L^1 solutions to first order hyperbolic equations in bounded domains

Alessio Porretta* Julien Vovelle†

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Abstract

We consider L^1 solutions to the initial-boundary value problem for first order hyperbolic equations. We detail the interactions between the boundary condition and the flux function and the way they influence the integrability of the solution. Following [BCW00], we define a notion of renormalized entropy solution in order to ensure the integrability of the non-linear term in the entropy inequalities. Existence, uniqueness and stability of such a solution is established.

Keywords: L^1 solution, hyperbolic equation, initial-boundary value problem, renormalized entropy solution

MSC: 35L60 (35D05 35F25 35F30 35L65)

1 Introduction

We are interested here in studying the initial boundary value problem for first order hyperbolic equations in a bounded domain $\Omega \subset \mathbb{R}^d$

$$\begin{cases} u_t(x,t) + \mathrm{div}_x f(u(x,t)) = g\,, \quad (x,t) \in \Omega \times (0,T)\,, \\ u(x,0) = u_0(x)\,, \quad x \in \Omega\,, \\ u(x,t) = \overline{u}(x,t)\,, \quad (x,t) \in \partial\Omega \times (0,T)\,, \end{cases} \tag{1.1}$$

in case of unbounded data g, u_0, $\overline{u}$.

In a recent work ([BCW00]) P. Benilan, J. Carrillo and P. Wittbold consider the Cauchy problem in the whole space ($x \in \mathbb{R}^d$) with data $g \in L^1$ and $u_0 \in L^1$. In order to deal with unbounded solutions they introduce the definition of renormalized entropy solutions which extends the classical definition of entropy solutions ([Vol67], [Kru70]) used in the L^∞ framework. They prove existence and uniqueness of such generalized solutions which are also proved to coincide with the mild solutions constructed via semigroup theory (see [Bén72], [Cra72]).

We deal here with the problem in a bounded domain. As it is well known in the study of boundary value problems for first order hyperbolic equations, the correct meaning of the boundary condition has to be precised since this condition may be not assumed pointwise but should be read as an entropy condition at the boundary. This latter condition (known as the BLN condition) has been defined in [BlRN79] for solutions u having bounded variation and data $\overline{u} \in C^2(\partial\Omega \times (0,T))$. Such a condition makes no sense if u and $\overline{u}$ are merely in L^∞, consequently a new formulation of the problem was needed in the L^∞ setting. It has been given by F. Otto ([Ott96]) who introduced an integral formulation of the boundary condition

*Dipartimento di Matematica, Universita' di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133 Roma -Italy, porretta@mat.uniroma2.it

†Universite de Provence, CMI, F-13453 Marseille, vovelle@cmi.univ-mrs.fr

and proved that (1.1) is well posed. Further results for bounded domains in case of boundary data $\overline{u}$ in L^∞ have been obtained in [Val00] and in [CW].

We focus our attention here on the role of the boundary data $\overline{u}$ in order to extend the previous results to an L^1 framework, namely to L^1 solutions of (1.1) in presence of possibly unbounded boundary data $\overline{u}$. To this aim it is quite natural to make use of the concept of renormalized entropy solution and try to adapt the definition given in [Ott96] to this setting. The main difficulty is related to understanding the influence of the boundary condition, while trying to find the assumptions the function $\overline{u}$ should satisfy in order to have suitable L^1–estimates on the solution. In fact, as far as the Cauchy problem is concerned, the L^1 norm of the solution is immediately estimated, thanks to the conservation of mass, in terms of the L^1 norm of the initial datum. In a bounded domain Ω, the L^1 norm of the solution also depends on the normal trace at the boundary of the flux $f(u)$, which may be different from $f(\overline{u})$.

Actually, we give a quite general and simple example, constructed together with J.L. Vazquez, showing that requiring $\overline{u}$ in $L^1(\partial\Omega \times (0,T))$ is not enough in general to ensure solutions in $L^1(\Omega \times (0,T))$. This feature had already been observed in [AM99], where the authors study the unidimensional case for a strictly convex and superlinear flux f (with $g = 0$), using the explicit formula given by P. Lefloch [LeF88] and constructing an example for the Burgers equation.

While in most cases the more natural assumption would be to have $f(\overline{u})$ in $L^1(\partial\Omega \times (0,T))$, this hypothesis may be not enough yet to have solutions in L^1, and we give here (in Section 2) an example showing that this situation may actually occur. Thus, we are led to introduce a suitable assumption on $\overline{u}$ allowing solutions of finite mass; actually we define a sort of "maximal effective flux"(the maximal flux which may be seen by the equation) as

$$\overline{f}(s) = \{\sup |f(t)|\,,\, t \in [-s^-, s^+]\}\,,$$

and we assume that $\overline{u}$ is a measurable function almost everywhere finite on Σ such that

$$\overline{f}(\overline{u}) \in L^1(\partial\Omega \times (0,T))\,. \tag{1.2}$$

If the function f is monotone at infinity then this assumption reduces to ask that $f(\overline{u})$ is in $L^1(\partial\Omega\times(0,T))$, and if the function f is concave, our assumption is even weaker than assuming $\overline{u}$ to be in L^1. It is interesting to note that a condition of similar kind on $\overline{u}$, but slightly stronger than ours, has also been given in [NOV99] to work in an L^1 framework from a kinetic point of view. Under hypothesis (1.2) we prove existence, uniqueness and continuous dependence of renormalized entropy solutions of (1.1). We leave to the next section the precise definition of solutions, further comments and main examples concerning assumption (1.2).

2 Statement of the results, remarks and counterexamples.

2.1 Definition of renormalized entropy solution

Let us fix some notations. For fixed $T > 0$, we set $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. We assume that Ω is a bounded strongly Lipschitz subset of $\mathbb{R}^d$, that

$$g \in L^1(Q), \tag{2.3}$$

$$u_0 \in L^1(\Omega), \tag{2.4}$$

and $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function. We introduce the modified flux

$$\overline{f}(s) = \{\sup |f(t)|\,,\, t \in [-s^-, s^+]\}\,,$$

and we assume that $\overline{u}$ is a measurable function satisfying

$$\overline{u} \text{ is finite almost everywhere on } \Sigma, \text{ and } \quad \overline{f}(\overline{u}) \in L^1(\Sigma)\,. \tag{2.5}$$

Actually, the flux $\overline{f}$ can be seen as the "maximal effective flux" which may affect the equation, as it can be observed by solving local Riemann problems, for which solutions are determined by convex or concave envelopes of the flux; this will be made more clear in our Example 2.6, which motivates assumption (2.5).

For real numbers a, b, we denote $(u \perp a) = \min(a,b)$ and $a \top b = \max(a,b)$, and we define the truncation function $T_l(s) = (u \perp l) \top (-l)$. The Lipschitz constant of f on $[-l,l]$ will be denoted as

$$\mathrm{Lip}_l \;:=\; \mathrm{Lip}(f|_{[-l,l]}).$$

Given a bounded measure μ on $\overline{Q}$, we denote by $||\mu||$ its total mass, that is to say the norm of μ viewed as an element of the dual space $\mathcal{C}(\overline{Q})'$, or, still, $||\mu|| = \mu(\overline{Q})$ if the measure μ is non-negative. Here is the definition of renormalized entropy solution, which adapts the one introduced in [BCW00] to the formulation for boundary value problems given in [Ott96] (see also [Vov02]).

Definition 2.1 *A function u of $L^1(Q)$ is said to be a renormalized entropy solution of problem (1.1) if there exist some non-negative bounded measures μ_l and ν_l on $\overline{Q}$ whose masses vanish with l, that is to say*

$$||\mu_l||\,,\; ||\nu_l|| \underset{l\to+\infty}{\longrightarrow} 0\,,$$

and such that the following entropy inequalities are satisfied: for all $\kappa \in \mathbb{R}$, for all $l \in \mathbb{R}$ such that $l \geq |\kappa|$, for all $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega} \times [0,T); \mathbb{R}_+)$,

$$\begin{aligned} &\int_0^T \int_\Omega \big((u\perp l - \kappa)^+ \varphi_t + \Phi^+(u\perp l, \kappa)\cdot \nabla\varphi\big)\, dx\, dt + \int_\Omega (u_0 \perp l - \kappa)^+ \varphi(x,0)\, dx \\ &+\mathrm{Lip}_l \int_0^T \int_{\partial\Omega} (\overline{u}\perp l - \kappa)^+ \varphi(x,t)\, d\gamma(x)\, dt \;+ \int_0^T \int_\Omega g \,\mathrm{sgn}^+(u\perp l - \kappa)\,\varphi\, dx\, dt \geq -\langle \mu_l, \varphi\rangle\,, \end{aligned} \tag{2.6}$$

and, similarly,

$$\begin{aligned} &\int_0^T \int_\Omega \big((u\top(-l) - \kappa)^- \varphi_t + \Phi^-(u\top(-l), \kappa)\cdot \nabla\varphi\big)\, dx\, dt + \int_\Omega (u_0 \top(-l) - \kappa)^- \varphi(x,0)\, dx \\ &+\mathrm{Lip}_l \int_0^T \int_{\partial\Omega} (\overline{u}\top(-l) - \kappa)^- \varphi(x,t)\, d\gamma(x)\, dt + \int_0^T \int_\Omega g \,\mathrm{sgn}^-(u\top(-l) - \kappa)\,\varphi\, dx\, dt \;\geq -\langle \nu_l, \varphi\rangle\,, \end{aligned} \tag{2.7}$$

where we denoted by $\langle\cdot,\cdot\rangle$ the duality between $\mathcal{C}(\overline{Q})'$ and $\mathcal{C}(\overline{Q})$ and by $\Phi^\pm(u,\kappa)$ the entropy-flux function

$$\Phi^\pm(u,\kappa) = \mathrm{sgn}^\pm(u-\kappa)(f(u) - f(\kappa))\,,$$

having defined $\mathrm{sgn}^\pm(s-\kappa) = 0$ if $s = k$.

Note that, as in the purpose of renormalization, all integrals appearing in (2.6) and (2.7) are restricted to sets where u and $\overline{u}$ are bounded, so as to make sense. On the other hand, the conditions on the asymptotic behavior of the measures μ_l, ν_l allow to recover informations on the zone where u is unbounded.

Remark 2.2 *In case of bounded data $\overline{u}$, g, u_0, choosing $\varphi = T - t$, $|k|$ great enough and letting l go to infinity in (2.6) and (2.7), one easily checks that any renormalized entropy solution is bounded and then coincides with the unique entropy solution u in the sense of [Ott96], which of course can be seen as a renormalized entropy solution where the measures μ_l, ν_l are taken to be zero for l large enough.*

2.2 Statement of the results

For such a notion of renormalized solution, we prove existence and uniqueness, as described in the following theorem.

Theorem 2.3 *Let assumptions (2.3), (2.4), (2.5) hold true. Then a renormalized entropy solution u exists and is unique. Moreover, $u \in C^0([0,T]; L^1(\Omega))$.*

Besides, we also prove stability of solutions with respect to convergence of the data.

Theorem 2.4 *Let (u^i) be a sequence of renormalized entropy solutions of Problem (1.1) with data $(u_0^i, \overline{u}^i, g^i)$. Assume that*

$$g^i \to g \qquad \text{strongly in } L^1(Q),$$
$$u_0^i \to u_0 \qquad \text{strongly in } L^1(\Omega),$$

and

$$\overline{u}^i \to \overline{u} \qquad \text{a.e. on } \Sigma \text{ and } \overline{f}(\overline{u}^i) \to \overline{f}(\overline{u}) \qquad \text{strongly in } L^1(\Sigma).$$

Then the sequence (u^i) strongly converges in $C^0([0,T]; L^1(\Omega))$ to the unique solution u of Problem (1.1) with data $(u_0, \overline{u}, g)$.

