mathbb{E}\left[R_1R_0(\bar{E}(0))\mathrm{div}_x\left(\rho\bar{E}(0)\right)\right]. \tag{6.12}$$

This can be rewritten as

$$\langle\psi, \mu_\rho\rangle = \mathrm{div}_x(K_\sharp\nabla_x\rho + \Psi\rho), \tag{6.13}$$

where $K_\sharp$ and Ψ are given in (1.18) and (1.19) respectively. Note that this is consistent with the result (5.30) obtained for a $\bar{E}$ independent on x (indeed, the drift coefficient Ψ vanishes if $\bar{E}$ is independent on x). To compute the second-order part in (6.10), we have two terms to consider: $\langle J(f)\otimes J(f),\mu_\rho\rangle$ and $\langle R_0(\mathbf{e})\otimes J(f),\mu_\rho\rangle$. We have already established

$$\langle R_0(\mathbf{e})\otimes J(f),\mu_\rho\rangle=\mathbb{E}\left[R_1R_0(\bar{E}(0))\otimes(\rho\bar{E}(0))\right].$$

By (5.24) and (5.27), we have also

$$\langle J(f)\otimes J(f),\mu_\rho\rangle=\mathbb{E}\left[(\rho R_1(\bar{E}(0)))\otimes(\rho\bar{E}(0))\right].$$

It follows by the resolvent identity $R_1R_0=R_0-R_1$ that

$$\int_{E\times F}D^2_\rho\varphi(\rho)\cdot(\mathrm{div}_x(H(f)),\mathrm{div}_x(J(f)))d\mu_\rho(f,\mathbf{e})\\ =\mathbb{E}D^2_\rho\varphi(\rho)\cdot(\mathrm{div}_x(\rho R_0(\bar{E}(0))),\mathrm{div}_x(\rho\bar{E}(0))). \quad (6.14)$$

Remark 6.1 (Energy estimate). From the explicit expression of the limit generator $\mathscr{L}$, we can draw a first consequence (in an informal way, in a first step; see the proof of Theorem 6.8 for more details), which is an energy estimate for the SPDE associated to $\mathscr{L}$. Indeed, let (ρ_t) be a solution of this SPDE (see Equation (1.24)). We have

$$\frac{1}{2}\mathbb{E}\|\rho_t\|^2_{L^2(\mathbb{T}^N)}=\frac{1}{2}\mathbb{E}\|\rho_0\|^2_{L^2(\mathbb{T}^N)}+\int_0^t\mathbb{E}\mathscr{L}\varphi(\rho_s)ds,\quad \varphi(\rho):=\frac{1}{2}\|\rho\|^2_{L^2(\mathbb{T}^N)}.$$

If we compute $\mathscr{L}\varphi(\rho)$, the first-order term $(\langle\psi,\mu_\rho\rangle,D_\rho\varphi(\rho))$ gives

$$-\langle K_\sharp\nabla_x\rho,\nabla_x\rho\rangle_{L^2(\mathbb{T}^N)}-\langle\Psi\rho,\nabla_x\rho\rangle_{L^2(\mathbb{T}^N)}. \quad (6.15)$$

Using Lemma 5.6, the fact that K is the identity and the Cauchy-Schwarz inequality give us

$$(6.15)\geq-\|\nabla_x\rho\|^2_{L^2(\mathbb{T}^N)}-\mathbb{E}\langle R_0(\bar{E}(0))\cdot\nabla_x\rho,\bar{E}(0)\cdot\nabla_x\rho\rangle_{L^2(\mathbb{T}^N)}\\ -\|\Psi\|_{L^\infty(\mathbb{T}^N)}\|\rho\|_{L^2(\mathbb{T}^N)}\|\nabla_x\rho\|_{L^2(\mathbb{T}^N)}. \quad (6.16)$$

The contribution of the second order term (6.14) to $\mathscr{L}\varphi(\rho)$ is

$$\mathbb{E}\langle\mathrm{div}_x(R_0(\bar{E}(0))\rho),\mathrm{div}_x(\bar{E}(0)\rho)\rangle_{L^2(\mathbb{T}^N)}. \quad (6.17)$$

By developing the divergence terms in (6.17), we recognize one of the terms in (6.16) (with the opposite sign) and some lower order terms (lower with respect to the order of differentiability in x). Therefore we have a bound from below

$$(6.15)+(6.17)\geq-\|\nabla_x\rho\|^2_{L^2(\mathbb{T}^N)}-C(\mathtt{R})\|\rho\|_{L^2(\mathbb{T}^N)}\|\nabla_x\rho\|_{L^2(\mathbb{T}^N)}-C(\mathtt{R})\|\rho\|^2_{L^2(\mathbb{T}^N)},$$

where, using (2.15), $C(\mathtt{R})$ is a constant which depends on $\mathtt{R}$. It follows that

$$(6.15)+(6.17)\geq-\frac{1}{2}\|\nabla_x\rho\|^2_{L^2(\mathbb{T}^N)}-C(\mathtt{R})\|\rho\|^2_{L^2(\mathbb{T}^N)},$$

for a different constant $C(\mathtt{R})$. We conclude to the energy estimate

$$\mathbb{E}\|\rho_t\|^2_{L^2(\mathbb{T}^N)} + \int_0^t \mathbb{E}\|\nabla_x\rho_s\|^2_{L^2(\mathbb{T}^N)}ds \leq \mathbb{E}\|\rho_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^N)} + C(\mathtt{R})\int_0^t \mathbb{E}\|\rho_s\|^2_{L^2(\mathbb{T}^N)}ds.$$

Using the Gronwall inequality and the initial condition $\rho_0 = \rho_{\mathrm{in}}$, this gives, for $0 \leq t \leq T$,

$$\mathbb{E}\|\rho_t\|^2_{L^2(\mathbb{T}^N)} + \int_0^t \mathbb{E}\|\nabla_x\rho_s\|^2_{L^2(\mathbb{T}^N)}ds \leq C(\mathtt{R},T)\mathbb{E}\|\rho_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^N)}, \tag{6.18}$$

where $C(\mathtt{R},T)$ depends on $\mathtt{R}$ and T.

6.1.3 First and second correctors

Recall (see (1.16), (1.17)) that

$$\bar{J}_m(f) = \iint_{\mathbb{T}^N\times\mathbb{R}^N} |v|^m f(x,v)dxdv, \quad G_m = \left\{f \in L^1(\mathbb{T}^N\times\mathbb{R}^N); \bar{J}_m(f) < +\infty\right\}.$$

Recall also that $F = C^1(\mathbb{T}^N)$. Let us introduce the following notations. We write $a \lesssim b$ with the meaning that $a \leq Cb$, where the constant C may depend on $\mathtt{R}$ (*cf.* (2.6)), on $C^0_{\mathtt{R}}$ (*cf.* (2.16)), on various irrelevant constants, and on the dimension N.

Proposition 6.1. *Let φ be of the form (6.3), with $\xi \in C^3(\mathbb{T}^N)$ and ψ a Lipschitz function on $\mathbb{R}$ such that $\psi' \in C^\infty_b(\mathbb{R})$. Let φ_1, φ_2 be the correctors defined by (6.4), (6.8) respectively. Then φ_1, φ_2 satisfy $\mathscr{L}_\sharp\varphi_i(f,\mathbf{e}) < +\infty$, $\mathscr{L}_\flat\varphi_i(f,\mathbf{e}) < +\infty$ for all $f \in G_3$, $\mathbf{e} \in F$ and are in the domain of $\mathscr{L}^\varepsilon$. We have the estimates*

$$|\varphi_1(f,\mathbf{e})| \lesssim \|\psi\|_{W^{1,\infty}(\mathbb{R})}\|\xi\|_{C^1(\mathbb{T}^N)}(\bar{J}_0(f)+\bar{J}_1(f)), \tag{6.19}$$

and

$$|\mathscr{L}_\flat\varphi_1(f,\mathbf{e})| \lesssim \|\psi\|_{W^{2,\infty}(\mathbb{R})}\|\xi\|^2_{C^2(\mathbb{T}^N)}(|\bar{J}_0(f)|^2+|\bar{J}_2(f)|^2), \tag{6.20}$$

on φ_1 and the following estimates on φ_2:

$$|\varphi_2(f,\mathbf{e})| \lesssim \|\psi\|_{W^{2,\infty}(\mathbb{R})}\|\xi\|^2_{C^2(\mathbb{T}^N)}(|\bar{J}_0(f)|^2+|\bar{J}_2(f)|^2), \tag{6.21}$$

and

$$|\mathscr{L}_\flat\varphi_2(f,\mathbf{e})| \lesssim \|\psi\|_{W^{3,\infty}(\mathbb{R})}\|\xi\|^3_{C^3(\mathbb{T}^N)}(|\bar{J}_0(f)|^3+|\bar{J}_3(f)|^3), \tag{6.22}$$

for all $f \in G_3$, for all $\mathbf{e} \in F$ with $\|\mathbf{e}\|_F \leq \mathtt{R}$.

Proof of Proposition 6.1. Let us focus on the estimate (6.21) on $|\varphi_2(f,\mathbf{e})|$. Since

$$(h, D_f\varphi(\rho)) = \psi'\left(\langle\rho,\xi\rangle_{L^2(\mathbb{T}^N)}\right)\langle\rho(h),\xi\rangle_{L^2(\mathbb{T}^N)},$$

the equation (6.6) gives a first corrector

$$\varphi_1(f,e) = \psi'\left(\langle\rho,\xi\rangle_{L^2(\mathbb{T}^N)}\right)\langle J(f)+\rho(f)R_0(\mathbf{e}),\nabla_x\xi\rangle_{L^2(\mathbb{T}^N)}. \tag{6.23}$$

For simplicity, let us denote by $\psi', \psi'', \ldots$ the derivatives of ψ evaluated at the point $\langle \rho, \xi\rangle_{L^2(\mathbb{T}^N)}$. By (6.9), we have

$$\mathscr{L}_\flat \varphi_1(f,\mathbf{e}) = \psi' \int_{\mathbb{T}^N} K(f){:}D_x^2\xi + J(f)\cdot\nabla_x[R_0(\mathbf{e})\cdot\nabla_x\xi]dx \\ + \psi''\langle J(f),\nabla_x\xi\rangle_{L^2(\mathbb{T}^N)}\langle J(f)+\rho(f)R_0(\mathbf{e}),\nabla_x\xi\rangle_{L^2(\mathbb{T}^N)}, \quad (6.24)$$

and

$$\varphi_2(f,\mathbf{e}) = \int_0^\infty \mathbb{E}_{(f,\mathbf{e})}\left[\mathscr{L}_\flat\varphi_1(f_t,E_t) - \langle \mathscr{L}_\flat\varphi_1, \mu_\rho\rangle\right]dt, \quad (6.25)$$

where f_t is obtained either by (3.3) or (3.4) with $s=0$. Consider the LB-case. There are two terms in f_t and three terms in $\mathscr{L}_\flat\varphi_1$, which makes at least six terms to consider. We find out more than six terms actually, because of the translations in v. Consider the first term in (3.3). By (5.22), and for

$$w_t := \int_0^t E_s(\mathbf{e})ds,$$

we have

$$\begin{aligned} K(f(\cdot - w_t)) =& K(f) + J(f)\otimes w_t + w_t\otimes J(f) + \rho(f)w_t^{\otimes 2}, \\ J(f(\cdot - w_t)) =& J(f) + \rho(f)w_t. \end{aligned}$$

