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Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains

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Summary. This paper is devoted to the study of the finite volume methods used in the discretization of conservation laws defined on bounded domains. General assumptions are made on the data: the initial condition and the boundary condition are supposed to be measurable bounded functions. Using a generalized notion of solution to the continuous problem (namely the notion of entropy process solution, see [9]) and a uniqueness result on this solution, we prove that the numerical solution converges to the entropy weak solution of the continuous problem in L^p_{loc} for every $p \in [1, +\infty)$. This also yields a new proof of the existence of an entropy weak solution.

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1 Introduction

1.1 The initial-boundary value problem

Let Ω be an open bounded polyhedral subset of $\mathbb{R}^d$. Let us denote by Γ its boundary, by n the unit normal to Γ outward to Ω, by γ the measure on Γ, by Q the set $Q =]0, +\infty[\times\Omega$ and by Σ the set $\Sigma =]0, +\infty[\times\Gamma$.

We consider the following scalar conservation law:

$$(1) \qquad u_t(t,x) + \mathrm{div}_x f(t,x,u(t,x)) = 0, \quad (t,x) \in Q\,,$$

with the initial condition

$$(2) \qquad u(0,x) = u_0(x), \quad x \in \Omega\,,$$

and the boundary condition

$$u(t,r) = u^b(t,r), \quad (t,r) \in \Sigma\,. \tag{3}$$

The way the boundary condition is satisfied has to be precised. Indeed, lest the problem (1)-(2)-(3) should be overdetermined, Equality (3) cannot be required to be assumed pointwise, even if the solution to (1) is a regular function (see [12] for a complete description of an intuitive approach to the nature of the boundary condition). Supposing that u_0 is BV and that u^b is $\mathcal{C}^2$-regular, Bardos, Le Roux and Nedelec [2] prove the existence and uniqueness of a solution to (1)-(2)-(3), explaining the way the boundary condition must be understood and detailing an inequality on the boundary now known as the BLN condition (see Remark 3).

Following the work of DiPerna [4], Szepessy defines a notion of measure-valued solution to (1) and, assuming the existence of a weak entropy solution to the problem, proves the uniqueness of the measure-valued solution. The existence of such a weak entropy solution is ensured by the work of Bardos, Le Roux and Nedelec. Notice that the "BLN condition " does make sense only if the solution u of (1)-(2)-(3) admits a trace on Σ. When handling the BLN condition we thus need the solution to be BV, which implies, in general, that the initial condition u_0 is BV and the minimum regularity required on the data is of BV type.

At any rate, the existence of a measure-valued solution is obtained through weak estimates on approximate solutions of the problem (1)-(2)-(3) and, under the hypotheses $u_0 \in BV(\Omega)$ and $u^b \in \mathcal{C}^2(\Sigma)$, this measure-valued solution gives rise to a weak entropy solution; this allows several authors to study the convergence of numerical schemes associated to the continuous problem. In [15], Szepessy proves the convergence of a streamline diffusion finite elements method; in [8], Cockburn, Coquel and Lefloch prove the convergence of the monotone finite volume method; in [3], Benharbit, Chalabi and Vila prove the convergence of a class of E-schemes.

We will use here a generalized notion of solution, similar to the one of measure-valued solution: the notion of entropy process solution introduced by Eymard, Gallouët and Herbin for the Cauchy Problem in [9]. The aim here is to adapt the method of [9] in order to obtain the same results as Eymard, Gallouët and Herbin in the case of the initial-boundary value problem. We deduce from a theorem of uniqueness (Theorem 2 in this paper) that an entropy process solution is actually a weak entropy solution. Let us highlight a difference between the way measure-valued solution and entropy process solution are handled: working in the framework of measure-valued solution, it is necessary to suppose the existence of a weak entropy solution in order to prove that any measure-valued solution is merely a weak entropy solution (see [4]), while this hypothesis is no longer required to prove that an entropy process solution is an entropy weak solution. This is why, here, existence and uniqueness of a solution is established for a flux function $f \in \mathcal{C}^1$ (or locally Lipchitz continuous, under an additionnal hypothesis, see Remark 1).