The main step in the proof of Theorems 2.3 and 2.4 is proving an L^1–contraction result for renormalized solutions only depending on the L^1 norm of $\overline{f}(\overline{u})$; to this aim we follow the basic lines of the proof given in [Ott96] improving the estimates with respect to the boundary data and taking into account the presence of the extra terms given by measures μ_l and ν_l. The contraction argument allows us to obtain uniqueness and continuous dependence, which also implies compactness of bounded approximated solutions corresponding to truncated data. The existence is then obtained by passing to the limit in the problem with truncated data; here the definition of the measures μ_l and ν_l appears quite naturally as in [BCW00] and our main task is proving the estimate on the masses of those measures in terms of possibly singular boundary data.

2.3 Comment on the assumption on $\overline{u}$.

Example 2.5 This example, which was constructed together with J. L. Vazquez, shows that assuming the boundary datum $\overline{u}$ in $L^1(\Sigma)$ is not enough in order to have *a priori* estimates in $L^1(Q)$, so that one can have an integrable boundary datum but solutions with infinite mass, in sharp contrast with what happens for the Cauchy problem, where there is conservation of the mass. Let here $f(s) = s^{p+1}$ with $p > 1$, $\Omega = (0,1)$, $u_0 = 0$. First of all we build a solution u which blows up at a given fixed time T in any point x; this is already enough to see that *a priori* estimates on the $L^1(\Omega)$ norm of solutions at time t are not available. In order to build such a solution u we fix the line $x = \zeta(t) = \frac{T^\alpha - (T-t)^\alpha}{T^\alpha}$ as a line of shock for the solution, obtain the value of u from the Rankine–Hugoniot condition and transport this value through the characteristics in the domain $x < \zeta(t)$. To this purpose, for $\alpha \in (0,1)$, let $\hat{u}(s) = \frac{\alpha^{\frac{1}{p}} (T-s)^{\frac{\alpha-1}{p}}}{T^{\frac{\alpha}{p}}}$ be the limit value, from the left, of the solution u on the line $x = \zeta(s)$. For any fixed $s \in (0,T)$ the characteristic line starting from the point $(\zeta(s), s)$ is defined by the equation $x = f'(\hat{u}(s))(t-s) + \zeta(s)$. It is easy to check that, since the flux is convex, these characteristic lines do not meet one another in the domain $x < \zeta(t)$; in fact for any s and $x < \zeta(s)$ there is only one value $t < s$ such that the point (x,t) belongs to the characteristic line starting at $(\zeta(s), s)$. Rephrasing we can also say that the equation $f'(\hat{u}(s))(t-s) + \zeta(s) - x = 0$ implicitly defines $s = s(x,t)$ as a function of (x,t), hence the function $u(x,t) = \hat{u}(s(x,t)) = \frac{\alpha^{\frac{1}{p}} (T-s(x,t))^{\frac{\alpha-1}{p}}}{T^{\frac{\alpha}{p}}}$ is well defined for $x < \zeta(t)$. Since u is defined through the method of characteristics, it is a classical solution in the domain $x < \zeta(t)$. Indeed, a straightforward calculation gives, for $x < \zeta(t)$,

$$u_t + f(u)_x = \alpha^{\frac{1}{p}} \left(\frac{1-\alpha}{p}\right) \frac{(T-s)^{\frac{\alpha-1}{p}-1}}{T^{\frac{\alpha}{p}}} \left[\frac{\partial s}{\partial t} + (p+1)\alpha \frac{(T-s)^{\alpha-1}}{T^\alpha} \frac{\partial s}{\partial x}\right],$$

and computing the partial derivatives of $s(x,t)$ in terms of the function $A(x,t,s) = f'(\hat{u}(s))(t-s) + \zeta(s) - x$ and the equation $A(x,t,s) = 0$, we obtain

$$\left[\frac{\partial s}{\partial t} + (p+1)\alpha \frac{(T-s)^{\alpha-1}}{T^\alpha} \frac{\partial s}{\partial x}\right] = -\left(\frac{\partial A}{\partial s}\right)^{-1} \left[f'(\hat{u}(s)) - (p+1)\alpha \frac{(T-s)^{\alpha-1}}{T^\alpha}\right] = 0\,.$$

Define $u = 0$ for $x > \zeta(t)$; since by construction u satisfies the Rankine–Hugoniot condition on $x = \zeta(t)$ and since the entropy condition is ensured from the fact that the flux is convex and the shock decreasing, we have that u is an entropy solution of the problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } (0,1) \times (0,T), \\ u(x,0) = 0 & \text{in } (0,1), \\ u(0,t) = \hat{u}(s(0,t)), \quad u(1,t) = 0 & \text{in } (0,T). \end{cases} \tag{2.8}$$

As far as the boundary datum is concerned, note that since the flux is an increasing function the solution coincides with the boundary datum at $x = 0$, while any boundary datum is admissible at $x = 1$ so that we have taken a trivial choice. Computing the $L^1(0,T)$ norm of the boundary datum gives

$$\int_0^T u(0,t)\,dt = \alpha^{\frac{1}{p}} \int_0^T \frac{(T-s)^{\frac{\alpha-1}{p}}}{T^{\frac{\alpha}{p}}} \left[\frac{p}{p+1} + \frac{\zeta(s)(1-\alpha)}{(p+1)\alpha} \frac{T^\alpha}{(T-s)^\alpha}\right],$$

and since $p > 1$ and $\alpha \in (0,1)$ we get

$$\int_0^T u(0,t)\,dt \leq C\,T^{1-\frac{1}{p}}, \tag{2.9}$$

where the constant C does not depend on T. Thus the boundary datum has a finite L^1 norm, but the solution blows up at $t = T$ and this is due to the fact that $f(u(0,t))$ is not in $L^1(0,T)$, as easily checked.

The solution u previously found blows up at $t = T$ but still belongs to $L^1(Q)$. However, the example can be slightly modified in order to build a solution which has infinite mass on $(0,1) \times (0,T)$.

Let $\tau_n = \frac{1}{n^{\frac{2p}{p-1}}}$; define time intervals $E_n = (e_{n-1}, e_n)$ such that $e_0 = 0$, $e_{2n} - e_{2n-1} = \frac{\sqrt{\tau_n}}{p+1}$ and $e_{2n-1} - e_{2n-2} = \tau_n$, for $n \geq 1$. We denote by $u(x,t;T)$ the solution of (2.8) previously constructed in the interval $(0,T)$, and we define $\overline{u}(0,t) = \sum_{n=1}^{\infty} u(0, t - e_{2n-2}; \tau_n)\chi_{E_{2n-1}}$. Thus in all odd intervals we have iterated the example built before and in all even intervals we have put $\overline{u}(0,t) = 0$ and we can use that at the initial time of the even intervals the solution is infinite. Precisely, let v be any entropy solution of

$$\begin{cases} v_t + f(v)_x = 0 & \text{in } (0,1) \times (0,T^*), \\ v(x,0) = 0 & \text{in } (0,1), \\ v(0,t) = \overline{u}(t),\ v(1,t) = 0 & \text{in } (0,T^*), \end{cases} \tag{2.10}$$

where $T^* = \lim_{n\to\infty} e_n = \sum_{n=1}^{\infty} \left(\frac{1}{n^{\frac{2p}{p-1}}} + \frac{1}{p+1} \frac{1}{n^{\frac{p}{p-1}}} \right)$, which is finite since $p > 1$. Clearly v is nonnegative so that, in each interval (e_{2n-2}, e_{2n-1}), v is greater than $u(x, t - e_{2n-2}; \tau_n)$, which is the solution of

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } (0,1) \times (e_{2n-2}, e_{2n-1}), \\ u(x, e_{2n-2}) = 0 & \text{in } (0,1), \\ u(0,t) = \overline{u}(t), \quad u(1,t) = 0 & \text{in } (e_{2n-2}, e_{2n-1}). \end{cases}$$

Since $u(x, t - e_{2n-2}; \tau_n)$ blows up at e_{2n-1}, in particular this implies that $v(x, e_{2n-1}) \geq n^{\frac{1}{p-1}}$, hence in each interval (e_{2n-1}, e_{2n}) we have that v is greater than the solution of

$$\begin{cases} w_t + f(w)_x = 0 & \text{in } (0,1) \times (e_{2n-1}, e_{2n}), \\ w(x, e_{2n-1}) = n^{\frac{1}{p-1}} & \text{in } (0,1), \\ w(0,t) = 0, \quad w(1,t) = 0 & \text{in } (e_{2n-1}, e_{2n}). \end{cases}$$

This function w is obtained by solving a Riemann problem with data 0 and $n^{\frac{1}{p-1}}$, and standard considerations (the flux is convex) imply that $w = n^{\frac{1}{p-1}}$ if $x > f'(n^{\frac{1}{p-1}})(t - e_{2n-1}) = (p+1)n^{\frac{p}{p-1}}(t - e_{2n-1})$. We deduce that $v \geq n^{\frac{1}{p-1}}$ for $x > (p+1)n^{\frac{p}{p-1}}(t - e_{2n-1})$ and $t \in (e_{2n-1}, e_{2n})$ (note that $e_{2n} - e_{2n-1} = \frac{1}{(p+1)n^{\frac{p}{p-1}}}$). We conclude that

$$\|v\|_{L^1((0,1)\times(0,T^*))} \geq \frac{1}{2(p+1)} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

so that any entropy solution of (2.10) has infinite mass, whereas the boundary datum $\overline{u}$ belongs to $L^1(0,T)$ since, by (2.9),

$$\|\overline{u}\|_{L^1(0,T)} \le C \sum_{n=1}^{\infty} \tau_n^{1-\frac{1}{p}} = C \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty .$$

■

The previous example proves that the L^1 norm of the boundary datum $\overline{u}$ does not control, in general, the L^1 norm of the solution, but an assumption concerning the L^1 norm of $f(\overline{u})$ is needed. In some cases, this can even lead to weaker assumptions; for instance, for all positive even concave flux functions f, we have $|f(\overline{u})| \le |\overline{f}(\overline{u})| \le C|\overline{u}|$, in this case the L^1 norm of $\overline{u}$ controls the L^1 norm of the solution but assuming (2.5) we still obtain L^1 solutions under a weaker assumption (as an example one can take $f(s) = \sqrt{1+|s|}$, (2.5) only requires $|\overline{u}|^{\frac{1}{2}}$, and not $\overline{u}$, to be integrable).

It is also worth pointing out that for a large class of functions f, our hypothesis (2.5) reduces to ask that $f(\overline{u}) \in L^1(\Sigma)$; for example, if f is monotone for $|s| \ge s_0$, then clearly

$$\overline{f}(\overline{u}) \le \max_{[-s_0,s_0]} |f| \; + |f(\overline{u})| , \qquad \text{a.e. in } \Sigma,$$

and since by definition $|f(\overline{u})| \le \overline{f}(\overline{u})$ it follows that the two assumptions are equivalent. However, in the following example we show that assuming $f(\overline{u}) \in L^1(\Sigma)$ may be not enough to ensure L^1 solutions, so that requiring the stronger hypothesis (2.5) seems to be necessary.