In (6.24)-(6.25), and regarding the linear terms with factor ψ', this gives the contributions

$$\Phi_{2,a} = \psi'\int_{\mathbb{T}^N}\int_0^\infty e^{-t}\mathbb{E}\left[K(f)+2J(f)\otimes w_t + \rho(f)w_t^{\otimes 2}\right]{:}D_x^2\xi dtdx,$$

and

$$\Phi_{2,b} = \psi'\int_{\mathbb{T}^N}\int_0^\infty e^{-t}\mathbb{E}\left[(J(f)+\rho(f)w_t)\cdot\nabla_x[R_0(E_t(\mathbf{e}))\cdot\nabla_x\xi]\right]dtdx.$$

Using the bound $\|w_t\|_F \le t\sup_{s\in[0,t]}\|E_s(\mathbf{e})\|_F$ and (2.6), (2.15), we have

$$|\Phi_{2,a}|, |\Phi_{2,b}| \lesssim \|\psi'\|_{L^\infty(\mathbb{R})}\|\xi\|_{C^2(\mathbb{T}^N)}(\bar{J}_0(f)+\bar{J}_1(f)+\bar{J}_2(f)).$$

Since $\bar{J}_1(f)\le \frac{1}{2}\bar{J}_0(f)+\frac{1}{2}\bar{J}_2(f)$, this gives us a bound by $\|\psi'\|_{L^\infty(\mathbb{R})}\|\xi\|_{C^2(\mathbb{T}^N)}(\bar{J}_0(f)+\bar{J}_2(f))$. Using (5.22) again, and still regarding the linear terms with factor ψ' only, we see that the second term in the expansion (3.3) of f_t^{LB} has the contributions

$$\Phi_{2,c} = \psi'\int_{\mathbb{T}^N}\int_0^\infty(\theta_c(t)-\theta_c(+\infty))dtdx, \quad \Phi_{2,d} = \psi'\int_{\mathbb{T}^N}\int_0^\infty(\theta_d(t)-\theta_d(+\infty))dtdx,$$

where

$$\theta_c(t) = \rho(f)\int_0^t e^{-(t-\sigma)}\left[K+\mathbb{E}\left(\int_\sigma^t E_s(\mathbf{e})ds\right)^{\otimes 2}\right]{:}D_x^2\xi d\sigma, \quad (6.26)$$

$$\theta_d(t) = \rho(f)\int_0^t e^{-(t-\sigma)}\mathbb{E}\left[\int_\sigma^t E_s(\mathbf{e})ds\cdot\nabla_x[R_0(E_t(\mathbf{e}))\cdot\nabla_x\xi]\right]d\sigma, \quad (6.27)$$

By standard manipulations on the integrals in (6.26), we have

$$\theta_c(t) = \rho(f)(1-e^{-t})K{:}D_x^2\xi + 2\rho(f)\int_0^t e^{-\sigma}\int_0^\sigma\int_r^\sigma \Gamma_{\mathbf{e}}(t-r,t-s){:}D_x^2\xi dsdrd\sigma,$$

where the covariance $\Gamma_{\mathbf{e}}$ is defined by (2.17). The most delicate term to estimate in $\Phi_{2,c}$ is

$$\Phi_{2,c}^* = 2\int_{\mathbb{T}^N}\rho(f)\int_0^\infty\int_0^t e^{-\sigma}\int_0^\sigma\int_r^\sigma \left[\Gamma_{\mathbf{e}}(t-r,t-s) - \bar\Gamma(s-r)\right]{:}D_x^2\xi dsdrd\sigma dtdx.$$

The other terms are bounded by $\|\xi\|_{C^2(\mathbb{T}^N)}(\bar J_0(f)+\bar J_2(f))$ using (2.6). Using also (2.18), we have

$$\begin{aligned}|\Phi_{2,c}^*| &\lesssim 2\bar J_0(f)\|\xi\|_{C^2(\mathbb{T}^N)}\int_0^\infty\int_0^t e^{-\sigma}\int_0^\sigma\int_r^\sigma \gamma_{\mathrm{mix}}(t-s)dsdrd\sigma dt\\ &\lesssim 2\bar J_0(f)\|\xi\|_{C^2(\mathbb{T}^N)}\int_0^\infty\int_0^t s(e^{-s}-e^{-t})\gamma_{\mathrm{mix}}(t-s)dsdt.\end{aligned}$$

Neglecting the term $-e^{-t}$ and using (2.13) gives a bound $|\Phi_{2,c}^*|\lesssim 2\bar J_0(f)\|\xi\|_{C^2(\mathbb{T}^N)}$. We have also

$$\theta_d(t) = \rho(f)\int_0^t e^{-\sigma}\int_0^\sigma \mathbb{E}\left[E_{t-s}(\mathbf{e})\cdot\nabla_x[R_0(E_t(\mathbf{e}))\cdot\nabla_x\xi]\right]dsd\sigma.$$

Conditioning on $\mathcal{G}_{t-s}$, we see that

$$\mathbb{E}\left[E_{t-s}(\mathbf{e})\otimes R_0(E_t(\mathbf{e}))\right] = e^{(t-s)A}\left[\psi\otimes e^{sA}R_0\psi\right](\mathbf{e}),\quad \psi(\mathbf{e})=\mathbf{e}.$$

By (2.12), (2.15), (2.13), we obtain

$$\begin{aligned}&\Bigg\|\int_0^\infty\int_0^t e^{-\sigma}\int_0^\sigma\Big[e^{(t-s)A}\left[\psi\cdot\nabla_x(e^{sA}R_0\psi\cdot\nabla_x\xi)\right](\mathbf{e})\\ &\qquad\qquad - \langle\psi\cdot\nabla_x(P_sR_0\psi\cdot\nabla_x\xi),\nu\rangle\Big]dsd\sigma dt\Bigg\|_{C(\mathbb{T}^N)}\\ &\qquad\le \mathtt{R}^2\int_0^\infty\int_0^t e^{-\sigma}\int_0^\sigma \gamma_{\mathrm{mix}}(t-s)dsd\sigma dt\|\xi\|_{C^1(\mathbb{T}^N)}\\ &\qquad\le \mathtt{R}^2\|\xi\|_{C^1(\mathbb{T}^N)}.\end{aligned}$$

With this estimate, it is easy to prove that $|\Phi_{2,d}|\lesssim\|\xi\|_{C^2(\mathbb{T}^N)}\bar J_0(f)$. Let us look as the quadratic terms with factor ψ'' now. There are two terms in (3.3), so four terms $\Phi_{2,e},\dots,\Phi_{2,h}$ to consider here. The first term in (3.3) has a factor e^{-t}, like in Φ_a, Φ_b. There is no contribution from $\langle\mathscr{L}_\flat\varphi_1,\mu_\rho\rangle$ in $\Phi_{2,e},\Phi_{2,f},\Phi_{2,g}$ hence, and the convergence of the integral in (6.25) is clear. Therefore, using the same arguments as above, we obtain the estimates

$$|\Phi_{2,e}|,|\Phi_{2,f}|,|\Phi_{2,g}|\lesssim\|\psi''\|_{L^\infty(\mathbb{R})}\|\nabla_x\xi\|_{C^1(\mathbb{T}^N)}^2(|\bar J_0(f)|^2+|\bar J_1(f)|^2).\tag{6.28}$$

Let us illustrate this on the example of $\Phi_{2,g}$. We have

$$\Phi_{2,g} = \psi'' \int_0^\infty e^{-t} \int_0^t e^{-(t-\sigma)} \mathbb{E}\left[\int_\sigma^t \langle \rho(f) E_r(\mathbf{e}), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} dr \right. \\ \left. \times \langle J(f) + \rho(f) \int_0^t E_s(\mathbf{e}) ds + \rho(f) R_0(E_t), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} \right],$$

which gives (6.28). The last term $\Phi_{2,h}$ is

$$\Phi_{2,h} = \psi'' \int_0^\infty (\theta_h(t) - \theta_h(+\infty)) dt,$$

where

$$\theta_h(t) = \mathbb{E} \int_0^t \int_\sigma^t \int_0^t \int_{\sigma'}^t e^{-(t-\sigma)} e^{-(t-\sigma')} \langle \rho(f) E_s(\mathbf{e}), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} \\ \times \langle \rho(f) E_{s'}(\mathbf{e}) + c(t)^{-1} \rho(f) R_0(E_t(\mathbf{e})), \nabla_x \xi \rangle_{L^2(\mathbb{T}^N)} ds' ds d\sigma' d\sigma.$$

The coefficient $c(t)$ is

$$c(t) = \int_0^t \int_{\sigma'}^t e^{-(t-\sigma')} ds' d\sigma' = \int_0^t \sigma e^{-\sigma} = 1 - (t+1) e^{-t}.$$

The technique used to estimate the terms $\Phi_{2,c}$ and $\Phi_{2,d}$ applies here to give

$$|\Phi_{2,h}| \lesssim \|\psi''\|_{L^\infty(\mathbb{R})} \|\nabla_x \xi\|^2_{C^1(\mathbb{T}^N)} |\bar{J}_0(f)|^2.$$

This concludes the estimate on φ_2 in the LB-case. The estimate on φ_2 in the FP-case is obtained by the same arguments. This follows from the expressions for $K(f_t)$, $J(f_t)$, which involve various terms, similar to those estimated in the LB-case. For example, a careful computation based on (3.4) and (5.22) gives

$$K(f_t^{\mathrm{FP}}) = \rho(f) \left[(1 - e^{-2t}) K + \left(\int_0^t e^{-(t-\sigma)} E_\sigma(\mathbf{e}) d\sigma \right)^{\otimes 2} \right] + e^{-2t} K(f) \\ + e^{-t} \left[\int_0^t e^{-(t-\sigma)} E_\sigma(\mathbf{e}) d\sigma \otimes J(f) + J(f) \otimes \int_0^t e^{-(t-\sigma)} E_\sigma(\mathbf{e}) d\sigma \right].$$

A comparable expansion for $J(f_t^{\mathrm{FP}})$ gives the result, like in the LB-case. Using (2.16), a careful study of the terms composing φ_2 shows that φ_1 and φ_2 are of the form (4.28) with some ξ_i as in Remark 4.1. By Proposition 4.4, we deduce that $\mathscr{L}_\sharp \varphi_i(f, \mathbf{e}) < +\infty$, $\mathscr{L}_\flat \varphi_i(f, \mathbf{e}) < +\infty$ for all $f \in G_3$, $\mathbf{e} \in F$ and that φ_1 and φ_2 are in the domain of $\mathscr{L}^\varepsilon$. There remains to prove (6.22). Compared to the development of φ_2, when computing $\mathscr{L}_\flat \varphi_2$, still more terms appear, which combine the derivatives of ψ up to the order three. However, all the questions of convergence of the integrals with respect to t have been dealt with in the estimate of φ_2. Although lengthy, it is not problematic, to prove (6.22): we do not expound that part thus. □

Remark 6.2 (Linear test function). In Section 6.3, we apply Proposition 6.1 to a linear test-function $\varphi(\rho) = \langle \rho, \xi \rangle_{L^2(\mathbb{T}^N)}$, which means $\psi' = 1$, $\psi'' = 0$. In that case, the bounds on the first corrector is a little bit simpler: we have

$$|\varphi_1(f, \mathbf{e})| \lesssim \|\xi\|_{C^1(\mathbb{T}^N)}(\bar{J}_0(f) + \bar{J}_1(f)), \tag{6.29}$$

and

$$|\mathscr{L}_\flat \varphi_1(f, \mathbf{e})| \lesssim \|\xi\|_{C^2(\mathbb{T}^N)}(\bar{J}_0(f) + \bar{J}_2(f)), \tag{6.30}$$

for all $f \in G$, for all $\mathbf{e} \in F$ with $\|\mathbf{e}\|_F \leq \mathtt{R}$.