Moreover, we intend to deal only with essentially bounded measurable data. Consequently, a solution is sought in $L^\infty(\mathbb{R}_+ \times \Omega)$ and this functionnal context does not allow the definition of such a notion as the trace of the solution. In the L^∞ framework a notion of weak entropy solution has been given by F. Otto, who achieved this work in his PhD. Thesis so that little bibliography is available: a summary is presented in [13] and a more complete exposition appears in [12]. In this last reference, the existence of an entropy solution is established under the hypothesis $f \in \mathcal{C}^2$ and the uniqueness is proved under the hypothesis $f \in \mathcal{C}^1$. The work of Otto relies on the use of particular entropy-flux pairs, namely the boundary entropy-flux pairs. We give a similar definition of solution of the problem (1)-(2)-(3), but merely using the "semi Kruzkov entropies", as they already appear in the work of Carillo [5] and Serre [14] (see Sect. 2). These entropy functions admit very simple algebraic definition, so that the study of the discrete entropy inequalities satisfied by the numerical solution of the problem (1)-(2)-(3) defined by a monotone finite volume scheme is quite straightforward.

The discrete (and local) entropy inequalities satisfied by the numerical solution allows us to derive approximate continuous entropy inequalities. Notice that, in the course of the proof of this result, a "weak BV estimate" [9] on the numerical solution is needed. This weak BV estimate cannot yield any compactness property on a family of approximate numerical solution but is one of the key point of the proof of Theorem 3.

Notice also that monotone finite volume schemes are widely used in practical application. For example, in oil reservoir engineering, an IMPES scheme can be implemented to study the behaviour of the fluid in a column (see [1] or [10]) and, in this case, comes down to a monotone finite volume scheme.

1.2 Hypotheses and notations

We make the following hypotheses on the data and on the flux:

$$(4)\quad \begin{cases} (i) \quad u^b \in L^\infty(\Sigma) \text{ and } u_0 \in L^\infty(\Omega)\,, \\ (ii) \;\; f \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R}^d) \text{ and } \dfrac{\partial f}{\partial u} \\ \qquad\qquad \text{is locally Lipschitz continuous}\,, \\ (iii) \; \mathrm{div}_x\, f(t,x,u) = 0 \;\; for \;\; a.e. \;\; (t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}, \end{cases}$$

Remark 1 Assumption (iii) on f may be relaxed, and we can consider source terms in (1)-(2)-(3) (see [7]). Assumption (ii) on f may also be weakened, in particular it is enough to suppose that

$$(ii)_a \; f \in \mathrm{Lip}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R}^d)\,;$$

(in which case $\dfrac{\partial f}{\partial u}$ is defined a.e. on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$) provided that, for every compact $K_{t,x} \subset \mathbb{R}_+ \times \mathbb{R}$, for every compact $K_u \subset \mathbb{R}$, there exists $V_{K_{t,x},K_u} \geq 0$ such that

$$(ii)_b \begin{cases} \text{for a.e. } v \in K_u\,, \text{ for a.e. } (s,y) \in K_{t,x}\,, \text{ for a.e. } (\sigma,z) \in K_{t,x}\,, \\ \left|\dfrac{\partial f}{\partial u}(s,y,v) - \dfrac{\partial f}{\partial u}(\sigma,z,v)\right| \leq V_{K_{t,x},K_u}\left(|s-\sigma| + |y-z|\right). \end{cases}$$

Notice that conditions $(ii)_a$ and $(ii)_b$ are fulfilled if the function f can be written as

$$f(t,x,u) = \mathbf{v}(t,x)\, g(u)$$

with $\mathbf{v} \in \mathrm{Lip}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d\,;\, \mathbb{R}^d)$ and $g \in W^{1,\infty}_{loc}(\mathbb{R})$.

Notations: We denote by B and A the quantities

(5) $$B = \max(\operatorname*{ess\,sup}_{\Omega}(u_0)\,, \operatorname*{ess\,sup}_{\Sigma}(u^b))\,,$$

and

(6) $$A = \min(\operatorname*{ess\,inf}_{\Omega}(u_0)\,, \operatorname*{ess\,inf}_{\Sigma}(u^b))\,.$$

Thanks to assumption (4) on f, it is known that, for every $T > 0$, f is Lipschitz continuous on $[0,T] \times \Omega \times [A,B]$. Our work requires f to be Lipschitz continuous but, instead of fixing $T > 0$, then working on the set $[0,T] \times \Omega \times [A,B]$, and, at last, extending the solutions obtained (with the help of a theorem of uniqueness), we already suppose f to be Lipschitz continuous on $\mathbb{R}_+ \times \Omega \times [A,B]$. We set $Lip(f)$ to be its Lipschitz constant.