Example 2.6 This example underlines the accuracy of the hypothesis $\overline{f}(\overline{u}) \in L^1(\Sigma)$ in order to have $L^1(Q)$ solutions; more precisely, for a certain flux function f, we construct a solution of the problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } (0,1)\times(0,T), \\ u(x,0) = 0 & \text{in } (0,1), \\ u(0,t) = \overline{u}(t)\,,\; u(1,t) = \overline{\overline{u}}(t) & \text{in } (0,T), \end{cases} \tag{2.11}$$

such that $\overline{u}$, $\overline{\overline{u}}$ are almost everywhere finite and

$$f(\overline{u}) = f(\overline{\overline{u}}) = 0\,, \qquad \overline{f}(\overline{u}) \notin L^1(0,T)\,, \qquad \text{and } u \notin L^1((0,1)\times(0,T))\,. \tag{2.12}$$

The function u is estimated by using a principle of comparison and the resolution of a sequence of Riemann Problems, as in Example 2.5. We consider the flux function f defined in the following way: let (S_n) and (Y_n) be the sequences defined by $S_n = (n+1)^2$ and $Y_n = (n+1)^4$. Set $S_n^+ = S_n + 1$, $S_n^- = S_n - 1$, and

$$f(s) = \begin{cases} Y_n(s - S_n-) & \text{if } S_n^- \le s \le S_n\,, \\ Y_n(S_n^+ - s) & \text{if } S_n \le s \le S_n^+\,, \\ 0 & \text{if } S_n^+ \le s \le S_{n+1}^-\,. \end{cases} \tag{2.13}$$

Notice that the function f is defined as being piecewise affine in order to make the several resolutions of Riemann problems more readable. A regular approximation of this flux could be considered, without affecting the main lines of this counter-example.

Let a_n be the slope of the straight line joining the point $(S_n, f(S_n))$ to the point $(S_{n+1}, f(S_{n+1}))$ and define time intervals $E_n = (e_{n-1}, e_n)$ such that $e_0 = 0$, $e_{2n} - e_{2n-1} = \frac{1}{Y_n}$ and $e_{2n-1} - e_{2n-2} = \frac{1}{a_{n-1}}$, for $n \ge 1$. Notice that

$$|E_{2n}| = \mathcal{O}(\frac{1}{n^4}) \text{ and } |E_{2n+1}| = \mathcal{O}(\frac{1}{n^2}) \tag{2.14}$$

so that $T = \sum_{n=1}^{\infty} |E_n|$ is well defined. We set $\overline{u}(t) = \sum_{n=1}^{\infty} S_n^+ \chi_{E_{2n-1}}$ and $\overline{\overline{u}}(t) = \sum_{n=1}^{\infty} S_{n-1}^- \chi_{E_{2n-1}}$, and in all odd intervals E_{2n-1} we solve the Riemann problem

$$\begin{cases} (v_{2n-1})_t + f(v_{2n-1})_x = 0 & \text{in } (0,1)\times E_{2n-1}, \\ v_{2n-1}(x, e_{2n-2}) = S_{n-1}^- & \text{in } (0,1), \\ v_{2n-1}(0,t) = S_n^+\,, \quad v_{2n-1}(1,t) = S_{n-1}^- & \text{in } E_{2n-1} \end{cases} \tag{2.15}$$

starting from $n=1$. The solution v_{2n-1} consists of two shocks with positive slopes $1/a_{n-1} > 1/Y_{n-1}$ in the (x,t)-plane, separating three constant states which are (turning counter-clockwise): S^-_{n-1}, S_{n-1} and S_n. Note that the time e_{2n-1} is defined so that the line $t = e_{2n-2} + \frac{x}{a_{n-1}}$ touches the boundary $x=1$ at $t = e_{2n-1}$. In all even intervals, we solve the Riemann problem

$$\begin{cases} (v_{2n})_t + f(v_{2n})_x = 0 & \textit{in} \text{ in } (0,1)\times E_{2n}, \\ v_{2n}(x, e_{2n-1}) = v_{2n-1}(e_{2n-1}) & \text{in } (0,1), \\ v_{2n}(0,t) = 0\,, \quad v_{2n}(1,t) = 0 & \text{in } E_{2n} \end{cases} \tag{2.16}$$

with $n \geq 1$. Since $v_{2n-1}(e_{2n-1}) = S_n$, it appears, after examination of the convex and concave hulls of the function f between the points $(0,0)$ and $(S_n, f(S_n))$, that, cutting the rectangle $(0,1)\times E_{2n}$ along the diagonale with positive slope (equal to $1/Y_n$ in the (x,t)-plane), the function v_{2n} consists of two constant states: S^-_n above the diagonale and S_n below. Thus $v_{2n}(e_{2n}) = S^-_n$, which is consistent in order to iterate the construction and solve repeatedly the Riemann problems (2.15) and (2.16). We deduce that, for every $n \in \mathbb{N}$, the solution u of Problem (2.11) is greater than S_{n+1} on half the domain $(0,1)\times E_{2n+1}$. Then using the estimate (2.14), it is easy to see that u does not belong to $L^1(Q)$. Moreover, since

$$|\{\overline{u} > l\} \cup \{\overline{\overline{u}} > l\}| \leq C \sum_{n>\sqrt{l}} |E_{2n-1}| \to 0 \qquad \text{as } l \to \infty,$$

we have that $\overline{u}$ and $\overline{\overline{u}}$ are almost everywhere finite in virtue of (2.14). Clearly we also have $f(\overline{u}) = f(\overline{\overline{u}}) = 0$ and again (2.14) implies that $\overline{f}(\overline{u}) \notin L^1(0,T)$, so that (2.12) is indeed satisfied. ■

3 Proof of the results

3.1 Localization at the boundary

Here, we set the tools used to study different terms related to the boundary, which are localization and the introduction of a cut-off function.

Localization. The set Ω being a strong Lipschitz open subset of $\mathbb{R}^d$, there exists a finite open cover $(B_\alpha)_{0,N}$ of $\overline{\Omega}$ and a partition of unity $(\lambda_\alpha)_{0,N}$ on $\overline{\Omega}$ subordinate to $(B_\alpha)_{0,N}$ such that, up to a change of coordinates represented by an orthogonal matrix A_α, the set $\partial\Omega \cap B_\alpha$ is the graph of a Lipschitz continuous function h_α, that is to say:

$$\Omega \cap B_\alpha = \{x \in B_\alpha \,;\, (A_\alpha\, x)_d > h_\alpha(\overline{A_\alpha\, x})\}$$

and

$$\partial\Omega \cap B_\alpha = \{x \in B_\alpha \,;\, (A_\alpha\, x)_d = h_\alpha(\overline{A_\alpha\, x})\}\,,$$

where $\overline{y}$ stands for $(y_i)_{1,d-1}$ if $y \in \mathbb{R}^d$. The problem will be localized with the help of the functions λ_α. We drop the index α and, for the sake of clarity, suppose that the change of coordinates is trivial: $A = I_d$. We set $\Pi = \{\overline{x}\,;\, x \in B\}$ which, clearly, is an open subset of $\mathbb{R}^{d-1}$. If a function θ is defined on $\partial\Omega \cap B$, then we denote by $\overline{\theta}$ the function defined on Π by $\overline{\theta}(\overline{x}) = \theta(\overline{x}, h(\overline{x}))$. Then we have

$$\int_{\partial\Omega\cap B} \theta(x) d\gamma(x) = \int_\Pi \overline{\theta}(\overline{x}) \sqrt{1 + |\overline{\nabla} h(\overline{x})|^2}\, d\overline{x}, \tag{3.17}$$

where $\overline{\nabla}$ denotes the gradient operator in $\mathbb{R}^{d-1}$. Finally, when the function $\overline{\theta}$ appears in volume integrals on $\Omega \cap B$, it will be meant that we use the constant prolongation of $\overline{\theta}$ inside Ω, that is (with a slight abuse of notation) $\overline{\theta}(\overline{x}, x_d) = \theta(\overline{x}, h(\overline{x}))$ for all $(\overline{x}, x_d) \in \Omega \cap B$.

Cut-off function. Let (ρ_n) be a sequence of mollifiers on $\mathbb{R}$ defined by

$$\rho_n(t) = n\rho(nt) \tag{3.18}$$

where ρ is a non-negative function of $\mathcal{C}_c^\infty(-1,0)$ such that $\int_{-1}^0 \rho(t)\,dt = 1$. The function h is extended outside Π so as to get a Lipschitz continuous function. For ε a positive number, we also denote by ρ_ε the function $t \mapsto \rho_{1/\varepsilon}(t)$ and define the cut-off function ω_ε by

$$\omega_\varepsilon(x) = \int_{h(\overline{x})-x_d}^0 \rho_\varepsilon(\sigma)\,d\sigma\,. \tag{3.19}$$

On $\Omega \cap B$, the function ω_ε vanishes in a neighbourhood of $\partial\Omega$ and equals 1 if $\mathrm{dist}(x, \partial\Omega) > \varepsilon$. If $\psi \in H^1(\Omega)^N$ then

$$\int_\Omega \lambda\,\psi \cdot \nabla\omega_\varepsilon\,dx = -\int_\Omega \mathrm{div}(\lambda\,\psi)\,\omega_\varepsilon\,dx \to -\int_\Omega \mathrm{div}(\lambda\,\psi)\,dx = -\int_{\partial\Omega} \lambda\,\psi \cdot n\,d\gamma(x)\,, \tag{3.20}$$

Also notice that an estimate on the L^1 norm of the gradient of ω_ε (uniform with respect to ε) holds. Indeed, from the definition (3.19), we have

$$\nabla\omega_\varepsilon(x) = \rho_\varepsilon(h(\overline{x}) - x_d)\begin{pmatrix} -\overline{\nabla}h(\overline{x}) \\ 1 \end{pmatrix}, \tag{3.21}$$

and, in particular:

$$\forall \overline{x}\,, \int_{x_d > h(\overline{x})} |\nabla\omega_\varepsilon(x)|\,dx_d \le \sqrt{1 + |\overline{\nabla}h(\overline{x})|^2}\,. \tag{3.22}$$

Eventually, if ψ is a smooth function defined in a neighbourhood V of $\partial\Omega \cap B$ then we have, using also (3.22),

$$\int_{V\cap\Omega} \psi(x) \cdot \nabla\omega_\varepsilon(x)\,dx = \int_{V\cap\Omega} \overline{\psi}(\overline{x}) \cdot \nabla\omega_\varepsilon(x)\,dx + o(1) \quad [\varepsilon \to 0]\,. \tag{3.23}$$

3.2 L^1–contraction principle.

Here, we aim at proving the following result.

Theorem 3.1 *Let u and v be two renormalized entropy solutions of Problem (1.1) with data $(u_0, \overline{u}, g)$ and $(v_0, \overline{v}, \tilde{g})$ respectively. Then we have*

$$\int_\Omega |u - v|(t) \le \int_0^t \int_{\partial\Omega} S(\overline{u}, \overline{v})\,d\gamma(x)\,ds + \int_0^t \int_\Omega |g - \tilde{g}|\,dx\,ds + \int_\Omega |u_0 - v_0|\,dx\,, \tag{3.24}$$

for almost every $t \in (0, T)$, where

$$S(\overline{u}, \overline{v})(t, x) = \sup\left\{2\,|f(\sigma) - f(\tau)|\ ;\ \min(\overline{u}(t,x), \overline{v}(t,x)) \le \sigma\,,\ \tau \le \max(\overline{u}(t,x), \overline{v}(t,x))\right\}\,,\ (t,x) \in \Sigma\,.$$

This theorem extends the known contraction result in case the boundary data are not bounded, and the main novelty with respect to previous results is the possibility to obtain this estimate depending on $\overline{f}(\overline{u})$ and $\overline{f}(\overline{v})$ at the boundary. Its proof however is an adaptation of the proof given by Otto in case of bounded data, proof which lies itself on the tecnhique of doubling variables of Kruzkov-Kutnetzov. As in [Ott96], preliminary results are required, in order to deal with the boundary terms in an accurate way. To expose them, we localize again and refer to the notations previously introduced.