By Theorem 4.3, Remark 4.2 and Theorem B.3, we obtain the following corollary to Proposition 6.1.

Corollary 6.2. *Let φ be of the form (6.3), with $\xi \in C^3(\mathbb{T}^N)$ and ψ a Lipschitz function on $\mathbb{R}$ such that $\psi' \in C_b^\infty(\mathbb{R})$. Let φ_1, φ_2 be the correctors defined by (6.4), (6.8) respectively. Let θ be the correction of φ at order 0, 1 or 2:*

$$\theta \in \{\varphi, \varphi + \varepsilon\varphi_1, \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2\}.$$

Then

$$M_\theta^\varepsilon(t) := \theta(f_t^\varepsilon, \bar{E}_t^\varepsilon) - \theta(f_{\mathrm{in}}, \bar{E}_0) - \int_0^t \mathscr{L}^\varepsilon \theta(f_s^\varepsilon, \bar{E}_s^\varepsilon) ds \tag{6.31}$$

is a $(\mathcal{G}_{t/\varepsilon^2})$-martingale with predictible quadratic variation given by

$$\langle M_\theta^\varepsilon, M_\theta^\varepsilon \rangle_t = \int_0^t \left[\mathscr{L}^\varepsilon |\theta|^2 - 2\theta \mathscr{L}^\varepsilon \theta\right] (f^\varepsilon(s), \bar{E}^\varepsilon(s)) ds,$$

for all $t \geq 0$.

6.2 Bounds on the moments

Recall that $\bar{J}_m(f)$ denotes the m-th moment of f (see (1.16)) and that G_m is the state space of functions $f \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$ such that $\bar{J}_m(f) < +\infty$.

Proposition 6.3. *Let $f_0^\varepsilon \in G_m$. Let (f_t^ε) be the unique mild solution to (1.7) on $[0, T]$ given by Proposition 4.1 or 4.2. Then, for all $m \in \mathbb{N}$, almost-surely, for all $t \geq 0$,*

$$\bar{J}_m(f_t^\varepsilon) \leq C(\mathtt{R}, m, t)\left[\bar{J}_m(f_0^\varepsilon) + \bar{J}_0(f_0^\varepsilon)\right], \tag{6.32}$$

where $C(\mathtt{R}, m, t)$ is a constant which is bounded for t in a bounded set.

Proof of Proposition 6.3. By density, we can assume that $f_{\mathrm{in}} \in W^{2,1}(\mathbb{T}^N \times \mathbb{R}^N)$. We can also replace $v \mapsto |v|^m$ by $v \mapsto |v|^m \chi_\eta(v)$, where χ_η is a function with compact support which converges pointwise to 1 when $\eta \to 0$. By the results of propagation of regularity given in Proposition 4.1 and Proposition 4.2, the following computations are licit then. For simplicity, we take directly $\chi \equiv 1$. First, we have

$$\frac{d}{dt}\bar{J}_{2m}(f_t^\varepsilon) = \frac{1}{\varepsilon^2}\left[\bar{J}_{2m}(Qf_t^\varepsilon) + 2m \iint_{\mathbb{T}^N \times \mathbb{R}^N} |v|^{2(m-1)} v \cdot \bar{E}_t^\varepsilon f_t^\varepsilon(x, v) dx dv\right]. \tag{6.33}$$

If $m = 0$, then, for all $t \geq 0$, almost-surely, $\bar{J}_0(f_t^\varepsilon) = \bar{J}_0(f_0^\varepsilon)$ since the equation is conservative. If $m > 0$, then we use the following inequality (which is a consequence of Young's inequality)

$$2m|v|^{2m-1} \leq \frac{1}{2\mathtt{R}}|v|^{2m} + [2\mathtt{R}(2m-1)]^{2m-1},$$

to infer, by (6.33) and (2.8), that

$$\frac{d}{dt}\bar{J}_{2m}(f_t^\varepsilon) \leq \frac{1}{\varepsilon^2}\left[\bar{J}_{2m}(Qf_t^\varepsilon) + \frac{1}{2}\bar{J}_{2m}(f_t^\varepsilon) + \mathtt{R}[2\mathtt{R}(2m-1)]^{2m-1}\bar{J}_0(f_t^\varepsilon)\right].$$

We have, in the case $Q = Q_{\mathrm{LB}}$,

$$\bar{J}_{2m}(Q_{\mathrm{LB}}f) = \bar{J}_{2m}(M)\bar{J}_0(f) - \bar{J}_{2m}(f).$$

If $Q = Q_{\mathrm{FP}}$, then

$$\begin{aligned}\bar{J}_{2m}(Q_{\mathrm{FP}}f) =& -2m\iint_{\mathbb{T}^N\times\mathbb{R}^N}|v|^{2(m-1)}v\cdot(\nabla_v f(x,v) + vf(x,v))dxdv\\ =&(N+2(m-1))\bar{J}_{2(m-1)}(f) - 2m\bar{J}_{2m}(f).\end{aligned}$$

In the first case $Q = Q_{\mathrm{LB}}$, we obtain

$$\bar{J}_{2m}(f_t^\varepsilon) \leq e^{-\frac{t}{2\varepsilon^2}}\bar{J}_{2m}(f_0^\varepsilon) + 2(1-e^{-\frac{t}{2\varepsilon^2}})\left[\bar{J}_{2m}(M) + \mathtt{R}[2\mathtt{R}(2m-1)]^{2m-1}\right]\bar{J}_0(f_0^\varepsilon).$$

This gives (6.32). If $Q = Q_{\mathrm{FP}}$, we conclude similarly by a recursive argument on m. □

6.3 Tightness

For $\sigma > 0$, we denote by $H^{-\sigma}(\mathbb{T}^N)$ the dual space of $H^\sigma(\mathbb{T}^N)$. Let $J_1^\sigma = (\mathrm{Id} - \Delta_x)^{-\sigma}$. In the standard Fourier basis (w_k) of $L^2(\mathbb{T}^N)$, J_1^σ is given by

$$J_1^\sigma w_k = (1+\lambda_k)^{-\sigma}w_k, \quad \lambda_k = 4\pi^2|k|^2, \quad w_k(x) = \exp(2\pi i k\cdot x).$$

As $J_1^{\sigma/2}$ is an isometry $L^2(\mathbb{T}^N) \to H^\sigma(\mathbb{T}^N)$, the norm on $H^{-\sigma}(\mathbb{T}^N)$ is

$$\|\Lambda\|_{H^{-\sigma}(\mathbb{T}^N)} = \left[\sum_{k\in\mathbb{Z}^d}|\langle\Lambda, J_1^{\sigma/2}w_k\rangle_{L^2(\mathbb{T}^N)}|^2\right]^{1/2}. \tag{6.34}$$

Proposition 6.4 (Tightness). *Let $f_0^\varepsilon \in G_3$. Let (f_t^ε) be the unique mild solution to (1.7) on $[0,T]$ given by Proposition 4.1 or 4.2. Then $(\rho_t^\varepsilon)_{t\in[0,T]}$ is tight in the space $C([0,T];H^{-1}(\mathbb{T}^N))$.*

Proof of Proposition 6.4. Let us introduce the decomposition

$$\rho^\varepsilon = \theta^\varepsilon + \zeta^\varepsilon, \quad \theta^\varepsilon = \varepsilon \mathrm{div}_x(J(f^\varepsilon) + \rho(f^\varepsilon) R_0(\bar{E}^\varepsilon_t)). \tag{6.35}$$

Note that, contrary to ρ^ε, which has continuous trajectories, θ^ε and ζ^ε are, *a priori*, càdlàg processes, just like $\bar{E}^\varepsilon$. We show first that ρ^ε is close to ζ^ε in the norm of $C([0,T];H^{-1}(\mathbb{T}^N))$ and then prove in a second step that (ζ^ε) is tight in the Skorohod space $D([0,T];H^{-1}(\mathbb{T}^N))$. In the third last step, we show that $(\rho^\varepsilon_t)_{t\in[0,T]}$ is tight in $C([0,T];H^{-1}(\mathbb{T}^N))$.

Step 1. ρ^ε is close to ζ^ε. This is a straightforward consequence of the bound on the moments (6.32). Let us extend the notation $a \lesssim b$ to denote the inequality $a \le Cb$, where the factor C may depend on $\mathtt{R}$, on $C^0_{\mathtt{R}}$, on N and also on $\sup_{0<\varepsilon<1} \bar{J}_m(f^\varepsilon_0)$ for $m = 0, \dots, 3$ and on T. Note that C should not depend on ε, nor on ω. Then, by (6.32), we have $\sup_{t\in[0,T]} \|\theta^\varepsilon_t\|_{H^{-1}(\mathbb{T}^N)} \lesssim \varepsilon$.

Step 2. (ζ^ε) is tight in $D([0,T];H^{-1}(\mathbb{T}^N))$. The bound on the moments (6.32) shows that $\sup_{t\in[0,T]} \|\rho^\varepsilon_t\|_{L^2(\mathbb{T}^N)}$ and $\sup_{t\in[0,T]} \|\theta^\varepsilon_t\|_{H^{-1}(\mathbb{T}^N)}$ are almost-surely bounded. Since $\zeta^\varepsilon = \rho^\varepsilon - \theta^\varepsilon$, the quantity $\sup_{t\in[0,T]} \|\zeta^\varepsilon_t\|_{H^{-1}(\mathbb{T}^N)}$ is also almost-surely bounded. By [13, Theorem 3.1], it is sufficient therefore to prove that, for all $\xi \in C^2(\mathbb{T}^N)$, the family of real-valued processes $\langle \zeta^\varepsilon, \xi\rangle_{L^2(\mathbb{T}^N)}$ is tight in $D([0,T])$. Let us fix such a ξ, and let us set $\varphi(\rho) = \langle \rho, \xi\rangle_{L^2(\mathbb{T}^N)}$ and $\gamma^\varepsilon = \langle \zeta^\varepsilon, \xi\rangle_{L^2(\mathbb{T}^N)}$. Denote by

$$\varphi_1(f, \mathbf{e}) = \langle J(f) + \rho(f) R_0(\mathbf{e}), \xi\rangle_{L^2(\mathbb{T}^N)}$$

the first corrector associated to φ. To obtain an estimate on the time increments of γ^ε, we introduce the perturbed test function $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1$ and the martingale (*cf.* (6.31))

$$M^\varepsilon(t) = \varphi^\varepsilon(f^\varepsilon(t), \bar{E}^\varepsilon(t)) - \varphi^\varepsilon(f^\varepsilon(0), \bar{E}^\varepsilon(0)) - \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon(s), \bar{E}^\varepsilon(s)) ds. \tag{6.36}$$

We have thus

$$\gamma^\varepsilon_t = \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon(\sigma), \bar{E}^\varepsilon(\sigma)) d\sigma + M^\varepsilon(t). \tag{6.37}$$

To prove that (γ^ε_t) is tight in $D([0,T])$, we will use the Aldous criterion, [12, Theorem 4.5, p.356]. Let $1 > \theta > 0$. Let τ_1, τ_2 be some $(\mathcal{F}^\varepsilon_t)$-stopping times such that

$$\tau_1 \le \tau_2 \le \tau_1 + \theta, \quad \tau_2 \le T, \text{ a.s.} \tag{6.38}$$