1.3 Main results

In Sect. 2, we emphasize the definition of weak entropy solution; the class of entropy-flux pairs considered in the definition of weak entropy solution can be reduced to the one of the so-called “semi Kruzkov” entropies. It is one of the keys of the result of convergence of the scheme. As in [9] and [6], a notion of entropy process solution is defined.

In Sect. 3, we develop the proof of a uniqueness result (that is Theorem 2). This theorem allows us to show that an entropy process solution of the problem (1)-(2)-(3) is necessarily a weak entropy solution. It also ensures the uniqueness of the weak entropy solution. Notice that, in the course of the proof of this theorem, it is not necessary to suppose the existence of a weak entropy solution.

In Sect. 4, we define the finite volume scheme with monotone fluxes associated to the problem (1)-(2)-(3) and the corresponding numerical solution $u_{\mathcal{T},k}$. We prove that $(u_{\mathcal{T},k})$ converges towards an entropy process solution of problem (1)-(2)-(3). This yields, thanks to Theorem 2, the result of existence of a weak entropy solution of the problem (1)-(2)-(3). Then it is proved that $(u_{\mathcal{T},k})$ converges to the weak entropy solution in $L^p_{loc}(\mathbb{R}_+ \times \Omega)$ for every $p \in [\,1, +\infty[$.

2 Weak entropy solution

It is well-known that the concept of weak solution is not accurate in the study of hyperbolic problems, for uniqueness of such a solution may fail, even if the data are regular functions. Thus, we turn to the notion of weak entropy solution.

Notations: Let sgn^+ denote the application $\mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\mathrm{sgn}^+(s) = \begin{cases} 1 \ \text{ if } s > 0\,, \\ 0 \ \text{ if } s \leq 0\,, \end{cases}$$

and sgn^- the application $s \longmapsto -\mathrm{sgn}^+(-s)$. As usual, we set $s^+ = \mathrm{sgn}^+(s)\, s$ and $s^- = (-s)^+$.

Let $\kappa \in [\,A, B]$. The entropy-flux pair $(\eta^+_\kappa, \Phi^+_\kappa)$ (respectively $(\eta^-_\kappa, \Phi^-_\kappa)$) is defined by

$$\begin{cases} \eta^+_\kappa(s) = (s-\kappa)^+\,, \\ \Phi^+_\kappa(t,x,s) = \mathrm{sgn}^+(s-\kappa)(f(t,x,s) - f(t,x,\kappa))\,, \end{cases} \tag{7}$$

$$\left(\text{respectively } \begin{cases} \eta^-_\kappa(s) = (s-\kappa)^-\,, \\ \Phi^-_\kappa(t,x,s) = \mathrm{sgn}^-(s-\kappa)(f(t,x,s) - f(t,x,\kappa)) \end{cases}\right). \tag{8}$$

Definition 1 *Let u be in $L^\infty(Q)$. The function u is said to be a weak entropy solution of the problem (1)-(2)-(3) if it satisfies the following entropy inequalities: for all $\kappa \in [\,A, B]$, for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d,\ \mathbb{R}_+)$,*

$$\begin{aligned} &\iint_Q \left(\eta^+_\kappa(u)\,\varphi_t \,+\, \Phi^+_\kappa(t,x,u)\cdot\nabla\varphi\right)\,dxdt \\ &+\int_\Omega \eta^+_\kappa(u_0)\,\varphi(0,x)\,dx \\ &+Lip(f)\,\iint_\Sigma \eta^+_\kappa(u^b)\,\varphi(t,r)\,d\gamma(r)dt \geq 0\,, \end{aligned} \tag{9}$$

and

$$
\begin{aligned}
&\iint_Q \left(\eta_\kappa^-(u)\,\varphi_t \,+\, \Phi_\kappa^-(t,x,u)\cdot\nabla\varphi\right)\,dxdt \\
&\quad + \int_\Omega \eta_\kappa^-(u_0)\,\varphi(0,x)\,dx \\
&\quad + Lip(f)\,\iint_\Sigma \eta_\kappa^-(u^b)\,\varphi(t,r)\,d\gamma(r)dt \geq 0\,.
\end{aligned}
\tag{10}
$$