Lemma 3.2 *Let $u \in L^1(Q)$ be a renormalized entropy solution of the problem (1.1), let $\overline{v}$ be a measurable function on Σ and $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega} \times [0, T))$. Then*

1. *for every $l \ge |\kappa|$, the following limit is well defined and, if the test function φ is nonnegative, bounded as follows:*

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Phi^+(u \bot l, \kappa \top T_l(\overline{u})) \cdot \nabla\omega_\varepsilon\,\lambda\,\varphi\,dx\,dt \le \int_0^T \int_{\partial\Omega} \overline{\lambda\varphi}\,d\mu_l\,, \tag{3.25}$$

where, in the right hand-side, we still denoted by μ_l the restriction of the measure μ_l to $\partial\Omega \times [0, T]$.

2. Similarly, for every $l \geq 0$ the following limit

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\perp l, T_l(\overline{v}))\cdot\nabla\omega_\varepsilon\,\lambda\,\varphi\,dx\,dt$$

exists and only depends on the value $\overline{\varphi}=\varphi|_{\partial\Omega\times[0,T)}$. Actually, this limit defines a linear continuous form on $L^1(\Sigma,\overline{\lambda}\,d\gamma\,dt)$, which is represented by a function $\mathcal{K}_l^+(u,\overline{v})$ of $L^\infty(\Sigma,\overline{\lambda}\,d\gamma\,dt)$, that is we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\perp l, T_l(\overline{v}))\cdot\nabla\omega_\varepsilon\,\lambda\,\varphi\,dx\,dt=\int_0^T\int_{\partial\Omega}\mathcal{K}_l^+(u,\overline{v})\,\overline{\lambda}\,\overline{\varphi}\,d\gamma(x)\,dt\,. \tag{3.26}$$

Proof: Let us set

$$\begin{aligned}L(\varphi)=&\\ &\int_0^T\int_\Omega\left((u\perp l-\kappa)^+\varphi_t+\Phi^+(u\perp l,\kappa)\cdot\nabla\varphi\right)\,dx\,dt+\int_\Omega(u_0\perp l-\kappa)^+\varphi(x,0)\,dx\\ &\quad+\mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}\perp l-\kappa)^+\varphi(x,t)\,d\gamma(x)\,dt\;+\int_0^T\int_\Omega g\,\mathrm{sgn}^+(u\perp l-\kappa)\,\varphi\,dx\,dt+\langle\mu_l,\varphi\rangle\,.\end{aligned}$$

By definition of entropy solution we have that L is a linear nonnegative operator. Letting ω_ε be defined in (3.19), and $\varphi\geq 0$, since $(1-\omega_\varepsilon)\varphi\lambda$ is a nonincreasing sequence we deduce that $L((1-\omega_\varepsilon)\varphi\lambda)$ is nonincreasing and nonnegative, hence admits a limit as ε tends to zero. In particular this implies the following inequality

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\Phi^+(u\perp l,\kappa)\cdot\nabla\omega_\varepsilon\,\lambda\varphi\,dx\,dt\leq\mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}\perp l-\kappa)^+\lambda\varphi(x,t)\,d\gamma(x)\,dt\;+\lim_{\varepsilon\to 0}\langle\mu_l,(1-\omega_\varepsilon)\lambda\varphi\rangle\,.$$

Using the monotone convergence theorem we have

$$\lim_{\varepsilon\to 0}\langle\mu_l,(1-\omega_\varepsilon)\lambda\varphi\rangle=\int_0^T\int_{\partial\Omega}\overline{\lambda}\,\overline{\varphi}\,d\mu_l\,.$$

Moreover, since $\Phi^+(u\perp l,\kappa)$ is bounded for l fixed and $k\geq -l$, reasoning as in (3.23) we easily get that

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\Phi^+(u\perp l,\kappa)\cdot\nabla\omega_\varepsilon\,\lambda\varphi\,dx\,dt=\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\Phi^+(u\perp l,\kappa)\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,dx\,dt\,,$$

so that we obtain

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\Phi^+(u\perp l,\kappa)\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,dx\,dt\leq\mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}\perp l-\kappa)^+\overline{\lambda}\,\overline{\varphi}\,d\gamma(x)\,dt\;+\int_0^T\int_{\partial\Omega}\overline{\lambda}\,\overline{\varphi}\,d\mu_l\,. \tag{3.27}$$

Our main task is now replacing $\overline{\varphi}$ with $\overline{\varphi}\chi_{\overline{B}}$ in (3.27) for any borelian subset $\overline{B}\subset\partial\Omega\times[0,T)$. Indeed, given such a $\overline{B}$, for any $\delta>0$ there exists a function $\psi_\delta\in C_c^\infty(\overline{\Omega}\times[0,T))$ such that

$$\|\overline{\psi}_\delta-\chi_{\overline{B}}\|_{L^1(\Sigma)}+\|\overline{\psi}_\delta-\chi_{\overline{B}}\|_{L^1(\Sigma,d\mu_l)}\leq\delta\,, \tag{3.28}$$

where $\overline{\psi}_\delta=\psi_\delta|_{\partial\Omega\times[0,T)}$. To construct such a ψ_δ, it is enough to use Uryshon lemma applied to an open set U and to a compact set $K\subset U$ such that $\gamma(U\setminus K)+\mu_l(U\setminus K)\leq\delta$ (recall that γ is the $d-1$ dimensional Lebesgue measure), the existence of U and K being given by the regularity of μ_l and of the Lebesgue measure on Σ. Define $B=\{(\overline{x},x_d)\in\Omega\,:\,(\overline{x},h(\overline{x}))\in\overline{B}\}$, which is a thickening of $\overline{B}$ inside Ω. Since we have, by (3.22),

$$\left|\int_0^T\int_\Omega\Phi^+(u\perp l,\kappa)\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}(\chi_B-\psi_\delta)\,dx\,dt\right|\leq C(l)\|\overline{\varphi}\|_{L^\infty(\Sigma)}\int_0^T\int_{\partial\Omega}|\chi_{\overline{B}}-\overline{\psi_\delta}|\,\overline{\lambda}\,d\gamma(x)\,dt\,,$$

we deduce from (3.27) and (3.28)

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\bot l,\kappa)\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,\chi_B\,dx\,dt$$
$$\le \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\bot l,\kappa)\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,\overline{\psi}_\delta\,dx\,dt + C\,\delta$$
$$\le \mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}\bot l-\kappa)^+\overline{\lambda}\,\overline{\varphi}\,\overline{\psi}_\delta\,d\gamma(x)\,dt + \int_0^T\int_{\partial\Omega}\overline{\lambda}\,\overline{\varphi}\,\overline{\psi}_\delta\,d\mu_l + C\delta\,,$$

and then, as δ tends to zero, thanks to (3.28)

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\bot l,\kappa)\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,\chi_B\,dx\,dt \le \mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}\bot l-\kappa)^+\overline{\lambda}\,\overline{\varphi}\,\chi_{\overline{B}}\,d\gamma(x)\,dt + \int_0^T\int_{\partial\Omega}\overline{\lambda}\,\overline{\varphi}\,\chi_{\overline{B}}\,d\mu_l\,. \tag{3.29}$$

Let now $s_n = \sum_{i=1}^n w_i\chi_{\overline{B}_i}$ be a sequence of step functions strongly converging to $T_l(\overline{v})$ in $L^1(\Sigma)$, with $|w_i|\le l$. Choosing $\overline{B}=\overline{B}_i$, replacing κ with $\kappa\top w_i$ in (3.29) and summing over i we get

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\bot l,\kappa\top s_n)\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,dx\,dt \le \mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}\bot l-\kappa\top s_n)^+\overline{\lambda}\,\overline{\varphi}\,d\gamma(x)\,dt + \int_0^T\int_{\partial\Omega}\overline{\lambda}\,\overline{\varphi}\,d\mu_l\,. \tag{3.30}$$

Using again the following estimate (uniform on ε)

$$\int_0^T\int_\Omega \big|\big[\Phi^+(u\bot l,\kappa\top s_n)-\Phi^+(u\bot l,\kappa\top T_l(\overline{v}))\big]\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\big|\,dx\,dt$$
$$\le \mathrm{Lip}_l\,\|\overline{\varphi}\|_{L^\infty(\Sigma)}\int_0^T\int_{\partial\Omega}|s_n-T_l(\overline{v})|\,\overline{\lambda}\,d\gamma(x)\,dt\,,$$

we obtain from (3.30):

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\bot l,\kappa\top T_l(\overline{v}))\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,dx\,dt \le \mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}\bot l-\kappa\top T_l(\overline{v}))^+\overline{\lambda}\,\overline{\varphi}\,d\gamma(x)\,dt + \int_0^T\int_{\partial\Omega}\overline{\lambda}\,\overline{\varphi}\,d\mu_l\,. \tag{3.31}$$

Applying this result with $\overline{v}=\overline{u}$ (we use that $(\overline{u}\bot l-k\top T_l(\overline{u}))^+=0$) proves (i). The proof of (ii) is analogous in what concerns the fact that there exists

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\bot l,T_l(\overline{v}))\cdot\nabla\omega_\varepsilon\,\lambda\varphi\;dx\,dt = \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi^+(u\bot l,T_l(\overline{v}))\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,dx\,dt\,.$$

Now, define

$$\Lambda_\varepsilon(\overline{\varphi}) = \int_0^T\int_\Omega \Phi^+(u\bot l,T_l(\overline{v}))\cdot\nabla\omega_\varepsilon\,\overline{\lambda}\,\overline{\varphi}\,dx\,dt\,,$$

if $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega} \times [0,+\infty); \mathbb{R}_+)$. Since the function $\Phi^+(u\bot l, T_l(\overline{v}))$ is bounded a.e., say by M_l, we have, according to the estimate (3.22) on the gradient of the cut-off function ω_ε,

$$\begin{aligned}\Lambda_\varepsilon(\overline{\varphi}) &= \int_0^T \int_\Pi \int_{x_d > h(\overline{x})} \Phi^+(u\bot l, T_l(\overline{v}))(x,t) \cdot \nabla\omega_\varepsilon(x)\overline{\lambda}(\overline{x})\overline{\varphi}(\overline{x},t)\, dx_d\, d\overline{x}\, dt \\ &\leq M_l \int_0^T \int_\Pi \sqrt{1 + |\overline{\nabla} h(\overline{x})|^2}\,\overline{\lambda}(\overline{x},t)|\overline{\varphi}|(\overline{x})\, d\overline{x}\, dt \\ &= M_l \int_0^T \int_{\partial\Omega} \overline{\lambda}\,\overline{\varphi}\, d\gamma\, dt\,.\end{aligned} \tag{3.32}$$

Therefore, the function Λ_ε is well defined on $L^1(\Sigma, \overline{\lambda}\, d\gamma\, dt)$ and induces a linear form on this space. Its norm is bounded independently of ε and it is pointwise converging. Therefore, $\Lambda : \beta \mapsto \lim_{\varepsilon\to 0} \Lambda_\varepsilon(\beta)$ is a continuous linear form on $L^1(\Sigma, \overline{\lambda}\, d\gamma\, dt)$ which can be represented by an element $\mathcal{K}_l^+(u, \overline{v})$ of $L^\infty(\Sigma, \overline{\lambda}\, d\gamma\, dt)$. ∎

Remark 3.3 *The two conclusions of Lemma 3.2 together imply the existence, in a weak sense, of the normal trace of $\phi^+(u\bot l, k\top T_l(\overline{u}))$ at the boundary, this normal trace being represented by the L^∞ function $\mathcal{K}_l^+(u, k\top T_l(\overline{u}))$. Note that requiring, from the definition of renormalized solution, that $\|\mu_l\|$ goes to zero as l tends to infinity implies that $\mathcal{K}_l^+(u, k\top T_l(\overline{u}))$ converges to zero in $L^1(\Sigma)$ as l tends to infinity for any $|k| \leq l$.*

Working on the "negative" semi Kruzkov entropy functions $u \mapsto (u - \kappa)^-$, analogous results to those of Lemma 3.2 can be proved, that is:

1. for every $l \geq |\kappa|$, if the test function φ is nonnegative, there holds
$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Phi^-(u\top(-l), \kappa\bot T_l(\overline{u})) \cdot \nabla\omega_\varepsilon\, \lambda\, \varphi\, dx\, dt \leq \int_0^T \int_{\partial\Omega} \overline{\lambda}\,\overline{\varphi}\, d\nu_l\,. \tag{3.33}$$

2. There exists $\mathcal{K}_l^-(u, \overline{v}) \in L^\infty(\Sigma, \overline{\lambda}\, d\gamma\, dt)$ such that for every $l \geq 0$ and for every $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega} \times [0,T))$,
$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Phi^-(u\top(-l), T_l(\overline{v})) \cdot \nabla\omega_\varepsilon\, \lambda\, \varphi\, dx\, dt = \int_0^T \int_{\partial\Omega} \mathcal{K}_l^-(u, \overline{v})\, \overline{\lambda}\,\overline{\varphi}\, d\gamma(x)\, dt\,. \tag{3.34}$$

Gathering these two kind of results, we obtain an entropic formulation for the truncations of the solutions.