By the Doob optional sampling theorem, we have

$$\mathbb{E}\left[|M^\varepsilon(\tau_2) - M^\varepsilon(\tau_1)|^2\right] = \mathbb{E}\left[|M^\varepsilon(\tau_2)|^2 - |M^\varepsilon(\tau_1)|^2\right].$$

Let (A^ε_t) be defined by (B.14), where $\mathscr{L} = \mathscr{L}^\varepsilon$ and $\varphi = \varphi^\varepsilon$. By Theorem B.3, $|M^\varepsilon(t)|^2 - A^\varepsilon_t$ is a martingale. Consequently,

$$\mathbb{E}\left[|M^\varepsilon(\tau_2) - M^\varepsilon(\tau_1)|^2\right] = \mathbb{E}\left[A^\varepsilon_{\tau_2} - A^\varepsilon_{\tau_1}\right] \lesssim \theta,$$

We also have

$$\mathbb{E}\left|\int_{\tau_1}^{\tau_2}\mathscr{L}^\varepsilon\varphi^\varepsilon(f^\varepsilon(s),\bar{E}^\varepsilon(s))ds\right|^2\lesssim\theta^2.$$

Due to the decomposition (6.37), we conclude that the increments of γ^ε also satisfy the estimate

$$\mathbb{E}\left[|\gamma^\varepsilon_{\tau_2}-\gamma^\varepsilon_{\tau_1}|^2\right]\lesssim\theta.$$

By the Markov inequality, the Aldous criterion

$$\lim_{\theta\to 0}\limsup_{\varepsilon\in(0,1)}\sup_{\tau_1,\tau_2}\mathbb{P}(|\gamma^\varepsilon_{\tau_2}-\gamma^\varepsilon_{\tau_1}|>\eta)=0$$

is satisfied for all $\eta>0$, (the sup on τ_1,τ_2 being the sup over the stopping times satisfying (6.38)). This gives the desired conclusion.

Step 3. (ρ^ε) is tight in $C([0,T];H^{-1}(\mathbb{T}^N))$. Using Step 1. and [12, Lemma 3.31 p.352], we deduce that (X^ε_t) is tight in $D([0,T];\mathbb{R}^d)$. Since (X^ε_t) is in $C([0,T];\mathbb{R}^d)$, it is actually tight in $C([0,T];\mathbb{R}^d)$. To establish this fact, it is sufficient to use the relation $w_\rho(\delta)\leq 2w'_\rho(\delta)$ ($t\mapsto\rho(t)$ continuous) between the modulus of continuity of continuous functions and the modulus of continuity of càdlàg functions, see [3, (12.10) p. 123]. □

6.4 Convergence to the solution of a Martingale problem

Assume that the hypotheses of Proposition 6.4 are satisfied. Let $\varepsilon_\mathbb{N}=\{\varepsilon_n;n\in\mathbb{N}\}$, where $(\varepsilon_n)\downarrow 0$. By the Skorohod theorem [3, p. 70], there is a subset of $\varepsilon_\mathbb{N}$, which we still denote by $\varepsilon_\mathbb{N}$, a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, some random variables $\{\tilde\rho^\varepsilon;\varepsilon\in\varepsilon_\mathbb{N}\}$, $\tilde\rho$ on $C([0,T];H^{-1}(\mathbb{T}^N))$, such that

1. for all $\varepsilon\in\varepsilon_\mathbb{N}$, the laws of ρ^ε and $\tilde\rho^\varepsilon$ as $C([0,T];H^{-1}(\mathbb{T}^N))$-random variables coincide,
2. $\tilde{\mathbb{P}}$-a.s., $(\tilde\rho^\varepsilon)$ is converging to $\tilde\rho$ in $C([0,T];H^{-1}(\mathbb{T}^N))$ along $\varepsilon_\mathbb{N}$.

Let $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ be the natural filtration of $(\tilde\rho(t))_{t\in[0,T]}$. Our aim is to show that the process $(\tilde\rho(t))_{t\in[0,T]}$ is solution of the martingale problem associated to the limit generator $\mathscr{L}$.

Proposition 6.5 (Martingale)**.** *Let $\xi\in C^3(\mathbb{T}^N)$, and let φ be defined on the space $H^{-1}(\mathbb{T}^N)$ by $\varphi(\rho)=\psi\left(\langle\rho,\xi\rangle_{H^{-1},H^1}\right)$, where ψ is a Lipschitz function on $\mathbb{R}$ such that $\psi'\in C^\infty_b(\mathbb{R})$. Then the process*

$$\tilde{M}_\varphi(t):=\varphi(\tilde\rho(t))-\varphi(\tilde\rho(0))-\int_0^t\mathscr{L}\varphi(\tilde\rho(s))ds \tag{6.39}$$

is a continuous martingale with respect to $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$.

Proof of Proposition 6.5. Let $0\leq s\leq t\leq T$. Let $0\leq t_1<\cdots<t_n\leq s$ and let Θ be a continuous and bounded function on $[H^{-1}(\mathbb{T}^N)]^n$. Note that $\tilde{\mathcal{F}}_s$ is generated by the

random variables $\Theta(\tilde\rho(t_1),\dots,\tilde\rho(t_n))$, for $n\in\mathbb{N}^*$, $(t_i)_{1,n}$ and Θ as above. Our aim is therefore to prove that

$$\mathbb{E}\left[(\tilde M_\varphi(t)-\tilde M_\varphi(s))\Theta(\tilde\rho(t_1),\dots,\tilde\rho(t_n))\right]=0. \tag{6.40}$$

Let $\varphi^\varepsilon=\varphi+\varepsilon\varphi_1+\varepsilon^2\varphi_2$ be the second order correction of φ, with φ_1 and φ_2 given by Proposition 6.1. We start from the identity (see (6.31))

$$\mathbb{E}\left[(M^\varepsilon_\varphi(t)-M^\varepsilon_\varphi(s))\Theta(\rho^\varepsilon(t_1),\dots,\rho^\varepsilon(t_n))\right]=0, \tag{6.41}$$

where

$$M^\varepsilon_\varphi(t):=\varphi^\varepsilon(f^\varepsilon(t),\bar E^\varepsilon_t)-\varphi^\varepsilon(f_{\mathrm{in}},\bar E^\varepsilon_0)-\int_0^t \mathscr{L}^\varepsilon\varphi^\varepsilon(f^\varepsilon(s),\bar E^\varepsilon_s)ds, \tag{6.42}$$

Recall that $\mathscr{L}^\varepsilon\varphi^\varepsilon=\mathscr{L}\varphi+\varepsilon\mathscr{L}_\flat\varphi_2$. By (6.41), the estimates on the correctors (Proposition 6.1) and the uniform estimates on the moments of (f^ε_t) (Proposition 6.3), we have

$$\mathbb{E}\left[(X^\varepsilon_\varphi(t)-X^\varepsilon_\varphi(s))\Theta(\rho^\varepsilon(t_1),\dots,\rho^\varepsilon(t_n))\right]=\mathcal{O}(\varepsilon),$$

where the process $(X^\varepsilon_\varphi(t))$ is

$$X^\varepsilon_\varphi(t)=\varphi(\rho^\varepsilon(t))-\varphi(\rho_{\mathrm{in}})-\int_0^t\mathscr{L}\varphi(\rho^\varepsilon(s))ds.$$

By identities of the laws, it follows that

$$\tilde{\mathbb{E}}\left[\left(\varphi(\tilde\rho^\varepsilon(t))-\varphi(\tilde\rho^\varepsilon(s))-\int_s^t\mathscr{L}\varphi(\tilde\rho^\varepsilon(s))ds\right)\Theta(\tilde\rho^\varepsilon(t_1),\dots,\tilde\rho^\varepsilon(t_n))\right]=\mathcal{O}(\varepsilon). \tag{6.43}$$

We must examine the convergence of each terms in (6.43). By a.s convergence of $(\tilde\rho^\varepsilon)$ in $C([0,T];H^{-1}(\mathbb{T}^N))$ along $\varepsilon_{\mathbb{N}}$, we have

$$\left[\varphi(\tilde\rho^\varepsilon(t))-\int_0^t\mathscr{L}\varphi(\tilde\rho^\varepsilon(s))ds\right]\Theta(\tilde\rho^\varepsilon(t_1),\dots,\tilde\rho^\varepsilon(t_n))$$
$$\to\left[\varphi(\tilde\rho(t))-\int_0^t\mathscr{L}\varphi(\tilde\rho(s))ds\right]\Theta(\tilde\rho(t_1),\dots,\tilde\rho(t_n))$$

almost-surely when $\varepsilon\to0$ along $\varepsilon_{\mathbb{N}}$. Indeed, $\mathscr{L}\varphi$ is continuous on $H^{-1}(\mathbb{T}^N)$, in virtue of (6.10)-(6.13)-(6.14) and the fact that $\xi\in C^3(\mathbb{T}^N)$. Since Θ is bounded and $\varphi(\tilde\rho^\varepsilon(t))$ and $\mathscr{L}\varphi(\tilde\rho^\varepsilon(t))$ are a.s. bounded by a constant (a consequence of (6.32)), we can apply the dominated convergence theorem. This gives (6.40). □

6.5 Limit SPDE

6.5.1 Covariance

For $i, j \in \{1, \ldots, N\}$, $x, y \in \mathbb{T}^N$, we set

$$H(i, x, j, y) = \mathbb{E}\left(\left[R_0(\bar{E}_0(x))\right]_i \left[\bar{E}_0(y)\right]_j\right). \tag{6.44}$$

This defines a kernel on the product space $\left[L^2(\mathbb{T}^N)\right]^N$, and an associated operator S,

$$S\rho_i(x) = \sum_{j=1}^{N} \int_{\mathbb{T}^N} H(i, x, j, y)\rho_j(y)dy. \tag{6.45}$$

Proposition 6.6. *The operator S is symmetric, non-negative and trace-class on the space $\left[L^2(\mathbb{T}^N)\right]^N$.*

Proof of Proposition 6.6. That S is non-negative means $\langle S\rho, \rho\rangle \geq 0$, where $\langle \cdot, \cdot \rangle$ is the canonical scalar product on $\left[L^2(\mathbb{T}^N)\right]^N$ given by

$$\langle \rho, \rho' \rangle = \sum_{i=1}^{N} \langle \rho_i, \rho_i' \rangle_{L^2(\mathbb{T}^N)}.$$

By Lemma 5.6 indeed, we have

$$\langle S\rho, \rho\rangle = \lim_{\delta \to 0} \delta \mathbb{E} \left| \int_0^\infty e^{-\delta t} \langle \rho, \bar{E}_t \rangle dt \right|^2 \geq 0.$$

Let (ζ_k) be an orthonormal basis of $\left[L^2(\mathbb{T}^N)\right]^N$. Using the Bessel-Parseval identity, we have

$$\mathrm{Trace}(S) = \sum_k \langle S\zeta_k, \zeta_k \rangle = \mathbb{E}\langle R_0(\bar{E}_0), \bar{E}_0 \rangle \lesssim 1,$$

therefore S is trace-class. □

Proposition 6.7. *The operator S admits a square-root $S^{1/2}$ which is associated to a kernel $H^{(1/2)}$.*