The semi Kruzkov entropies have rather simple algebraic expressions that allows the study of the numerical problem associated to (1)-(2)-(3), while, working with “boundary entropy-flux pairs”, this study may be much more difficult. The boundary entropy-flux pairs are the entropy-flux pairs used by Otto to define the notion of weak entropy solution. They are defined in the following way:

Definition 2 *Let (H,Q) be in $\mathcal{C}^2(\mathbb{R}^2)\times(\mathcal{C}^2(\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^2))^d$. The pair (H,Q) is said to be a* boundary entropy-flux pair *(for the flux f) if:*

1. *for all $w\in\mathbb{R}$, $s\mapsto H(s,w)$ is a convex function,*
2. $\forall w\in\mathbb{R}\,,\ \partial_s Q(t,x,s,w)=\partial_s H(s,w)\dfrac{\partial f}{\partial s}(t,x,s)$*,*
3. $\forall w\in\mathbb{R}, H(w,w)=0\,,\ Q(.,.,w,w)=0\,,\ \partial_s H(w,w)=0$*.*

Thanks to the following lemma, Definition 1 of weak entropy solution gives rise to exactly the same notion of solution as defined by Otto.

Lemma 1 *Let $\eta\in\mathcal{C}^1(\mathbb{R},\mathbb{R})$ be a convex function such that: there exists $w\in[A,B]$ with $\eta(w)=0$ and $\eta'(w)=0$. Then η can be uniformly approximated on $[A,B]$ by applications of the kind*

$$
s\longmapsto\sum_{1,p}\alpha_i(s-\kappa_i)^- + \sum_{1,q}\beta_j(s-\widetilde{\kappa_j})^+
$$

where $\alpha_i\geq 0$, $\beta_j\geq 0$, $\kappa_i\in[A,B]$ and $\widetilde{\kappa_j}\in[A,B]$.

We conclude this section by making some comments on weak entropy solution.

Remark 2 (see [12]) If $u\in L^\infty(Q)$ is a weak entropy solution of the problem (1)-(2)-(3) then: for almost every $(t,x)\in Q$,

$$
A\leq u(t,x)\leq B\,.
$$

Remark 3 If $u\in L^\infty(Q)$ is a weak entropy solution of the problem (1)-(2)-(3) then u satisfies (see [12]): for all classical entropy-flux pair (η,Φ), for all $\varphi\in\mathcal{C}_c^\infty((0,+\infty)\times\Omega,\ \mathbb{R}_+)$,

$$
\iint_Q \eta(u)\,\varphi_t \,+\, \Phi(t,x,u)\cdot\nabla\varphi \geq 0 \tag{11}
$$

and

$$\text{(12)} \qquad \operatorname*{ess\,lim}_{t\to 0^+} \int_\Omega |u(t) - u_0|\, dx = 0;$$

moreover, the boundary condition is fulfilled in the following way: for all boundary entropy-flux pair (H, Q), for all $\beta \in L^1(\Sigma)$ such that $\beta \geq 0$ $a.e.$,

$$\text{(13)} \qquad \begin{aligned} &\operatorname*{ess\,lim}_{s\to 0^+} \int_0^{\mathrm{T}} \int_\Gamma Q(t, r, u(t, r - sn(r)), u^b(t, r)) \\ &\quad \cdot n(r)\, \beta(t, r)\, d\gamma(r) dt \geq 0. \end{aligned}$$

Reciprocally, if $u \in L^\infty(Q)$, with $A \leq u \leq B$ a.e., and u satisfies (11), (12), (13), then u is a weak entropy solution of the problem (1)-(2)-(3).