Proposition 3.4 *Let $u \in L^1(Q)$ be a renormalized entropy solution of the problem (1.1), let $\overline{v}$ be a measurable function on Σ and $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega} \times [0,T))$. Then*

1. *There exists a function $\mathcal{K}_l(u, \overline{v}) \in L^\infty(\Sigma, \overline{\lambda}\, d\gamma\, dt)$ such that*
$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Phi(T_l(u), T_l(\overline{v})) \cdot \nabla\omega_\varepsilon\, \lambda\, \varphi\, dx\, dt = \int_0^T \int_{\partial\Omega} \mathcal{K}_l(u, \overline{v})\, \overline{\lambda}\,\overline{\varphi}\, d\gamma(x)\, dt \tag{3.35}$$
for every l and every $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega} \times [0,T))$.

2. *For every $k \in \mathbb{R}$, $l \in \mathbb{R}_+$, with $|k| \leq l$, and for every $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega} \times [0,T)$,*
$$\begin{aligned}&\int_0^T \int_\Omega |T_l(u) - \kappa|(\lambda\varphi)_t + \Phi(T_l(u), \kappa) \cdot \nabla(\lambda\varphi)\, dx\, dt + \int_\Omega |T_l(u_0) - \kappa|\lambda(x)\, \varphi(x,0)\, dx \\ &- \int_0^T \int_{\partial\Omega} (\mathcal{K}_l(u, \overline{u}) - \Phi(\kappa, T_l(\overline{u})) \cdot n)\, \overline{\lambda}\,\overline{\varphi}\, d\gamma(x)\, dt \\ &+ \int_0^T \int_\Omega g\, \mathrm{sgn}(T_l(u) - \kappa)\, \lambda\, \varphi\, dx\, dt \geq -2\langle \mu_l + \nu_l, \lambda\, \varphi\rangle\,.\end{aligned} \tag{3.36}$$

PROOF: In view of the identity

$$(u\bot l-\kappa)^+ + (u\top(-l)-\kappa)^- = |T_l(u)-\kappa|\,, \tag{3.37}$$

which is valid for every u, l, $\kappa \in \mathbb{R}$ with $l \geq |\kappa|$, (3.35) follows from (3.26) and (3.34) by simply defining the function $\mathcal{K}_l(u,\overline{v})$ by

$$\mathcal{K}_l(u,\overline{v}) = \mathcal{K}_l^+(u,\overline{v}) + \mathcal{K}_l^-(u,\overline{v})\,.$$

Moreover, once the limit in (3.35) has been defined thanks to the function $\mathcal{K}_l(u,\overline{v})$, we have the following estimate: for all $\kappa \in \mathbb{R}$, for all $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega}\times[0,T))$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi(T_l(u),\kappa)\cdot\nabla\omega_\varepsilon\,\lambda\,\varphi\,dx\,dt\\ \leq\;& -\int_0^T\int_{\partial\Omega}(\mathcal{K}_l(u,\overline{u})-\Phi(\kappa,T_l(\overline{u}))\cdot n)\,\overline{\lambda}\,\overline{\varphi}\,d\gamma(x)\,dt + 2\int_0^T\int_{\partial\Omega}\overline{\lambda}\,\overline{\varphi}\,d(\mu_l+\nu_l)\,.\end{aligned} \tag{3.38}$$

The proof of (3.38) lies on the manipulation of algebraic formula. Indeed, defining as in [MNRR96] the function $\mathcal{F}$ by

$$\mathcal{F}(s,\kappa,w) = 2\left(\Phi^+(s,\kappa\top w)+\Phi^-(s,\kappa\bot w)\right)\,,$$

we have

$$\Phi(T_l(u),\kappa) = \mathcal{F}(T_l(u),\kappa,T_l(\overline{u})) - \Phi(T_l(u),T_l(\overline{u})) + \Phi(\kappa,T_l(\overline{u}))$$

which yields, by (3.35),

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Phi(T_l(u),\kappa)\cdot\nabla\omega_\varepsilon\,\lambda\,\varphi\,dx\,dt\\ \leq\;& \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \mathcal{F}(T_l(u),\kappa,T_l(\overline{u}))\cdot\nabla\omega_\varepsilon\,\lambda\,\varphi\,dx\,dt - \int_0^T\int_{\partial\Omega}(\mathcal{K}_l(u,\overline{u})-\Phi(\kappa,T_l(\overline{u}))\cdot n)\,\overline{\lambda}\,\overline{\varphi}\,d\gamma(x)\,dt\,.\end{aligned}$$

Moreover, one can check that, for $l \geq |\kappa|$, one has

$$\mathcal{F}(T_l(u),\kappa,T_l(\overline{u})) = 2\left(\Phi^+(u\bot l,\kappa\top T_l(\overline{u})) + \Phi^-(u\top(-l),\kappa\bot T_l(\overline{u}))\right)\,,$$

and deduce from (3.25) and (3.33) the estimate

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \mathcal{F}(T_l(u),\kappa,T_l(\overline{u}))\cdot\nabla\omega_\varepsilon\,\lambda\,\varphi\,dx\,dt \leq 2\int_0^T\int_{\partial\Omega}\overline{\lambda}\,\overline{\varphi}\,d(\mu_l+\nu_l).$$

Estimate (3.38) then follows. It is used to derive the entropy inequalities for the truncations $T_l(u)$. Indeed, adding entropy inequalities (2.6) and (2.7) where the function $\omega_\varepsilon\,\lambda\,\varphi$ has been chosen as a test function yields, thanks to formula (3.37): for $l \geq |\kappa|$,

$$\begin{aligned}&\int_0^T\!\!\int_\Omega \left(|T_l(u)-\kappa|(\lambda\varphi)_t + \Phi(T_l(u),\kappa)\cdot\nabla(\lambda\varphi)\right)\,dx\,dt + \int_\Omega |T_l(u_0)-\kappa|\lambda(x)\,\varphi(x,0)\,dx\\ &+\int_0^T\!\!\int_\Omega \Phi(T_l(u),\kappa)\cdot\nabla\omega_\varepsilon\,\lambda\varphi\,dxdt + \int_0^T\!\!\int_\Omega g\,\mathrm{sgn}(T_l(u)-\kappa)\,\lambda\,\varphi\,dxdt\\ &\geq -\int_0^T\!\!\int_\Omega \omega_\varepsilon\,\lambda\,\varphi\,d(\mu_l+\nu_l) + o(1)\,.\end{aligned}$$

Here $o(1)$ is a quantity which group together the terms in which the function $1-\omega_\varepsilon$ is integrated and, therefore, tends to zero with ε. Consequently, passing to the limit $[\varepsilon\to 0]$, using that $\omega_\varepsilon \leq 1$ and taking into account estimate (3.38) yields the entropy inequality (3.36). ■

We prove the fundamental L^1–contraction principle by using the doubling variable technique of Kruzkov-Kutnetsov.

Proof of Theorem 3.1.
Let u and $v \in L^1(Q)$ be two entropy solutions of Problem (1.1) associated to the data $(u_0, \overline{u}, g)$ and $(v_0, \overline{v}, \widetilde{g})$ respectively and let $\rho_{p,q,r}$ be the function defined on $\mathbb{R}^{d+1}$ by

$$\rho_{p,q,r}(t,x) = \rho_p(t)\rho_q(\overline{x})\rho_r(x_d)$$

where the mollifying function ρ_n is defined by (3.18). To compare u with v, choose $\kappa = T_l(v(y,s))$ in inequality (3.36) and $(x,t) \mapsto \rho_{p,q,r}(t-s, x-y)\psi(x,t)$ as a test function, where $\psi \in \mathcal{C}_c^\infty(\overline{\Omega} \times [0,T); \mathbb{R}_+)$, then integrate with respect to (y,s). This yields the inequality

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 \geq A_8 \tag{3.39}$$

where

$$A_1 = \int_0^T \int_\Omega \int_0^T \int_\Omega |T_l(u(x,t)) - T_l(v(y,s))| (\lambda\psi)_t \, \rho_{p,q,r}(t-s, x-y)$$

$$A_2 = \int_0^T \int_\Omega \int_0^T \int_\Omega |T_l(u(x,t)) - T_l(v(y,s))| \lambda\psi \, (\partial_t \rho_{p,q,r})(t-s, x-y) \, dx \, dt \, dy \, ds$$

$$A_3 = \int_0^T \int_\Omega \int_0^T \int_\Omega \Phi(T_l(u(x,t)), T_l(v(y,s))) \nabla(\lambda\psi) \, \rho_{p,q,r}(t-s, x-y) \, dx \, dt \, dy \, ds$$

$$A_4 = \int_0^T \int_\Omega \int_0^T \int_\Omega \Phi(T_l(u(x,t)), T_l(v(y,s))) \, \lambda\psi \, (\nabla \rho_{p,q,r})(t-s, x-y) \, dx \, dt \, dy \, ds$$

$$A_5 = \int_0^T \int_\Omega \int_\Omega |T_l(u_0) - T_l(v(y,s))| (\lambda\psi)(x,0) \, \rho_{p,q,r}(-s, x-y) \, dx \, dy \, ds$$

$$A_6 = -\int_0^T \int_\Omega \int_0^T \int_\Pi \left(\mathcal{K}_l(u, \overline{u})(\overline{x}, t) \sqrt{1 + |\overline{\nabla} h(\overline{x})|^2} - \Phi(T_l[v(y,s)], T_l[\overline{u}(\overline{x}, t)]) \cdot \begin{pmatrix} -\overline{\nabla} h(\overline{x}) \\ 1 \end{pmatrix} \right) \times \overline{\lambda\psi}(\overline{x}, t) \rho_p(t-s) \, \rho_q(\overline{x} - \overline{y}) \rho_r(h(\overline{x}) - y_d) \, d\overline{x} \, dt \, dy \, ds$$

$$A_7 = \int_0^T \int_\Omega \int_0^T \int_\Omega g \, \mathrm{sgn}(T_l(u(x,t)) - T_l(v(y,s))) \, \lambda \, \psi \, \rho_{p,q,r}(t-s, x-y) \, dx \, dt \, dy \, ds$$

$$A_8 = -2 < \mu_l + \nu_l, \lambda \, \psi > \; .$$

Notice that, in A_8, the mollifiers have been integrated with respect to (y,s) and that Identity (3.17) has been used to write the term A_6.