Proof of Proposition 6.7. By the spectral theorem, there exists an orthonormal basis (ζ_k) of $\left[L^2(\mathbb{T}^N)\right]^N$ and some non-negative eigenvalues $\lambda_k \geq 0$ such that

$$S = \sum_{k \in \mathbb{N}} \lambda_k \zeta_k \otimes \zeta_k, \text{ meaning that } S\rho = \sum_{k \in \mathbb{N}} \lambda_k \langle \rho, \zeta_k \rangle \zeta_k, \ \forall \rho.$$

It follows that the operator $S^{1/2} := \sum_{k \in \mathbb{N}} \lambda_k^{1/2} \zeta_k \otimes \zeta_k$ is well-defined, $S = \left[S^{1/2}\right]^2$. In addition, H is given by

$$H(i, x, j, y) = \sum_{k \in \mathbb{N}} \lambda_k \zeta_k(i, x) \zeta_k(j, y)$$

and $S^{1/2}$ is associated to the kernel $H^{(1/2)}$, with

$$H^{(1/2)}(i,x,j,y)=\sum_{k\in\mathbb{N}}\lambda_k^{1/2}\zeta_{k,i}(x)\zeta_{k,j}(y)$$

and

$$H(i,x,j,y)=\langle H^{(1/2)}(i,x,\cdot),H^{(1/2)}(j,y,\cdot)\rangle. \tag{6.46}$$

This concludes the proof. □

6.5.2 Limit equation

Let $(\beta_k(t))_{k\in\mathbb{N}}$ be some independent one-dimensional Wiener processes, let (λ_k,ζ_k) denote the spectral elements of S, as in the Proof of Proposition 6.7 and let

$$W(t)=\sum_{k\in\mathbb{N}}\beta_k(t)\zeta_k \tag{6.47}$$

be a cylindrical Wiener process on $\left[L^2(\mathbb{T}^N)\right]^N$. Consider the equation (1.24) with the cylindrical Wiener process $W(t)$ given by (6.47). In a first formal step, we test (1.24) against a test-function $\xi\in C^2(\mathbb{T}^N)$. This gives

$$d\langle\rho_t,\xi\rangle_{L^2(\mathbb{T}^N)}=b(\rho_t,\xi)dt+\sum_{k\in\mathbb{N}}\sigma_k(\rho_t,\xi)d\beta_k(t), \tag{6.48}$$

where

$$b(\rho,\xi)=\langle\rho,\mathrm{div}_x(K_\sharp\nabla_x\xi)-\Psi\nabla_x\xi\rangle_{L^2(\mathbb{T}^N)},\quad \sigma_k(\rho,\xi)=\sqrt{2}\lambda_k^{1/2}\langle\rho\nabla_x\xi,\zeta_k\rangle_{L^2(\mathbb{T}^N)}.$$

Since

$$\frac{1}{2}\sum_{k\in\mathbb{N}}|\sigma_k(\rho,\xi)|^2=\left\|S^{1/2}(\rho\nabla_x\xi)\right\|^2_{[L^2(\mathbb{T}^N)]^N}, \tag{6.49}$$

the equation (1.24) is indeed the equation associated to the limit generator $\mathscr{L}$.

6.5.3 Resolution of the limit equation

Definition 6.1. Let $\rho_{\mathrm{in}}\in L^2(\mathbb{T}^N)$. Let $W(t)$ be given by (6.47), let $(\mathcal{F}_t^W)$ be the filtration generated by W. A $(\mathcal{F}_t^W)$-adapted process $\rho\in C([0,T];L^2(\mathbb{T}^N))$ is said to be a weak solution to (1.24)-(1.25) in $L^2(\mathbb{T}^N)$ if $\rho\in L^2(0,T;H^1(\mathbb{T}^N))$ almost-surely and

$$\langle\rho_t,\xi\rangle_{L^2(\mathbb{T}^N)}=\langle\rho_{\mathrm{in}},\xi\rangle_{L^2(\mathbb{T}^N)}+\int_0^t b(\rho_s,\xi)ds+\sum_{k\in\mathbb{N}}\int_0^t\sigma_k(\rho_s,\xi)d\beta_k(s), \tag{6.50}$$

for all $\xi\in C^2(\mathbb{T}^N)$, for all $t\in[0,T]$.

Theorem 6.8. *Let $S^{1/2}$ be the Hilbert-Schmidt operator defined in Section 6.5.1. Let $W(t)$ be given by (6.47). Then there exists a unique weak solution to (1.24)-(1.25) in $L^2(\mathbb{T}^N)$.*

Proof of Theorem 6.8. The result follows essentially from the energy estimate (6.18) derived in Remark 6.1. To justify it rigorously, we use a Fourier decomposition. Let $(w_n)_{n\in\mathbb{Z}^N}$ denote the usual Fourier basis on $\mathbb{T}^N$, $w_n(x) = \exp(2\pi in\cdot x)$. We take $\xi = w_n$ in (6.50) and apply the Itô formula to obtain

$$\frac{1}{2}\mathbb{E}|\langle\rho_t, w_n\rangle_{L^2(\mathbb{T}^N)}|^2 = \frac{1}{2}|\langle\rho_{\mathrm{in}}, w_n\rangle_{L^2(\mathbb{T}^N)}|^2 + \mathbb{E}\int_0^t \mathrm{Re}(\bar{b}(\rho_s, w_n)\langle\rho_s, w_n\rangle_{L^2(\mathbb{T}^N)})ds \\ + \frac{1}{2}\sum_{k\in\mathbb{N}}\mathbb{E}\int_0^t |\sigma_k(\rho_s, w_n)|^2 ds. \quad (6.51)$$

Since $\nabla w_n = 2\pi in w_n$ and $\rho_t \in H^1(\mathbb{T}^N)$ a.s. in (ω, t), the product $\bar{b}(\rho_s, w_n)\langle\rho_s, w_n\rangle_{L^2(\mathbb{T}^N)}$ is

$$-2\pi in\left[\overline{\langle -K_\sharp\nabla_x\rho_s - \Psi\rho_s, w_n\rangle_{L^2(\mathbb{T}^N)}}\langle\rho_s, w_n\rangle_{L^2(\mathbb{T}^N)}\right] \\ = -\overline{\langle K_\sharp\nabla_x\rho_s + \Psi\rho_s, w_n\rangle_{L^2(\mathbb{T}^N)}}\langle\nabla_x\rho_s, w_n\rangle_{L^2(\mathbb{T}^N)}.$$

When summing over $n \in \mathbb{Z}^N$, the Bessel-Parseval identity yields then

$$\sum_{n\in\mathbb{Z}^N}\mathbb{E}\int_0^t \mathrm{Re}(\bar{b}(\rho_s, w_n)\langle\rho_s, w_n\rangle_{L^2(\mathbb{T}^N)})ds\mathbb{E}\int_0^t\langle K_\sharp\nabla_x\rho_s + \Psi\rho_s, \nabla\rho_s\rangle_{L^2(\mathbb{T}^N)}ds,$$

a term which corresponds to the term (6.15) in Remark 6.1. Similarly, the identity (6.49) gives

$$\frac{1}{2}\sum_{k\in\mathbb{N}}|\sigma_k(\rho_s, w_n)|^2 = \left\|S^{1/2}(\rho_s\nabla_x w_n)\right\|^2_{[L^2(\mathbb{T}^N)]^N} = \langle S(\rho_s\nabla_x w_n), \rho_s\nabla_x w_n\rangle_{L^2(\mathbb{T}^N)}.$$

The operator S is the operator with kernel H (*cf.* (6.45)). Integration by parts gives

$$\frac{1}{2}\sum_{k\in\mathbb{N}}|\sigma_k(\rho_s, w_n)|^2 = \langle\mathrm{div}_x(R_0(\bar{E}_0)\rho_s), w_n\rangle_{L^2(\mathbb{T}^N)}\overline{\langle\mathrm{div}_x(\bar{E}_0\rho_s), w_n\rangle_{L^2(\mathbb{T}^N)}}.$$

By summation over $n \in \mathbb{Z}^N$, the Bessel-Parseval identity gives

$$\sum_{n\in\mathbb{Z}^N}\frac{1}{2}\sum_{k\in\mathbb{N}}\mathbb{E}\int_0^t|\sigma_k(\rho_s, w_n)|^2 ds = \mathbb{E}\int_0^t\langle\mathrm{div}_x(R_0(\bar{E}_0)\rho_s), \mathrm{div}_x(\bar{E}_0\rho_s)\rangle_{L^2(\mathbb{T}^N)}ds,$$

which is precisely the term (6.16). We then conclude as in Remark 6.1 to the energy estimate

$$\mathbb{E}\|\rho_t\|^2_{L^2(\mathbb{T}^N)} + \int_0^t\mathbb{E}\|\nabla_x\rho_s\|^2_{L^2(\mathbb{T}^N)}ds \leq C(\mathtt{R}, T)\mathbb{E}\|\rho_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^N)}, \quad (6.52)$$

Uniqueness of weak solutions to (1.24)-(1.25) follows from (6.18), since the problem is linear. Existence can be proved using a Galerkin approximation. □

6.6 Uniqueness for the limit martingale problem

Denote by (ρ^*_t) the weak solution in $L^2(\mathbb{T}^N)$ to (1.24)-(1.25) given by Theorem 6.8. In the following proposition, we show that (ρ^*_t) is a Markov process with a generator that coincide with $\mathscr{L}$ on a sufficiently big set. We use this result to identify the law of the limit $(\tilde\rho_t)$ with the law of (ρ^*_t). This may seem surprising since the state space of $(\tilde\rho_t)$ is $H^{-1}(\mathbb{T}^N)$, while the state space of (ρ^*_t) is $L^2(\mathbb{T}^N)$. This can be interpreted as an information a posteriori on the support of the law $\mathbb{P}_{\tilde\rho}$ of $(\tilde\rho_t)$ however, indicating that $\mathbb{P}_{\tilde\rho}$ is supported in $L^2(\mathbb{T}^N)$.

Proposition 6.9 (Markov property)**.** *Let $(\mathcal{F}^W_t)$ be the filtration generated by the cylindrical Wiener process (6.47). For $\varphi\colon L^2(\mathbb{T}^N)\to\mathbb{R}$ measurable and bounded, define*

$$P^*_t\varphi(\rho_{\mathrm{in}})=\mathbb{E}\varphi(\rho^*_t). \tag{6.53}$$

Then $(P^*_t)_{t\geq 0}$ is a Markov semi-group and

$$\mathbb{E}\left[\varphi(\rho^*_{t+s})|\mathcal{F}^W_s\right]=(P^*_t\varphi)(\rho^*_s),\ a.s., \tag{6.54}$$

for all $t,s\geq 0$, $t+s\leq T$. Furthermore, let D_0 be the set of functions $\varphi(\rho)=\psi\left(\langle\rho,\xi\rangle_{L^2(\mathbb{T}^N)}\right)$, where ψ is a Lipschitz function on $\mathbb{R}$ such that $\psi'\in C^\infty_b(\mathbb{R})$ and $\xi\in C^2(\mathbb{T}^d)$. Let $\mathscr{L}^*$ be the generator of (P^*_t). Then $D_0\subset D(\mathscr{L}^*)$ and $\mathscr{L}^*\varphi=\mathscr{L}\varphi$ for all $\varphi\in D_0$.