Besides, if $u \in L^\infty(Q)$ is a weak entropy solution of the problem (1)-(2)-(3) that admits a trace, meaning there exists u^τ in $L^\infty(\Sigma)$ such that

$$\operatorname*{ess\,lim}_{s\to 0^+} \int_\Sigma |u(t, r - sn(r)) - u^\tau(t, r)|\, d\gamma(r) dt = 0\,,$$

then (13) is equivalent to the equation

$$Q(u^\tau, u^b) \cdot n \geq 0 \text{ a.e. on } \Sigma\,.$$

Choosing $Q(s, w) = \Phi^+(s, \max(w, k)) + \Phi^-(s, \min(w, k))$ yields the BLN condition ([2]), that is:

for a.e. $(t, r) \in \Sigma$, $\forall k \in [\, u^\tau(t, r), u^b(t, r)]$,

$$sgn(u^\tau(t, r) - u^b(t, r))\, (f(u^\tau(t, r) - f(k)) \cdot n(r) \geq 0\,.$$

Notice that, in the case where $\Omega = \mathbb{R}^d$, it is well-known that the class of Kruzkov entropies is wide enough to ensure the uniqueness of the solution. It is the same here, except that we have to consider the semi Kruzkov entropies and that working with the mere Kruzkov entropies would not be sufficient, for uniqueness would be lacking. Indeed, the classical Kruzkov entropy-flux pairs are defined by:

$$\text{(14)} \qquad \begin{aligned} &\eta_\kappa(s) = |s - \kappa|\,, \\ &\Phi_\kappa(t, x, s) = \operatorname{sgn}(s - \kappa)(f(t, x, s) - f(t, x, \kappa))\,. \end{aligned}$$

Now, suppose that $\Omega =]\,0, +\infty[$ and define the flux-function $f : [\,0, 1] \longmapsto \mathbb{R}$ by

$$f(u) = u(1 - u)\,;$$

then consider the solution u of the Riemann problem on $\mathbb{R}$ associated to the equation $u_t + (f(u))_x = 0$ and to the datum (u_-, u_+). Let $u_0 = u_+$ and u^b

be constant. Then $u \in L^\infty(Q)$ and satisfies: for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R},\ \mathbb{R}_+)$, for all $\kappa \in [\,A, B]$,

$$\iint_Q \left(|u-\kappa|\,\varphi_t \,+\, \mathrm{sgn}(u-\kappa)(f(u)-f(\kappa))\varphi_x\right)\,dxdt \\ +\int_{\mathbb{R}_+} |u_0-\kappa|\,\varphi(0,x)\,dx \,+ Lip(f)\int_{\mathbb{R}_+} |u^b-\kappa|\,\varphi(t,0)\,dt \geq 0\,,$$

if, and only if, for all $\kappa \in [\,A, B]$, for all $t > 0$,

$$-\mathrm{sgn}(u(t,0+)-\kappa)(f(u(t,0+)-f(\kappa)) + Lip(f)\,|u^b-\kappa| \geq 0\,. \tag{15}$$

Now, choosing $u_0 = u_+ = 0$ and $u^b = 1$, the data $u_-^1 = 1/4$ and $u_-^2 = 1/2$ define, through the Riemann problem, two distinct measurable bounded functions which both satisfy (15).

2.1 Entropy process solution

The proof of the existence of a weak entropy solution to the problem (1)-(2)-(3) lies in the study of the numerical solution $u_{\mathcal{T},k}$ defined by the finite volume scheme associated to (1)-(2)-(3). Here $\mathcal{T}$ denotes the mesh, h its "size" and k the time step (see Sect. 4). Theorem 3 states that the numerical solution satisfies the following approximate entropy inequalities:

$$\begin{cases} \forall\kappa \in [\,A,B]\,,\ \forall\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)\,, \\ \displaystyle\iint_Q \Big(\eta_\kappa^+(u_{\mathcal{T},k})\varphi_t + \Phi_\kappa^+(t,x,u_{\mathcal{T},k})\cdot\nabla\varphi\Big)dx\,dt \\ \quad \displaystyle +\int_\Omega \eta_\kappa^+(u_0)\varphi(0)dx \\ \displaystyle +\int_\Sigma \eta_\kappa^+(u^b)\varphi(t,x)d\gamma(x)\,dt \geq -\varepsilon_{\mathcal{T},k}(\varphi)\,, \end{cases} \tag{16}$$

where

$$\forall\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)\,,\ \varepsilon_{\mathcal{T},k}(\varphi) \longrightarrow 0 \text{ when } h \to 0.$$

The same result holds when the entropy-flux pair $(\eta_\kappa^-, \Phi_\kappa^-)$ is considered.