The function v also satisfies the entropy inequality

$$\begin{aligned} &\int_0^T \int_\Omega |T_l(v) - \kappa| \theta_s + \Phi(T_l(v), \kappa) \cdot \nabla\theta \, dy \, ds + \int_\Omega |T_l(v_0(y)) - \kappa| \theta(y,0) \\ &\qquad \int_0^T \int_\Omega \int_0^T \int_\Omega \overline{f}(\overline{u})(s,y) \, \mathrm{sgn}(T_l(v(y,s)) - \kappa) \, \theta \, dy \, ds \geq -\int_0^T \int_\Omega \theta \, d(\tilde{\mu}_l + \tilde{\nu}_l) \end{aligned} \tag{3.40}$$

for every $\theta \in \mathcal{C}_c^\infty(\Omega \times [0,T); \mathbb{R}_+)$ and $l \geq |\kappa|$. In (3.40), choose $\kappa = T_l(u(x,t))$ and $\theta : (y,s) \mapsto \rho_{p,q,r}(t-s, x-y)\lambda(x)\psi(x,t)$ as a test function (notice that for $y \in \partial\Omega \times [0,T]$ one has $x_d - y_d \geq h(\overline{x}) - h(\overline{y}) \geq -\mathrm{Lip}\, h \, |\overline{x} - \overline{y}|$, hence, for q small enough with respect to r, the function θ vanishes, because the function ρ, through which the mollifier ρ_n is defined, is supported on the left of zero). Integrating the result with respect to (x,t) yields the inequality

$$-A_2 - A_4 + B_7 \geq -\int_0^T \int_\Omega \theta\, d(\tilde{\mu}_l + \tilde{\nu}_l)$$

which, added to (3.39) gives

$$A_1 + A_3 + A_5 + A_6 + A_7 + B_7 \geq A_8 - \int_0^T \int_\Omega \theta\, d(\tilde{\mu}_l + \tilde{\nu}_l)\,, \tag{3.41}$$

where

$$B_7 = \int_0^T \int_\Omega \int_0^T \int_\Omega \widetilde{g}(s,y)\, \mathrm{sgn}(T_l(v(y,s)) - T_l(u(x,t)))\, \lambda\, \psi\, \rho_{p,q,r}(t-s, x-y)\, dx\, dt\, dy\, ds\,.$$

To deal with the term A_5 , let us denote by R_p the function defined by

$$R_p(s) = \int_{-\infty}^{-s} \rho_p(z)\, dz$$

(notice that (R_p) is bounded and converges to zero everywhere on $(0,+\infty)$) and let us choose $\kappa = T_l(u_0(x))$, $\theta(y,s) = R_p(s)\rho_{q,r}(x-y)\lambda(x)\psi(x,0)$ in the entropy inequality (3.40). Integrating the result with respect to x yields the following upper bound on A_5:

$$A_5 \leq C_2 + C_5 + C_7 + C_8$$

where

$$\begin{aligned}
C_2 &= -\int_\Omega \int_0^T \int_\Omega \Phi(T_l(v(y,s)), T_l(u_0(x))) \cdot (\nabla \rho_{q,r})(x-y)\, R_p(s)(\lambda\psi)(x,0) dx\, dy\, ds\,,\\
C_5 &= \int_\Omega \int_\Omega |T_l(v_0(y)) - T_l(u_0(x))| \rho_{q,r}(x-y)(\lambda\psi)(x,0) dx\, dy\,,\\
C_7 &= \int_\Omega \int_0^T \int_\Omega \widetilde{g}\, \mathrm{sgn}(T_l(v(y,s)) - T_l(u_0(x)))\, \lambda(x)\rho_{q,r}(x-y)\, R_p(s)\psi(x,0) dx\, dy\, ds\,,\\
C_8 &= \int_\Omega \int_0^T \int_\Omega \lambda(x)\rho_{q,r}(x-y)\, R_p(s)\psi(x,0)\, d(\tilde{\mu}_l + \tilde{\nu}_l)(y,s)\, dx\,.
\end{aligned}$$

The theorem of convergence in means ensures $\lim_{p\to+\infty} C_2 = 0$ and $\lim_{p\to+\infty} C_7 = 0$ (because the sequence (R_p) converges to 0 in L^1) and $|C_8| \leq \|\psi\|_{L^\infty(Q)}(\|\tilde{\mu}_l\| + \|\tilde{\nu}_l\|)$.

Besides, owing to the continuity of the translations in L^1, we have

$$B_7 = \int_0^T \int_\Omega \int_0^T \int_\Omega \widetilde{g}(x,t)\, \mathrm{sgn}(T_l(v(y,s)) - T_l(u(x,t)))\, \lambda\, \psi\, \rho_{p,q,r}(t-s, x-y)\, dx\, dt\, dy\, ds + \eta_{p,q,r}$$

where $\eta_{p,q,r}$ tends to zero when p, q, and r tend to ∞. Since $\mathrm{sgn}(-a) = -\mathrm{sgn}(a)$, $a \in \mathbb{R}$, there holds:

$$A_7 + B_7 \leq F_7 = \int_0^T \int_\Omega |g - \widetilde{g}|\, \lambda\, \psi\, dx\, dt + \eta_{p,q,r}\,.$$

Therefore, first passing to the limit $p \to +\infty$, then $q \to +\infty$ in inequality (3.41) yields

$$\overline{A_1} + \overline{A_3} + \overline{C_5} + \overline{A_6} + F_7 + \eta_r \geq A_8 - \|\psi\|_{L^\infty(Q)}(\|\tilde{\mu}_l\| + \|\tilde{\nu}_l\|) \tag{3.42}$$

where $\lim_{r\to+\infty} \eta_r = 0$ and

$$\overline{A_1} = \int_\Omega \int_0^T \int_{y_d>h(\overline{x})} |T_l(u(x,t)) - T_l(v(\overline{x}, y_d, t))|(\lambda\psi)_t \, \rho_r(x_d - y_d) dx\, dt\, dy_d$$
$$\overline{A_3} = \int_\Omega \int_0^T \int_{y_d>h(\overline{x})} \Phi(T_l(u(x,t)), T_l(v(\overline{x}, y_d, t)))\nabla(\lambda\psi)\, \rho_r(x_d - y_d)\, dx\, dt\, dy_d$$

$$\overline{A_6} = -\int_0^T \iint_{\Pi} \int_{y_d>h(\overline{x})} \Big(\mathcal{K}_l(u, \overline{u})(\overline{x}, t)\sqrt{1 + |\overline{\nabla} h(\overline{x})|^2} - \Phi(T_l[v(\overline{x}, y_d, t)], T_l[\overline{u}(\overline{x}, t)]) \cdot \begin{pmatrix} -\overline{\nabla} h(\overline{x}) \\ 1 \end{pmatrix} \Big) \times \overline{\lambda\psi}(\overline{x}, t)\rho_r(h(\overline{x}) - y_d)\, d\overline{x}\, dt\, dy_d$$

$$\overline{C_5} = \int_\Omega \int_{y_d>h(\overline{x})} |T_l(v_0(\overline{x}, y_d)) - T_l(u_0(x))|\rho_r(x_d - y_d)(\lambda\psi)(x, 0) dx\, dy_d\,.$$

Replacing y_d by x_d in the term $\overline{A_6}$ and using Formula (3.21) and Identity (3.17), we have

$$\overline{A_6} = -\int_0^T \int_{\partial\Omega} \mathcal{K}_l(u, \overline{u})\overline{\lambda\psi} d\gamma(x)\, dt + \int_0^T \int_\Omega \Phi(T_l[v(\overline{x}, x_d, t)], T_l[\overline{u}(\overline{x}, t)]) \cdot \nabla\omega_{1/r}(x)\overline{\lambda\psi}\, dx\, dt\,,$$

so that

$$\lim_{r\to+\infty} \overline{A_6} = -\int_0^T \int_{\partial\Omega} \big(\mathcal{K}_l(u, \overline{u}) - \mathcal{K}_l(v, \overline{u})\big)\, \overline{\lambda\,\psi} d\gamma(x)\, dt\,.$$

From the theorem of convergence in means we derive the limit as r tends to $+\infty$ of $\overline{A_1}$, $\overline{A_3}$ and $\overline{C_5}$ respectively to deduce from inequality (3.42) the following result:

$$\int_\Omega \int_0^T |T_l(u) - T_l(v)|(\lambda\psi)_t + \Phi(T_l(u), T_l(v))\nabla(\lambda\psi)\, dx\, dt + \int_\Omega |T_l(u_0) - T_l(v_0)|\, \lambda\, \psi(0)\, dx$$
$$- \int_0^T \int_{\partial\Omega} \big(\mathcal{K}_l(u, \overline{u}) - \mathcal{K}_l(v, \overline{u})\big)\, \overline{\lambda\,\psi} d\gamma(x)\, dt \tag{3.43}$$
$$+ \int_0^T \int_\Omega |g - \widetilde{g}|\, \lambda\, \psi\, dx\, dt \geq -C\|\psi\|_{L^\infty(Q)} \left(\|\mu_l\| + \|\nu_l\| + \|\tilde{\mu}_l\| + \|\tilde{\nu}_l\|\right).$$

Changing the role of the solutions u and v in (3.43), then adding the result (side by side) to (3.43) and dividing the result by 2, recalling that the function λ is actually one of the functions of localizations $(\lambda_\alpha)_{0,N}$ and summing the previous result over $\alpha \in \{0, \ldots, N\}$ we get:

$$\begin{aligned} &\int_\Omega \int_0^T |T_l(u) - T_l(v)|\psi_t + \Phi(T_l(u), T_l(v))\nabla\psi\, dx\, dt + \int_\Omega |T_l(u_0) - T_l(v_0)|\, \psi(0)\, dx \\ &- \int_0^T \int_{\partial\Omega} \big(\mathcal{K}_l(u, \overline{u}) - \mathcal{K}_l(v, \overline{u}) + \mathcal{K}_l(v, \overline{v}) - \mathcal{K}_l(u, \overline{v})\big)\, \overline{\psi}\, d\gamma(x)\, dt \\ &+ \int_0^T \int_\Omega |g - \widetilde{g}|\, \psi\, dx\, dt \geq -\frac{C}{2}\|\psi\|_{L^\infty(Q)} \left(\|\mu_l\| + \|\nu_l\| + \|\tilde{\mu}_l\| + \|\tilde{\nu}_l\|\right). \end{aligned} \tag{3.44}$$

Coming back to the definition of $\mathcal{K}_l(u, \overline{v})$, we have

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} \big(\mathcal{K}_l(u, \overline{u}) - \mathcal{K}_l(v, \overline{u}) + \mathcal{K}_l(v, \overline{v}) - \mathcal{K}_l(u, \overline{v})\big)\, \overline{\psi}\, d\gamma(x)\, dt \\ &= \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \big(\Phi(T_l(u), T_l(\overline{u})) - \Phi(T_l(v), T_l(\overline{u})) + \Phi(T_l(v), T_l(\overline{v})) - \Phi(T_l(u), T_l(\overline{v}))\big)\, \nabla\omega_\varepsilon \overline{\psi}\, dx\, dt\,. \end{aligned}$$

Discussing the respective positions of u, v, $\overline{u}$, $\overline{v}$, we have a pointwise estimate:

$$\big|\Phi(T_l(u),T_l(\overline{u})) - \Phi(T_l(v),T_l(\overline{u})) + \Phi(T_l(v),T_l(\overline{v})) - \Phi(T_l(u),T_l(\overline{v}))\big| \leq S(\overline{u},\overline{v})\,,$$

where

$$S(\overline{u},\overline{v})(t,x) = \sup\{2\;|f(\sigma)-f(\tau)|\;;\;\min(\overline{u}(t,x),\overline{v}(t,x)) \leq \sigma\,,\,\tau \leq \max(\overline{u}(t,x),\overline{v}(t,x))\}\;,\;(t,x)\in\Sigma\,.$$