Proof of Proposition 6.9. We only give the sketch of the proof. First, assume that we regularize the equation (1.24) into the following SPDE:

$$d\rho^\delta=\mathrm{div}_x(K_\sharp\nabla_x\rho^\delta+\Psi\rho^\delta)dt+\sqrt{2}J_\delta\mathrm{div}_x(\rho^\delta S^{1/2}dW(t)), \tag{6.55}$$

where $J_\delta=(\mathrm{Id}-\delta\Delta_x)^{-1/2}$, and let us solve (6.55) starting from ρ_{in} (we apply Theorem 7.4 in [6]). We claim that the energy estimate (6.18) can be adapted to establish that $\rho^\delta\to\rho^*$ in $L^2(\Omega;C([0,T];L^2(\mathbb{T}^N)))$. The process $(\rho^\delta_t)_{t\in[0,T]}$ is Markov and satisfies the analog to (6.54) (see Theorem 9.8 in [6]). By taking the limit $[\varepsilon\to 0]$, we obtain (6.54) for φ continuous and bounded on $L^2(\mathbb{T}^N)$. The remaining arguments to conclude to the Markov property are given by Proposition B.2. The last statement about the generator follows from (6.50) and Itô's formula. □

Conclusion: end of the proof of Theorem 1.2. Using Proposition 6.5, we can apply Theorem B.4 (see Remark B.3). By (B.15) with $t=0$, we have $\tilde{\mathbb{E}}\varphi(\tilde\rho_t)=P^*_t(\rho_{\mathrm{in}})=\mathbb{E}\varphi(\rho^*_t)$. This means that $(\tilde\rho_t)$ and (ρ^*_t) have the same law. By uniqueness in law of the limit $(\tilde\rho_t)$, we also deduce that the whole sequence (ρ^ε_t) is converging in law. This concludes the proof of Theorem 1.2.

Remark 6.3 (Identification of the limit by representation of martingale)*.* There is an alternative approach to the identification of $(\tilde\rho_t)$. It consists in computing the quadratic variation of the martingale (6.39). Indeed, using Theorem B.3, we can show that

$$\langle\tilde M,\tilde M\rangle_t=2\int_0^t\left\|S^{1/2}(\tilde\rho_s\nabla_x\xi)\right\|^2_{[L^2(\mathbb{T}^N)]^N}ds.$$

Then we use a theorem of representation for martingales (see, *e.g.*, Theorem 8.2 p. 220 in [6]) to introduce the cylindrical Wiener process $W(t)$ and the equation (1.24). The interest of this method is that one only needs to show the limit martingale property of Proposition 6.5 for linear and quadratic test-functions. Also, uniqueness of the limit law of ρ^ε follows from Yamada-Watanabe theorem since it states that pathwise uniqueness implies uniqueness of the law of solutions of (1.24). However, in the approach we have adopted here, we do not need any theorem of representation for martingales, and it may be considered more consistent than this alternative one, since we remain focused on the Martingale Problem.

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A Resolution of the unperturbed equation

Consider the LB case first. By integration with respect to v in the equation

$$\partial_t f_t + E(t,s;\mathbf{e})\cdot\nabla_v f_t + f_t = \rho(f_t)M, \tag{A.1}$$

one checks that $\rho(f_t)=\rho(f)$ for all $t\geq 0$. Therefore, the formula (3.3) is simply the Duhamel formula associated to the PDE (A.1). In the FP case, instead of working on the PDE

$$\partial_t f_t + E(t,s;\mathbf{e})\cdot\nabla_v f_t = Q_{\mathrm{FP}} f_t, \tag{A.2}$$

we work on the solution V_t to the equation

$$dV_t = (-V_t + E(t,s;\mathbf{e}))dt + \sqrt{2}dB_t,\quad t\geq s. \tag{A.3}$$

If V_s has the law of density f with respect to the Lebesgue measure on $\mathbb{R}^N$, then by (1.12) (with no dependence on x here), we obtain, by explicit integration in (A.3),

$$\begin{aligned}&\int_{\mathbb{R}^N}\varphi(v)f^{\mathrm{FP}}_{s,t}(v)dv\\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi\left(e^{-(t-s)}v+\int_s^t e^{-(t-\sigma)}E(\sigma,s;\mathbf{e})ds+\sqrt{1-e^{-2(t-s)}}\,w\right)M(w)f(v)dwdv.\end{aligned}$$

A change of variable gives (3.4) then.

B Martingale characterization of Markov processes

B.1 Semi-groups

Let us first give our definition of Markov semi-group. Let E be a Polish space. We denote by $\mathrm{BM}(E)$ the set of bounded measurable functions on E with norm

$$\|\varphi\|_{\mathrm{BM}}(E)=\sup_{x\in E}|\varphi(x)|.$$

We denote by $\mathrm{BC}(E)$ the subspace of $\mathrm{BM}(E)$ constituted of bounded continuous functions.

Definition B.1. A family $\{P_t; t \geq 0\}$ of operators $\mathrm{BM}(E) \to \mathrm{BM}(E)$ is said to be a *Markov semi-group* if

$$P_t\varphi(x) = \int_E \varphi(y) Q_t(x, dy), \quad \varphi \in \mathrm{BM}(E), \tag{B.1}$$

where Q_t has the following properties:

1. for all $t \geq 0$, for all $x \in E$, $Q_t(x, \cdot)$ is a Borel probability measure on E,
2. for all $x \in E$, $Q_0(x, \cdot) = \delta_x$, the Dirac mass at x,
3. for all $A \in \mathcal{B}(E)$, $(t, x) \mapsto Q_t(x, A)$ is Borel measurable on $\mathbb{R}_+ \times E$,
4. the following Chapman-Kolmogorov relation is satisfied:

$$Q_{t+s}(x, A) = \int_E Q_s(y, A) Q_t(x, dy) \tag{B.2}$$

for all $0 \leq s, t$, $x \in E$, $A \in \mathcal{B}(E)$.

We assume besides that, for all $\varphi \in \mathrm{BC}(E)$, for all $x \in E$,

$$P_t\varphi(x) \to \varphi(x) \tag{B.3}$$

when $t \to 0$.

Note the condition of continuity (B.3) ("stochastic continuity"), which is not always required for Markov semi-groups. Let us now introduce the two following notions: π-convergence, π-semi-group, taken from [18].

Definition B.2 (π-convergence)**.** A sequence (φ_n) in $\mathrm{BM}(E)$ is said to π-converge to a function $\varphi \in \mathrm{BM}(E)$ if $\sup_n \|\varphi_n\|_{\mathrm{BM}(E)} < +\infty$ and $\varphi_n(x) \to \varphi(x)$ for all $x \in E$.

Note that this mode of convergence, denoted $\varphi_n \xrightarrow{\pi} \varphi$ in what follows, has no standard denomination (it is called the bp-convergence in [9] for example, [9, p. 111]).

Definition B.3 (π-contraction semi-group, [18])**.** A semi-group of operators $(P_t)_{t \geq 0}$ on $\mathrm{BM}(E)$ is said to be a π-contraction semi-group if $P_0\varphi = \varphi$ and

1. for all $\varphi \in \mathrm{BC}(E)$, for all $x \in E$, $t \mapsto P_t\varphi(x)$ is continuous from the right on $\mathbb{R}_+$,
2. for all $t \geq 0$, P_t is continuous $(\mathrm{BM}(E), \pi) \to (\mathrm{BM}(E), \pi)$, in the sense that

$$\varphi_n \xrightarrow{\pi} \varphi \quad \Rightarrow \quad P_t\varphi_n \xrightarrow{\pi} P_t\varphi, \tag{B.4}$$

3. for all $t \geq 0$, $\|P_t\| \leq 1$ in operator norm.

Note that, in [18], semi-groups $P_t\colon \mathrm{BC}(E) \to \mathrm{BC}(E)$ are considered and $t \mapsto P_t\varphi(x)$ is assumed to be continuous (not just continuous from the right) on $\mathbb{R}_+$. We have modified slightly the notion of π-semi-group introduced in [18], because it gets easier then to compare π-contraction semi-groups and Markov semi-groups. This is the object of the following Proposition B.1.

Proposition B.1. *Markov semi-groups are exactly the π-contraction semi-groups that preserve the positivity ($\varphi \geq 0$ implies $P_t\varphi \geq 0$) and fix the constants ($P_t\mathbf{1} = \mathbf{1}$).*

Proof of Proposition B.1. Assume that $(P_t)_{t\geq 0}$ is a π-contraction semi-group that preserves the positivity and fixes the constants. Set $Q_t(x, A) = P_t\mathbf{1}_A(x)$ for $t \in \mathbb{R}_+$, $x \in E$, $A \in \mathcal{B}(E)$. We have several points to consider.
Probability measure. The set function $A \mapsto Q_t(x, A)$ is a probability measure. Indeed $Q_t(x, A) \geq 0$ since $\mathbf{1}_A \geq 0$, $Q_t(x, E) = 1$ since $P_t\mathbf{1} = \mathbf{1}$ and we will see that the property of σ-additivity is satisfied. Let $A_1, A_2, \dots$ be disjoint Borel subsets of E. We have then

$$Q_t(x, A_1 \cup \cdots \cup A_N) = P_t(\mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_N})(x) = \sum_{n=1}^{N} Q_t(x, A_n). \tag{B.5}$$

The right-hand side of (B.5) is converging to $\sum_n Q_t(x, A_n)$ when $N \to +\infty$. The left-hand side of (B.5) is $P_t\varphi_N(x)$, where $\varphi_N = \mathbf{1}_{A_1\cup\cdots\cup A_N}$ is π-converging to $\mathbf{1}_A$, $A = \cup_n A_n$. Therefore $P_t\varphi_N \xrightarrow{\pi} P_t\mathbf{1}_A$ by (B.4), and we obtain the countable additivity. Similarly, using the continuity property (B.4), and approaching $\varphi \in \mathrm{BM}(E)$ by a sequence of simple functions, we deduce from the relation $Q_t(\cdot, A) = P_t\mathbf{1}_A$ that (B.1) is satisfied. We have also $Q_0(x, \cdot) = \delta_x$ since $P_0\varphi = \varphi$.
Measurability. Let $A \in \mathcal{B}(E)$. We want to show that $(t, x) \mapsto Q_t(x, A)$ is measurable. The Radon measure $Q_t(x, \cdot)$ is inner regular [3, Theorem 1.1]: $Q_t(x, A) = \sup Q_t(x, F)$, where the supremum is taken over closed sets $F \subset A$. Therefore it is sufficient to consider the case A closed. If A is closed, there is a sequence (φ_k) of Lipschitz bounded functions that π-converges to $\mathbf{1}_A$ (consider the sup-convolution $\varphi_k(x) = \sup_{y\in E}(\mathbf{1}_A(y) + kd(x, y))$). Consequently $Q_t(x, A)$ is the limit of $P_t\varphi_k(x)$ when $k \to +\infty$, and that $(t, x) \mapsto Q_t(x, A)$ is measurable follows from the fact that $(t, x) \mapsto P_t\varphi(x)$ is measurable when $\varphi \in \mathrm{BC}(E)$. Indeed, the map $h\colon (t, x) \mapsto P_t\varphi(x)$ is continuous from the right in t and measurable in x. Consider a regular partition of $\mathbb{R}_+ \setminus \{0\}$ in intervals $(a, b]$ of length N^{-1} and approximate $h(\cdot, x)$ on $(a, b]$ by the value $h(b, x)$ at the right of the interval, we obtain[1] a sequence of $\mathcal{B}(\mathbb{R}_+ \times E)$- measurable functions h_N that π-converges to h.
Chapman-Kolmogorov property. The Chapman-Kolmogorov property (B.2) follows from the semi-group property of $(P_t)_{t\geq 0}$ and (B.1).
Conversely, assume now that $(P_t)_{t\geq 0}$ is a Markov semi-group. By dominated convergence, (P_t) satisfies (B.4). The semi-group property and the stochastic continuity (B.3) imply that $t \mapsto P_t\varphi(x)$ is continuous from the right when $\varphi \in \mathrm{BC}(E)$. This concludes the proof. □

[1] we also set $h_N(0, x) = h(0, x)$

Several times in the paper, modulo Remark B.1 (see Theorem 4.3, Proposition 6.9, Theorem B.4), we encounter the following situation: we have a family of E-valued stochastic processes $(X_t(x))$ indexed by the starting point x: $X_0(x) = x$ a.s. We define the family of operators $P_t\colon \mathrm{BM}(E) \to \mathrm{BM}(E)$ by

$$P_t\varphi(x) = \langle \varphi, \mu_t(x,\cdot)\rangle = \mathbb{E}\varphi(X_t(x)), \tag{B.6}$$

where $(X_t(x))$ is a known stochastic process and $\mu_t(x,\cdot)$ denote the law of $X_t(x)$. Assume that we have

$$\mathbb{E}[\varphi(X_{t+s}(x))|\mathcal{F}_s] = (P_t\varphi)(X_s(x)) \text{ a.s.}, \tag{B.7}$$

for all $0 \leq s \leq t$, for all $x \in E$, for all $\varphi \in \mathrm{BC}(E)$, where $(\mathcal{F}_t)$ is a given filtration. Then we want to show that $(X_t(x))$ is an $(\mathcal{F}_t)$-Markov process, with Markov semi-group (P_t). Under few additional hypotheses, this will be is a consequence of the following proposition.