The numerical approximate solution $(u_{\mathcal{T},k})$ is also known to be bounded in $L^\infty(Q)$ but it is not enough to pass to the limit in Inequation (16). Thus, owing to the non-linearity of the equation and to the lack of estimate on the approximate solution, we have to turn to the notion of measure-valued solution (see DiPerna, [4], Szepessy, [15]) or, equivalently, to the notion of entropy process solution defined by Eymard, Gallouët, Herbin in [9]. The interest of this notion lies in the following result, which generalizes the notion of weak-$\star$ convergence in L^∞ and free oneself from the problems of non-linearity.

Theorem 1 *Let $\mathcal{O}$ be a borelian subset of $\mathbb{R}^m$, let R be positive and (u^n) be a sequence of $L^\infty(\mathcal{O})$ such that, for all $n \in \mathbb{N}$, $||u^n||_{L^\infty} \leq R$. Then there exists a sub-sequence still denoted by (u^n) and $\mu \in L^\infty(\mathcal{O} \times (0,1))$ such that:*

$$\forall g \in \mathcal{C}(\mathbb{R})\,,\ g(u^n) \longrightarrow \int_0^1 g(\mu(.,\alpha))\,d\alpha \ \textit{ in } L^\infty(\mathcal{O}) \ \textit{weak-}\star\ .$$

Now the notion of entropy process solution can be defined.

Definition 3 *Let μ be in $L^\infty(Q \times (0,1))$. The function μ is said to be an* entropy process solution *to (1)-(2)-(3) if:*

1. *for a.e. $(t,x,\alpha) \in Q \times (0,1)$, $A \leq \mu(t,x,\alpha) \leq B$,*
2. *for all $\kappa \in [A,B]$, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\,,\ \varphi \geq 0$,*

$$\begin{aligned}\iint_Q \int_0^1 \Big[\eta_\kappa^+(\mu(t,x,\alpha))\,\varphi_t(t,x) \,+\, \Phi_\kappa^+(t,x,\mu(t,x,\alpha)) \cdot \nabla\varphi(t,x)\Big] \\ \times d\alpha\,dx\,dt + \int_\Omega \eta_\kappa^+(u_0)\,\varphi(0,x)\,dx \\ +Lip(f) \iint_\Sigma \eta_\kappa^+(u^b)\,\varphi(t,x)\,d\gamma(x)\,dt \geq 0\,,\end{aligned} \tag{17}$$

3. *the same entropy inequality holds when $(\eta_\kappa^-, \Phi_\kappa^-)$ is selected as an entropy-flux pair.*

Notice that if μ is an entropy process solution of the the problem (1)-(2)-(3) and if μ does not depend on its last variable, that is to say: there exists $u \in L^\infty(Q)$ such that

$$\text{for a.e. } (t,x,\alpha) \in Q \times (0,1)\,,\ \mu(t,x,\alpha) \,=\, u(t,x),$$

then u is a weak entropy solution to (1)-(2)-(3).

We will now prove that if $\mu \in L^\infty(Q \times (0,1))$ is an entropy process solution then, in fact, μ does not depend on its last variable and that the weak entropy solution is unique.

3 Uniqueness of the entropy process solution

Theorem 2 ("uniqueness" of the entropy process solution) *Let $\mu\,,\ \nu \in L^\infty(Q \times (0,1))$ be two entropy process solutions. Then there exists $u \in L^\infty(Q)$ such that:*

$$\mu(t,x,\alpha) \,=\, u(t,x) \,=\, \nu(t,x,\beta) \ \ for\ \ a.e.\ (t,x,\alpha,\beta) \in Q \times (0,1)^2\,.$$

Corollary 1 *The problem (1)-(2)-(3) admits at most one weak entropy solution.*

Let us first prove some lemmas that will entail Theorem (2). In order to clarify certain forthcoming expressions, the following notations will be used: for $(t,x)\in Q$, for s and $\kappa\in\mathbb{R}$,

$$\begin{cases} \Phi(t,x,s,\kappa) \;\; = \Phi_\kappa(t,x,s) \; = \operatorname{sgn}(s-\kappa)(f(t,x,s)-f(t,x,\kappa)\,, \\ \Phi^+(t,x,s,\kappa) = \Phi^+_\kappa(t,x,s) = \operatorname{sgn}^+(s-\kappa)(f(t,x,s)-f(t,x,\kappa)\,, \\ \Phi^-(t,x,s,\kappa) = \Phi^-_\kappa(t,x,s) = \operatorname{sgn}^-(s-\kappa)(f(t,x,s)-f(t,x,\kappa)\,. \end{cases}$$

Notice that $\Phi=(\Phi_1,\dots,\Phi_d)$ takes its values in $\mathbb{R}^d$.