Then we deduce, using also (3.22),

$$\Big|\int_0^T\int_{\partial\Omega}\big(\mathcal{K}_l(u,\overline{u}) - \mathcal{K}_l(v,\overline{u}) + \mathcal{K}_l(v,\overline{v}) - \mathcal{K}_l(u,\overline{v})\big)\,\overline{\psi}\,d\gamma(x)\,dt\Big| \leq \int_0^T\int_{\partial\Omega} S(\overline{u},\overline{v})\,\overline{\psi}\,d\gamma(x)\,dt\,. \tag{3.45}$$

Taking also into account the estimate (3.45) and passing to the limit $[l\to+\infty]$ we get from (3.44)

$$-\int_\Omega\int_0^T |u-v|\psi_t \leq \int_0^T\int_{\partial\Omega} S(\overline{u},\overline{v})\,\psi\,d\gamma(x)\,dt + \int_0^T\int_\Omega |g-\widetilde{g}|\,\psi\,dx\,dt + \int_\Omega |u_0-v_0|\,\psi(0)\,dx\,,$$

for every nonnegative $\psi\in C_c^\infty[0,T)$. Since $S(\overline{u},\overline{v})\in L^1(\Sigma)$ by (2.5), and since $|g-\widetilde{g}|\in L^1(Q)$, $|u_0-v_0|\in L^1(\Omega)$, standard arguments allow to obtain (3.24). ■

3.3 Existence and uniqueness

For $n\in\mathbb{N}$, we define the truncated data $g^n = T_n(g)$, $u_0^n = T_n(u_0)$ and $\overline{u}^n = T_n(\overline{u})$. There exists a unique entropy solution $u^n\in L^\infty(Q)$ of the problem (1.1) with data g^n, u_0^n and $\overline{u}^n$ [Ott96], moreover $|u^n|\leq n$ and u^n satisfies the following entropy inequalities: for every $\kappa\in\mathbb{R}$, for every $\varphi\in\mathcal{C}_c^\infty(\overline{\Omega}\times[0,T);\mathbb{R}_+)$,

$$\begin{aligned}&\int_0^T\int_\Omega((u^n-\kappa)^\pm\varphi_t + \Phi^\pm(u^n,k)\cdot\nabla\varphi)\,dx\,dt + \int_\Omega(u_0^n-\kappa)^\pm\varphi(x,0)\,dx\\ &+\mathrm{Lip}_n\int_0^T\int_{\partial\Omega}(\overline{u}^n-\kappa)^\pm\varphi(x,t)\,d\gamma(x)\,dt + \int_0^T\int_\Omega g^n\,\mathrm{sgn}^\pm(u^n-\kappa)\,\varphi\,dx\,dt \geq 0.\end{aligned} \tag{3.46}$$

Remark 3.5 *Ineq, (3.46) remains true if the test-function φ does not vanish on T, that is $\varphi\in\mathcal{C}^\infty(\overline{\Omega}\times[0,T];\mathbb{R}_+)$. This can be seen by considering $\varphi\,\xi_\alpha$ as a test function in that case, the function ξ_α being a cut-off function defined by $\xi_\alpha(t)=\int_{t-T}^0\rho_\alpha$ for example, and by noticing that the additional term*

$$\int_0^T\int_\Omega(u^n-\kappa)^\pm\,\varphi\,\xi_\alpha'(t)\,dx\,dt$$

is non-positive for every $\alpha>0$.

First, we will reformulate the entropy inequalities in order to prove the convergence of the sequence (u^n) to a solution of Problem (1.1). To that purpose, let us set (for $l\in\mathbb{R}$ and $\varphi\in\mathcal{C}^\infty(\overline{\Omega}\times[0,T])$, with λ the cut–off function defined in Section 3.1)

$$\begin{aligned}\mathcal{T}_\lambda^+(u^n,l,\varphi) = \int_0^T\int_\Omega((u^n-l)^+\,\lambda\,\varphi_t + \Phi^+(u^n,l)\cdot\nabla(\varphi\,\lambda)\,dx\,dt + \int_\Omega(u_0^n-l)^+\varphi(x,0)\,\lambda\,dx\\ +\int_0^T\int_\Omega g^n\,\mathrm{sgn}^+(u^n-l)\,\varphi\,\lambda\,dx\,dt\,,\end{aligned} \tag{3.47}$$

and

$$<\mu_{l,n}^\lambda,\varphi> = \mathcal{T}_\lambda^+(u^n,l,\varphi) + \lim_{\varepsilon\to0}\int_0^T\int_\Omega\big(\Phi^+(u^n,l\bot\overline{u}^n) - \Phi^+(u^n,\overline{u}^n)\big)\cdot\nabla\omega_\varepsilon\,\varphi\,\lambda\,dx\,dt\,. \tag{3.48}$$

Finally, recalling that the function λ is actually an element of a partition of unit $(\lambda_\alpha)_{0,N}$, we globally define the functional

$$< \mu_{l,n}, \varphi >= \sum_{\alpha=0}^{N} < \mu_{l,n}^{\lambda_\alpha}, \varphi > \ . \tag{3.49}$$

Lemma 3.6 *The linear functional $\mu_{l,n}$ is well defined; it is a non-negative measure on $\overline{\Omega} \times [0,T]$ and, for positive $l \leq n$, its mass can be estimated as follows:*

$$||\mu_{l,n}|| \leq \int_\Omega (u_0 - l)^+ \, dx + 2 \int_0^T \int_{\partial\Omega} \overline{f}(\overline{u}) \ \mathbb{1}_{\overline{u}>l} \, d\gamma \, dt + \int_0^T \int_\Omega \mathbb{1}_{u^n>l} \, |g| \, dx \, dt \, . \tag{3.50}$$

PROOF: Reasoning as in Lemma 3.2 we deduce that the functional $\mu_{l,n}^\lambda$ is well defined. Indeed, from (3.46), applied with $\kappa = l \in [-n,n]$, and Remark 3.5, we have that the linear functional

$$\varphi \mapsto \mathcal{T}_\lambda^+(u^n, l, \varphi) + \mathrm{Lip}_n \int_0^T \int_{\partial\Omega} (\overline{u}^n - l)^+ \varphi \, \lambda \, d\gamma(x) \, dt$$

is non-decreasing on $\mathcal{C}^\infty(\Omega \times [0,T])$. Consequently, if $\psi \in \mathcal{C}^\infty(\overline{\Omega} \times [0,T]; \mathbb{R}_+)$, then

$$\varepsilon \mapsto \mathcal{T}_\lambda^+(u^n, l, \psi(1-\omega_\varepsilon)) + \mathrm{Lip}_n \int_0^T \int_{\partial\Omega} (\overline{u}^n - l)^+ \psi \lambda \, d\gamma(x) \, dt$$

is a nonincreasing sequence of non-negative reals and, therefore, admits a limit as $\varepsilon \to 0$, which is non-negative. Since $\lim_{\varepsilon \to 0} (1 - \omega_\varepsilon) = 0$, the limit considered in the following inequality is well defined and we have:

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Phi^+(u^n, l) \cdot \nabla \omega_\varepsilon \, \psi \lambda \, dx \, dt \leq \mathrm{Lip}_n \int_0^T \int_{\partial\Omega} (\overline{u}^n - l)^+ \psi \lambda \, d\gamma(x) \, dt \, .$$

Moreover, this result is still true if l is replaced with $l \top \overline{u}^n$ (see [Ott96], [Vov02], but the result is also contained in Lemma 3.2 if $\mu_l = 0$). This proves that the following limit exists and is non positive:

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Phi^+(u^n, l \top \overline{u}^n) \cdot \nabla \omega_\varepsilon \, \psi \lambda \, dx \, dt \leq 0 \, . \tag{3.51}$$

Using equation(3.46) with $\varphi = \psi \lambda \omega_\varepsilon$ as a test function we get, as ε tends to zero,

$$\mathcal{T}_\lambda^+(u^n, l, \psi) + \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Phi^+(u^n, l) \cdot \nabla \omega_\varepsilon \, \psi \lambda \, dx \, dt \geq 0 \, . \tag{3.52}$$

Combining this last inequality with Inequation (3.51) and the algebraic identity

$$\Phi^+(u^n, l) = \Phi^+(u^n, l \top \overline{u}^n) + \Phi^+(u^n, l \perp \overline{u}^n) - \Phi^+(u^n, \overline{u}^n)$$

we obtain that $\mu_{l,n}^\lambda$ is well defined and it is a non-negative measure on $\overline{\Omega} \times [0,T]$; besides it has a finite mass. Indeed we have

$$\begin{aligned} ||\mu_{l,n}^\lambda|| =< \mu_{l,n}^\lambda, 1 >= \quad & \int_\Omega (u_0^n - l)^+ \lambda \, dx + \int_0^T \int_\Omega \Phi^+(u^n, l) \cdot \nabla \lambda \, dx \, dt \\ & + \int_0^T \int_\Omega g^n \, \mathrm{sgn}^+(u^n - l) \, \lambda \, dx \, dt \\ & + \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \big(\Phi^+(u^n, l \perp \overline{u}^n) - \Phi^+(u^n, \overline{u}^n) \big) \cdot \nabla \omega_\varepsilon \, \lambda \, dx \, dt \, . \end{aligned} \tag{3.53}$$

Suppose $0 < l \leq n$. If $l \geq \overline{u}^n$, then $\Phi^+(u^n, l \perp \overline{u}^n) - \Phi^+(u^n, \overline{u}^n) = 0$. If $l < \overline{u}^n$ then

$$\Phi^+(u^n, l \perp \overline{u}^n) - \Phi^+(u^n, \overline{u}^n) = \begin{cases} f(u^n) - f(l) & \text{if } l \leq u^n \leq \overline{u}^n \, , \\ f(\overline{u}^n) - f(l) & \text{if } \overline{u}^n < u^n \, , \end{cases}$$

and in both cases one has $|\Phi^+(u^n, l\bot\overline{u}^n) - \Phi^+(u^n,\overline{u}^n)| \leq 2\,\overline{f}(\overline{u})$. Therefore we have

$$\begin{aligned}||\mu^\lambda_{l,n}|| \leq\ & \int_\Omega (u_0^n - l)^+\lambda\,dx + 2\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \overline{f}(\overline{u})\ \mathbb{1}_{\overline{u}>l}|\nabla\omega_\varepsilon|\,\lambda\,dx\,dt\\ &+\int_0^T\int_\Omega \mathbb{1}_{u^n>l}\,|g^n|\,\lambda\,dx\,dt + \int_0^T\int_\Omega \Phi^+(u^n,l)\cdot\nabla\lambda\,dx\,dt\,,\end{aligned}$$

and taking into account the estimate (3.22) and the fact that $u_0^n = T_n(u_0)$, $g^n = T_n(g)$, we get

$$\begin{aligned}||\mu^\lambda_{l,n}|| \leq\ & \int_\Omega (u_0 - l)^+\lambda\,dx + 2\int_0^T\int_{\partial\Omega} \overline{f}(\overline{u})\ \mathbb{1}_{\overline{u}>l}\,\lambda\,d\gamma\,dt\\ &+\int_0^T\int_\Omega \mathbb{1}_{u^n>l}\,|g|\,\lambda\,dx\,dt + \int_0^T\int_\Omega \Phi^+(u^n,l)\cdot\nabla\lambda\,dx\,dt\,,\end{aligned}$$

Since $||\sum_{\alpha=0}^N \mu^{\lambda_\alpha}_{l,n}|| \leq \sum_{\alpha=0}^N ||\mu^{\lambda_\alpha}_{l,n}||$, we end up with the estimate (3.50).