Proposition B.2. *Let $\{P_t; t \geq 0\}$ be a family of operators defined by (B.6) and satisfying (B.7). Assume that, for all $t \geq 0$, $(\omega, x) \mapsto X_t(\omega, x)$ is measurable $\Omega \times E \to E$. Assume the stochastic continuity at 0: $P_t\varphi(x) \to \varphi(x)$ when $t \to 0$, for all $x \in E$, for all $\varphi \in \mathrm{BC}(E)$. Then $(X_t(x))$ is an $(\mathcal{F}_t)$-Markov process, with Markov semi-group (P_t).*

Proof of Proposition B.2. The fact that $(\omega, x) \mapsto X_t(\omega, x)$ is measurable $\Omega \times E \to E$ ensures that $P_t\colon \mathrm{BM}(E) \to \mathrm{BM}(E)$. It is clear that (P_t) satisfies the criteria 1, 2, 3 in Definition B.3. Besides, P_t preserves positivity and fixes the constants. However, we do not know at that stage that (P_t) is a semi-group. Indeed, (B.7) is satisfied for φ continuous and bounded only. Nevertheless, taking $s = 0$ and taking expectation in (B.7), we see that $P_{t+s}\varphi(x) = (P_s \circ P_t\varphi)(x)$ for all $\varphi \in \mathrm{BC}(E)$. This means that, for all $x \in E$, for all $\varphi \in \mathrm{BC}(E)$,

$$\langle \varphi, \mu_{t+s}(x,\cdot)\rangle = \langle \varphi, \nu_{s,t}(x,\cdot)\rangle, \quad \nu_{s,t}(x, A) := (P_s \circ P_t\mathbf{1}_A)(x).$$

The set function $\nu_{s,t}(x,\cdot)$ is a measure (same proof as for $Q_t(x,\cdot)$ in the proof of Proposition B.1). Since $\mathrm{BC}(E)$ is a separating class, we deduce that $\mu_{t+s}(x,\cdot) = \nu_{s,t}(x,\cdot)$, *i.e.* $P_{t+s}\varphi(x) = (P_s \circ P_t\varphi)(x)$ for all $\varphi \in \mathrm{BM}(E)$. This shows that (P_t) is a semi-group. By Proposition B.1, (P_t) is a Markov semi-group. There remains to prove (B.7) for $\varphi \in \mathrm{BM}(E)$. We want to establish

$$\frac{1}{\mathbb{P}(B)}\mathbb{E}[\mathbf{1}_B\varphi(X_{t+s}(x))] = \frac{1}{\mathbb{P}(B)}\mathbb{E}[\mathbf{1}_B(P_t\varphi)(X_s(x))], \tag{B.8}$$

where $B \in \mathcal{F}_s$ is non-trivial ($\mathbb{P}(B) > 0$). Again, both sides of (B.8) define probability measures. Since $\mathrm{BC}(E)$ is a separating class, the result follows. □

Remark B.1. If, instead of $(X_t(x), \mathbb{P})_{x\in E}$ we are given $(X_t, \mathbb{P}_x)_{x\in E}$ with $\mathbb{P}_x(X_0 = x) = 1$, then Proposition B.2 holds true under the hypothesis that $x \mapsto \mathbb{P}_x(A)$ is measurable for all Borel subset $A \subset E$. Indeed, this ensures that P_t maps $\mathrm{BM}(E)$ into $\mathrm{BM}(E)$.

B.2 Martingale characterization of Markov processes

The generator $\mathscr{L}$ associated to a π-semi-group (P_t) is defined by

$$D(\mathscr{L}) = \left\{\varphi \in \mathrm{BC}(E); \exists \psi \in \mathrm{BC}(E), \Delta_t\varphi \xrightarrow{\pi} \psi \text{ when } t \to 0+\right\}, \tag{B.9}$$

$$\mathscr{L}\varphi(x) = \lim_{t\to 0+} \Delta_t\varphi(x), \tag{B.10}$$

where

$$\Delta_t = \frac{P_t - \mathrm{Id}}{t}.$$

With this notion of generator of generator, we have the following characterization of Markov processes (Theorem B.3 & Theorem B.4).

Theorem B.3 (Martingale characterization of Markov processes)**.** *Let E be a Polish space, let $(\mathcal{G}_t)$ be a filtration. Let (X_t) be an E-valued time-homogeneous Markov process with respect to a filtration $(\mathcal{F}_t)$, with Markov semi-group (P_t) of generator $\mathscr{L}$: for all $\varphi \in \mathrm{BM}(E)$*

$$\mathbb{E}\left[\varphi(X_{t+s})|\mathcal{F}_t\right] = (P_s\varphi)(X_t). \tag{B.11}$$

Assume that $t \mapsto P_t\varphi(x)$ is continuous, for all $\varphi \in \mathrm{BC}(E)$, $x \in E$. Assume that $(\omega, t) \mapsto X_t(\omega)$ is measurable $\Omega \times \mathbb{R}_+ \to E$. Then, for all φ in the domain of $\mathscr{L}$,

$$M_\varphi(t) := \varphi(X_t) - \varphi(X_0) - \int_0^t \mathscr{L}\varphi(X_s)ds \tag{B.12}$$

is a $(\mathcal{F}_t)$-martingale. Assume furthermore that $|\varphi|^2$ is in the domain of $\mathscr{L}$. Then the process (Z_t) defined by

$$Z_t := |M_\varphi(t)|^2 - \int_0^t (\mathscr{L}|\varphi|^2 - 2\varphi\mathscr{L}\varphi)(X_s)ds, \tag{B.13}$$

is a martingale.

Remark B.2. Assume that (X_t) is càdlàg. Then the process

$$A_t := \int_0^t (\mathscr{L}|\varphi|^2 - 2\varphi\mathscr{L}\varphi)(X_s)ds \tag{B.14}$$

is continuous and adapted, and thus predictable. Consequently, (A_t) is the predictable quadratic variation $\langle M_\varphi, M_\varphi\rangle_t$, [12, p.38], of M_φ: this is the compensator, [12, p.32], of the quadratic variation $[M_\varphi, M_\varphi]_t$, [12, p.51], of M_φ.

Note that we assume also continuity from the left of $t \mapsto P_t\varphi(x)$ in Theorem B.3. If $\varphi \in D(\mathscr{L})$, this ensures that $t \mapsto P_t\varphi(x)$ is differentiable, with $\frac{d}{dt}P_t\varphi(x) = P_t\mathscr{L}\varphi(x)$, [18, Proposition 3.2]. Conversely, we have the following result.

Theorem B.4. *Let E be a Polish space. Let $(P_t)_{t\geq 0}$ be a π-contraction semi-group with generator $\mathscr{L}$. Assume that $t\mapsto P_t\varphi(x)$ is continuous for all $\varphi\in \mathrm{BC}(E)$, for all $x\in E$. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, let $(X_t)_{t\geq 0}$ be an E-valued process such that $(\omega,t)\mapsto X_t(\omega)$ is measurable $\Omega\times\mathbb{R}_+\to E$. Assume that, for all $\varphi\in D_0\subset D(\mathscr{L})$, the process $(M_\varphi(t))_{t\geq 0}$ defined by (B.12) is an $(\mathcal{F}_t)$-martingale. Then $(X_t)_{t\geq 0}$ satisfies*

$$\mathbb{E}\Big[\varphi(X_{s+t})\Big|\mathcal{F}_t\Big]=P_s\varphi(X_t), \tag{B.15}$$

for all $\varphi\in D_0$, for all $t,s\geq 0$.

Remark B.3. Theorem B.4 is the full reciprocal statement to Theorem B.3 in the following situation:

1. we are given an indexed process $(X_t(x),\mathbb{P})_{x\in E}$ such that $\mathbb{P}(X_0(x)=x)=1$ (or, equivalently, a process (X_t) and indexed probability measures $(\mathbb{P}_x)_{x\in E}$ such that $\mathbb{P}_x(X_0=x)=1$)
2. D_0 is a separating class.

Indeed, we can apply Proposition B.2 then, to infer that (X_t) is a Markov process with Markov semi-group (P_t).

Proof of Theorem B.3. Let $0\leq s\leq t$. By the Markov property (B.11), we have

$$\begin{aligned}\mathbb{E}[M_\varphi(t)|\mathcal{F}_s]-M_\varphi(s)&=\mathbb{E}[M_\varphi(t)-M_\varphi(s)|\mathcal{F}_s]\\&=P_{t-s}\varphi(X_s)-\varphi(X_s)-\int_s^t[P_{\sigma-s}\mathscr{L}\varphi](X_s)d\sigma.\end{aligned}$$

We use the relation $\frac{d}{dt}P_t\varphi(x)=P_t\mathscr{L}\varphi(x)$ to obtain the martingale property. Indeed, this gives

$$P_{t-s}\varphi-\varphi=\int_s^t P_{\sigma-s}\mathscr{L}\varphi d\sigma,$$

and thus $\mathbb{E}[M_\varphi(t)|\mathcal{F}_s]-M_\varphi(s)=0$. The proof of the martingale property for (B.13) is divided in several steps. By $C(\varphi)$, we will denote any constant that depend on φ and may vary from lines to lines. We fix a subdivision $\sigma=(t_i)_{0,n}$ of $[0,T]$. In a first step, we show that

$$A_t=\lim_{|\sigma|\to 0}\sum_{i=0}^{n-1}\mathbb{E}\left[A_{t\wedge t_{i+1}}-A_{t\wedge t_i}|\mathcal{F}_{t_i}\right], \tag{B.16}$$

with a convergence in $L^2(\Omega)$. Indeed, we have

$$A_t=\sum_{i=0}^{n-1}A_{t\wedge t_{i+1}}-A_{t\wedge t_i}, \tag{B.17}$$

and $\zeta(t_{i+1}):=A_{t\wedge t_{i+1}}-A_{t\wedge t_i}-\mathbb{E}\left[A_{t\wedge t_{i+1}}-A_{t\wedge t_i}|\mathcal{F}_{t_i}\right]$ satisfies

$$\mathbb{E}[\zeta(t_i)\zeta(t_j)]=0,\quad i\neq j,\quad |\zeta(t_{i+1})|\leq C(\varphi)(t_{i+1}-t_i), \tag{B.18}$$

where $C(\varphi) = \|\mathscr{L}\varphi^2\|_{BM(E)} + 2\|\varphi\|_{BM(E)}\|\mathscr{L}\varphi\|_{BM(E)}$. It follows that

$$\mathbb{E}\left|\sum_{i=0}^{n-1}\zeta(t_{i+1})\right|^2 = \mathbb{E}\sum_{i=0}^{n-1}|\zeta(t_{i+1})|^2 \leq C(\varphi)T|\sigma|,$$

which tends to 0 when $|\sigma| \to 0$. Using (B.17), we obtain (B.16). In a second step we prove that

$$|M_\varphi(t_{i+1}) - M_\varphi(t_i)|^2 = |\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2 + R_{t_i,t_{i+1}}, \tag{B.19}$$

with

$$\mathbb{E}\sum_{i=0}^{n-1}|R_{t_i,t_{i+1}}| = \mathcal{O}(|\sigma|^{1/2}). \tag{B.20}$$

By definition of $M_\varphi(t)$, (B.19) is satisfied with a remainder term

$$R_{t_i,t_{i+1}} = \left|\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds\right|^2 - 2(\varphi(X_{t_{i+1}}) - \varphi(X_{t_i}))\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds. \tag{B.21}$$

Using the fact that $\varphi^2 \in D(\mathscr{L})$, we have also

$$\begin{aligned}|\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2 = M_{\varphi^2}(t_{i+1}) - M_{\varphi^2}(t_i) - 2\varphi(X_{t_i})(M_\varphi(t_{i+1}) - M_\varphi(t_i))\\ + \int_{t_i}^{t_{i+1}}\mathscr{L}\varphi^2(X_s)ds - 2\varphi(X_{t_i})\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds.\end{aligned}$$

It follows that

$$\mathbb{E}[|\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2|\mathcal{F}_{t_i}] = \int_{t_i}^{t_{i+1}}\mathbb{E}\left[\left(\mathscr{L}\varphi^2(X_s) - 2\varphi(X_{t_i})\mathscr{L}\varphi(X_s)\right)|\mathcal{F}_{t_i}\right]ds. \tag{B.22}$$

Taking expectation in (B.22), we get the following bound.

$$\mathbb{E}[|\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2] \leq C_\varphi(t_{i+1} - t_i). \tag{B.23}$$

Consider now the cross-product term in the right-hand side of (B.21). Using Young's inequality with a parameter $\eta > 0$, we see that the term $\mathbb{E}|R_{t_i,t_{i+1}}|$ can be bounded by

$$(1+\eta^{-1})\mathbb{E}\left|\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds\right|^2 + \eta\mathbb{E}[|\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})|^2],$$

and thus, taking $\eta = (t_{i+1} - t_i)^{1/2}$, bounded from above by $C(\varphi)(t_{i+1} - t_i)^{3/2}$. This gives (B.20). The third step establishes the limit

$$A_t = \lim_{|\sigma|\to 0}\sum_{i=0}^{n-1}\mathbb{E}\left[|M_\varphi(t_{i+1}) - M_\varphi(t_i)|^2|\mathcal{F}_{t_i}\right], \tag{B.24}$$

with a convergence in $L^1(\Omega)$. To that purpose, we note that (B.19) shows that we can replace the increment $M_\varphi(t_{i+1}) - M_\varphi(t_i)$ by the increment $\varphi(t_{i+1}) - \varphi(t_i)$ in the right-hand side of (B.24). This gives an error term $\varepsilon_1(|\sigma|)$ which converges to 0 in $L^1(\Omega)$, due to (B.20). By (B.16) and(B.22), we deduce that

$$A_t - \sum_{i=0}^{n-1} \mathbb{E}\left[|M_\varphi(t_{i+1}) - M_\varphi(t_i)|^2 | \mathcal{F}_{t_i}\right] = \varepsilon_2(|\sigma|) + r(t,\sigma), \tag{B.25}$$

where $\varepsilon_2(|\sigma|)$ converges to 0 in $L^1(\Omega)$ and

$$|r(t,\sigma)| \leq 2\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |(\varphi(X_{t_i}) - \varphi(X_s))\mathscr{L}\varphi(X_s)|\, ds.$$

We have in particular

$$|r(t,\sigma)| \leq C(\varphi)\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\varphi(X_{t_i}) - \varphi(X_s)|\, ds$$

and an estimate similar to (B.23) (obtained by working on the increment $\varphi(X_s) - \varphi(X_{t_i})$ instead of $\varphi(X_{t_{i+1}}) - \varphi(X_{t_i})$) shows that

$$\mathbb{E}|\varphi(X_s) - \varphi(X_{t_i})|^2 \leq C(\varphi)(s - t_i). \tag{B.26}$$

We deduce that $r(t,\sigma)$ is converging to 0 in $L^2(\Omega)$ when $|\sigma| \to 0$. At last, let us show that $Z_t = |M_\varphi(t)|^2 - A_t$ is a martingale. Let $0 \leq s < t$. Set $t_{n+1} = \min\{t_i; t_i \geq t\}$, $t_{l+1} = \min\{t_i; t_i \geq s\}$. We may assume $t_n \geq s$. Then $\mathbb{E}[Z_t - Z_s|\mathcal{F}_s]$ is the limit when $|\sigma| \to 0$ of the quantity

$$\mathbb{E}\Big[|M_\varphi(t)|^2 - |M_\varphi(s)|^2 - \sum_{i=l}^{n-1} \mathbb{E}\left[|M_\varphi(t_{i+1}) - M_\varphi(t_i)|^2 | \mathcal{F}_{t_i}\right] \Big| \mathcal{F}_s\Big] \tag{B.27}$$

By the tower property $\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_{t_i}]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s]$ if $t_i \geq s$, and the usual cancellation properties for martingales, (B.27) is equal to

$$\mathbb{E}\Big[|M_\varphi(t) - M_\varphi(t_n)|^2 + \mathbb{E}[|M_\varphi(s) - M_\varphi(t_l)|^2|\mathcal{F}_{t_l}]\Big|\mathcal{F}_s\Big]. \tag{B.28}$$

Using (B.26), we see that (B.28) tends to zero in $L^1(\Omega)$. This gives the desired result $\mathbb{E}[Z_t - Z_s|\mathcal{F}_s] = 0$. □

The proof of Theorem B.4 is very similar to the proof of Theorem 4.1 p. 181-184 in [9], but is also different in many anecdotal aspects, so we expound on it.

Proof of Theorem B.4. Let $\varphi \in D_0$. Let us first show that the martingale property for (B.12) implies the martingale property for

$$M_\psi(t) := \psi(t, X_t) - \psi(0, X_0) - \int_0^t \left[(\partial_s + \mathscr{L})\psi\right](s, X_s) ds \tag{B.29}$$

with $\psi(t,x) = \theta(t)\varphi(x)$, $\theta \in C^1(\mathbb{R}_+)$. If $(M_t)_{t\geq 0}$ is a martingale, then

$$t \mapsto M_t\theta(t) - \int_0^t M_\sigma \theta'(\sigma)d\sigma$$

is a martingale. For M_t given by (B.12), using the Fubini theorem we obtain (B.29). Taking now $\theta(t) = e^{-\lambda t}$, $\lambda > 0$ gives us

$$e^{-\lambda(t+T)}\mathbb{E}[\varphi(X_{t+T})|\mathcal{F}_t] = e^{-\lambda t}\varphi(X_t) + \mathbb{E}\left[\int_t^{t+T} \lambda e^{-\lambda s}(\lambda^{-1}\mathscr{L} - \mathrm{Id})\varphi(X_s)ds\middle|\mathcal{F}_t\right]. \tag{B.30}$$

Doing the change of variable $s = s' + t$ in the integral shows that

$$\varphi(X_t) = e^{-\lambda T}\mathbb{E}[\varphi(X_{t+T})|\mathcal{F}_t] - \mathbb{E}\left[\int_0^T \lambda e^{-\lambda s}(\lambda^{-1}\mathscr{L} - \mathrm{Id})\varphi(X_{s+t})ds\middle|\mathcal{F}_t\right].$$

We let $T \to +\infty$ to obtain

$$\varphi(X_t) = \mathbb{E}\left[\int_0^{+\infty} \lambda e^{-\lambda s}(\mathrm{Id} - \lambda^{-1}\mathscr{L})\varphi(X_{s+t})ds\middle|\mathcal{F}_t\right]. \tag{B.31}$$

The convergence is easy to justify since φ and $\mathscr{L}\varphi$ are bounded. Compare (B.31) to Formula (B.32) for the resolvent. Actually, both the formula

$$R_\lambda\varphi = \int_0^\infty e^{-\lambda t}P_t\varphi dt, \tag{B.32}$$

and (B.31) can be written more concisely by introducing a random variable independent τ with exponential distribution of parameter λ. We may work on the probability space $(\Omega, \mathcal{F})$ (it suffices to assume independence of τ and $(\mathcal{F}_t)_{t\geq 0}$). However, the lines below will be more explicit if we consider that τ is defined on a probability space $(\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$. Let $J_\lambda\varphi := \lambda R_\lambda\varphi$. We rewrite (B.32) and (B.31) as

$$J_\lambda\varphi = \mathbb{E}^\sharp P_\tau\varphi, \quad \varphi(X_t) = \mathbb{E}^\sharp\mathbb{E}\left[(J_\lambda^{-1}\varphi)(X_{\tau+t})\middle|\mathcal{F}_t\right],$$

respectively. By iteration of the two formulas (we apply it to $J_\lambda\varphi$, $J_\lambda^2\varphi$, etc.), we obtain, for $k \geq 1$,

$$J_\lambda^k\varphi = \mathbb{E}^\sharp P_{\sigma_k}\varphi, \quad J_\lambda^k\varphi(X_t) = \mathbb{E}^\sharp\mathbb{E}\left[\varphi(X_{\sigma_k+t})\middle|\mathcal{F}_t\right], \tag{B.33}$$

where $\sigma_k = \tau_1 + \cdots + \tau_k$ for $\tau_1, \ldots, \tau_k$ some i.i.d. $\mathcal{E}(\lambda)$ independent random variables. from $(\mathcal{F}_t)_{t\geq 0}$. Take now $\lambda = N$, where $N \to +\infty$ and $k = [Ns]$ for a given $s > 0$. By the weak law of large numbers, we have $\sigma_k \to s$ in probability (for $\mathbb{P}^\sharp$). Consequently, the limit $[N \to +\infty]$ of the first equality in (B.33) gives

$$J_N^{[ns]}\varphi \xrightarrow{\pi} P_s\varphi. \tag{B.34}$$

The map $\theta\colon \sigma \mapsto \mathbb{E}[\varphi(X_{\sigma+t})|\mathcal{F}_t]$ is continuous since, for $\sigma' \geq \sigma$, and by the martingale property,

$$\left|\theta(\sigma') - \theta(\sigma)\right| = \left|\int_{t+\sigma}^{t+\sigma'} \mathbb{E}[\mathscr{L}\varphi(X_s)|\mathcal{F}_t]ds\right| \leq \|\mathscr{L}\varphi\|_{\mathrm{BM}(E)}(\sigma' - \sigma).$$

Consequently, at the limit $[N \to +\infty]$ in the second identity in (B.33), we get (B.15). □

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