Proposition 3.7 *Let u^n be the solution of problem (1.1) with data g^n, u_0^n and $\overline{u}^n$. Then, for all κ, l, with $n \geq l \geq |\kappa|$, for all non-negative functions φ in $\mathcal{C}_c^\infty(\overline{\Omega}\times[0,T))$, the function u^n satisfies the following entropy inequality:*

$$\begin{aligned}&\int_0^T\int_\Omega ((u^n\bot l - \kappa)^+\varphi_t + \Phi^+(u^n\bot l,\kappa)\cdot\nabla\varphi)\,dx\,dt + \int_\Omega (u_0^n\bot l - \kappa)^+\varphi(x,0)\,dx\\ &+\mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}^n\bot l - \kappa)^+\varphi(x,t)\,d\gamma(x)\,dt\ + \int_0^T\int_\Omega g^n\,\mathrm{sgn}^+(u^n\bot l-\kappa)\,\varphi\,dx\,dt \geq -<\mu_{l,n},\varphi>\,,\end{aligned} \tag{3.54}$$

where $\mu_{l,n}$ is defined in (3.49).

Proof: the proof is quite directly deduced from the formula

$$(u\bot l - \kappa)^+ = (u-\kappa)^+ - (u-l)^+\,.$$

Indeed, this implies

$$\begin{aligned}&\int_0^T\int_\Omega (u^n\bot l - \kappa)^+\lambda\varphi_t + \Phi^+(u^n\bot l,\kappa)\cdot\nabla(\varphi\,\lambda)\,dx\,dt + \int_\Omega (u_0^n\bot l - \kappa)^+\varphi(x,0)\,\lambda\,dx\\ &\qquad + \int_0^T\int_\Omega g^n\,\mathrm{sgn}^+(u^n\bot l - \kappa)\,\varphi\,\lambda\,dx\,dt = \mathcal{T}^+_\lambda(u^n,\kappa,\varphi) - \mathcal{T}^+_\lambda(u^n,l,\varphi)\,.\end{aligned} \tag{3.55}$$

By definition of $\mu^\lambda_{l,n}$ (Eq.(3.48)), we have

$$\mathcal{T}^+_\lambda(u^n,\kappa,\varphi) - \mathcal{T}^+_\lambda(u^n,l,\varphi) =<\mu^\lambda_{\kappa,n},\varphi> - <\mu^\lambda_{l,n},\varphi> + \delta_\varepsilon(l,\kappa) \tag{3.56}$$

where $\delta_\varepsilon(l,\kappa) = -\lim_{\varepsilon\to 0}\left(\int_0^T\int_\Omega \left(\Phi^+(u^n,\kappa\bot\overline{u}^n) - \Phi^+(u^n,l\bot\overline{u}^n)\right)\cdot\nabla\omega_\varepsilon\,\varphi\lambda\,dx\,dt\right)$.

Since $|\kappa| \leq l$, we have $\Phi^+(u^n,\kappa\bot\overline{u}^n) - \Phi^+(u^n,l\bot\overline{u}^n) = 0$ if $\overline{u}^n \leq \kappa$ and, therefore,

$$|\Phi^+(u^n,\kappa\bot\overline{u}^n) - \Phi^+(u^n,l\bot\overline{u}^n)| \leq \mathrm{Lip}_l(l\bot\overline{u}^n - k)^+\,,$$

so that $|\delta_\varepsilon(l,\kappa)| \leq \mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}^n\bot l - \kappa)^+\varphi\,\lambda\,d\gamma\,dt$. Also noticing that $<\mu^\lambda_{\kappa,n},\varphi>\geq 0$ (see Lemma 3.6), we get from (3.55) and (3.56)

$$\begin{aligned}&\int_0^T\int_\Omega (u^n\bot l - \kappa)^+\lambda\,\varphi_t + \Phi^+(u^n\bot l,\kappa)\cdot\nabla(\varphi\,\lambda)\,dx\,dt + \int_\Omega (u_0^n\bot l - \kappa)^+\varphi(x,0)\,\lambda(x)\,dx\\ &+\mathrm{Lip}_l\int_0^T\int_{\partial\Omega}(\overline{u}^n\bot l - \kappa)^+\varphi\,\lambda\,d\gamma(x)\,dt\ + \int_0^T\int_\Omega g^n\,\mathrm{sgn}^+(u^n\bot l - \kappa)\,\varphi\,\lambda\,dx\,dt \geq -<\mu^\lambda_{l,n},\varphi>\ .\end{aligned}$$

Summing the previous inequality over $\alpha \in \{0,\dots,N\}$ and recalling the definition of $\mu_{l,n}$ in (3.49) we get (3.54). ∎

We are now in the position to conclude the proof of the existence of a solution, and then to prove Theorem 2.3.

Proof of Theorem 2.3.

We prove first the existence of a renormalized entropy solution as limit of the sequence u^n of bounded solutions satisfying (3.46). Since bounded entropy solutions are in particular renormalized entropy solutions (see Remark 2.2), we can apply Theorem 3.1 to u_n and u_m, obtaining:

$$\begin{aligned}\int_\Omega |u^n - u^m|(t) \le \int_0^T \int_{\partial\Omega} S(T_n(\overline{u}), T_m(\overline{u}))\, d\gamma(x)\, ds \\ + \int_0^T \int_\Omega |T_n(g) - T_m(g)|\, dx\, ds + \int_\Omega |T_n(u_0) - T_m(u_0)|\, dx\,,\end{aligned} \tag{3.57}$$

for all $t \in [0,T]$, where

$$S(T_n(\overline{u}), T_m(\overline{u})) = \sup\{2\ |f(\sigma) - f(\tau)|\ ;\ \min(T_n(\overline{u}), T_m(\overline{u})) \le \sigma\,,\ \tau \le \max(T_n(\overline{u}), T_m(\overline{u}))\}\ ,$$

Since

$$S(T_n(\overline{u}), T_m(\overline{u})) \le 2|\overline{f}(\overline{u})|\ \mathbb{1}_{|\overline{u}|>n} \quad \text{a.e. on } \Sigma,$$

using that $\overline{u}$ is almost everywhere finite on Σ and $\overline{f}(\overline{u})$ belongs to $L^1(\Sigma)$ we deduce that the quantity $S(T_n(\overline{u}), T_m(\overline{u}))$ converges to zero in $L^1(\Sigma)$ as n and m tend to infinity. Similarly since g and u_0 are in L^1 the remaining terms in the right hand side of (3.57) converge to zero, so that we conclude from (3.57) that u_n is a Cauchy sequence in $C^0([0,T];L^1(\Omega))$ and strongly converges in $C^0([0,T];L^1(\Omega))$ towards a function u as n tends to infinity. This implies that we can pass to the limit as n tends to infinity in the left hand side of the entropy inequality (3.54) by simply using Lebesgue's Theorem. Moreover, since $0 \le \mathbb{1}_{u^n>l} \le 1$, the estimate (3.50) shows that the sequence of measures $(\mu_{n,l})_n$ is bounded independently of n. Therefore, there exists a subsequence still denoted by $(\mu_{n,l})_n$ and μ_l a bounded positive measure on $\overline{Q}$ such that $\mu_{n,l} \longrightarrow \mu_l$ in $\mathcal{C}(\overline{Q})'$ for the weak–$*$ topology. Then passing to the limit in (3.54) we obtain that u satisfies the renormalized entropy inequality (2.6). Finally, since (u^n) converges to u in $L^1(Q)$ then, in view of the estimate (3.50), we have

$$\begin{aligned}||\mu_l|| &\le \liminf_{n\to+\infty} ||\mu_{n,l}|| \\ &\le \int_\Omega (u_0 - l)^+\, dx + 2\int_0^T \int_{\partial\Omega} \overline{f}(\overline{u})\ \mathbb{1}_{\overline{u}>l}\, d\gamma\, dt + \int_0^T \int_\Omega \mathbb{1}_{u>l}\, |g|\, dx\, dt\,,\end{aligned}$$

which ensures $\lim\limits_{l\to+\infty} ||\mu_l|| = 0$. Clearly, reasoning as in Lemma 3.6 and Proposition 3.7, one also proves that u^n satisfies, for all $|k| \le l < n$,

$$\begin{aligned}\int_0^T \int_\Omega \big((u^n\top(-l) - \kappa)^- \varphi_t + \Phi^-(u^n\top(-l), \kappa) \cdot \nabla\varphi\big)\ dx\, dt + \int_\Omega (u_0^n\top(-l) - \kappa)^- \varphi(x,0)\, dx \\ + \mathrm{Lip}_l \int_0^T \int_{\partial\Omega} (\overline{u}^n\top(-l) - \kappa)^- \varphi(x,t)\, d\gamma(x)\, dt \\ + \int_0^T \int_\Omega g^n \operatorname{sgn}^-(u^n\top(-l) - \kappa)\, \varphi\, dx\, dt \ \ge - < \nu_{l,n}, \varphi >\,,\end{aligned}$$

where $\nu_{l,n}$ is a nonnegative bounded measure on $\overline{Q}$ whose mass is estimated as follows

$$||\nu_{l,n}|| \le \int_\Omega (u_0 + l)^-\, dx + 2\int_0^T \int_{\partial\Omega} \overline{f}(\overline{u})\ \mathbb{1}_{\overline{u}<-l}\, d\gamma\, dt + \int_0^T \int_\Omega \mathbb{1}_{u^n<-l}\, |g|\, dx\, dt\,.$$

Then passing to the limit we conclude that u also satisfies (2.7) with ν_l satisfying $\lim_{l\to+\infty} ||\nu_l|| = 0$. This concludes the proof of the existence of a renormalized entropy solution.

The uniqueness of the renormalized entropy solution of Problem (1.1) is a straightforward consequence of Theorem 3.1. ∎

3.4 Continuous dependence

Proof of Theorem 2.4 Let u be the unique renormalized solution of (1.1). We use (3.24) with $v = u^i$, $v_0 = u_0^i$, $\overline{v} = \overline{u}^i$, $\widetilde{g} = g^i$ so as to have

$$\int_\Omega |u^i - u|(t) \leq \int_0^T \int_{\partial\Omega} S(\overline{u}^i, \overline{u})\, d\gamma(x)\, dt + \int_0^T \int_\Omega |g^i - g|\, dx\, dt + \int_\Omega |u_0^i - u_0|\, dx\,, \tag{3.58}$$

for all $t \in [0,T]$, where, for $(t,x) \in \Sigma$

$$S(\overline{u}^i, \overline{u})(t,x) = \sup\left\{2\,|f(\sigma) - f(\tau)|\ ;\ \min(\overline{u}^i(t,x), \overline{u}(t,x)) \leq \sigma\,,\ \tau \leq \max(\overline{u}^i(t,x), \overline{u}(t,x))\right\}\,.$$

Since $\overline{u}^i$ almost everywhere converges to $\overline{u}$ on Σ we have that $S(\overline{u}^i, \overline{u})$ converges to zero almost everywhere on Σ. Moreover

$$|S(\overline{u}^i, \overline{u})| \leq 2(|\overline{f}(\overline{u}^i)| + |\overline{f}(\overline{u})|)\,.$$

Since the sequence $\overline{f}(\overline{u}^i)$ is strongly converging in $L^1(\Sigma)$ we deduce, from Vitali theorem, that $S(\overline{u}^i, \overline{u})$ strongly converges to zero in $L^1(\Sigma)$. Since (u_0^i) also converges to u_0 in $L^1(\Omega)$ and (g^i) converges to g in $L^1(Q)$, we can pass to the limit in (3.58) and deduce that the sequence (u^i) strongly converges to u in $C^0([0,T]; L^1(\Omega))$ (in particular in $L^1(Q)$). ∎

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