Lemma 2 *Let $\mu\,,\ \nu\in L^\infty(Q\times(\,0,1))$ be two entropy process solutions. Then:*

$$(18)\quad \begin{cases} \forall\psi\in\mathcal{C}^\infty_c(\mathbb{R}_+\times\mathbb{R}^d_+)\,,\ \psi\geq 0, \\ \displaystyle\iint_Q\int_0^1\int_0^1\Big[|\mu(t,x,\alpha)-\nu(t,x,\beta)|\,\psi_t \\ \quad +\Phi(t,x,\mu(t,x,\alpha),\nu(t,x,\beta))\cdot\nabla\psi\Big]\,d\beta\,d\alpha\,dx\,dt\;\geq 0\,. \end{cases}$$

The set Ω was supposed to be an open polyhedral subset of $\mathbb{R}^d$. Notice that the following proof would still be correct if Ω were an open set with $\mathcal{C}^1$ boundary. Indeed, working locally (thanks to local maps covering Ω), we can suppose $\Omega=\mathbb{R}^d$ or $\Omega=\mathbb{R}^d_+$. What really requires care in the proof of Lemma 2 is the study of the behaviour of an entropy process solution near the boundary, so that we already suppose

$$\Omega=\mathbb{R}^d_+=\{x=(\overline{x},x_d)\in\mathbb{R}^d\,,x_d>0\}$$

and detail the following lemmas.

Lemma 3 *Let $a\top b$ denotes the maximum value between two reals a and b and $a\bot b$ denotes their minimum value. Let μ be an entropy process solution to (1)-(2)-(3) and κ be in $[\,A,B]$. Then:*

1. *there exists $\theta^+_{\mu,\kappa}\in L^\infty(\Sigma)$ such that: for all $\beta\in L^1(\Sigma)$,*

$$-\operatorname*{ess\,lim}_{x_d\to 0^+}\iint_\Sigma\int_0^1\Phi^+_d(t,x,\mu(t,x,\alpha),u^b(t,\overline{x})\top\kappa)\,\beta(t,\overline{x})\,d\alpha\,d\overline{x}\,dt \\ =\iint_\Sigma\theta^+_{\mu,\kappa}(t,\overline{x})\,\beta(t,\overline{x})\,d\overline{x}\,dt\,,$$

and $\theta^+_{\mu,\kappa}\geq 0$ a.e.

2. *there exists* $\theta^-_{\mu,\kappa} \in L^\infty(\Sigma)$ *such that: for all* $\beta \in L^1(\Sigma)$,

$$-\operatorname*{ess\,lim}_{x_d \to 0^+} \iint_\Sigma \int_0^1 \Phi_d^-(t,x,\mu(t,x,\alpha),u^b(t,\overline{x})\bot\kappa)\,\beta(t,\overline{x})\,d\alpha\,d\overline{x}\,dt = \iint_\Sigma \theta^-_{\mu,\kappa}(t,\overline{x})\,\beta(t,\overline{x})\,d\overline{x}\,dt\,,$$

and $\theta^-_{\mu,\kappa} \ge 0$ *a.e.*

3. *there exists* $\theta_\mu \in L^\infty(\Sigma)$ *such that: for all* $\beta \in L^1(\Sigma)$,

$$-\operatorname*{ess\,lim}_{x_d \to 0^+} \iint_\Sigma \int_0^1 \Phi_d(t,x,\mu(t,x,\alpha),u^b(t,\overline{x}))\,\beta(t,\overline{x})\,d\alpha\,d\overline{x}\,dt = \iint_\Sigma \theta_\mu(t,\overline{x})\,\beta(t,\overline{x})\,d\overline{x}\,dt\,.$$

Lemma 4 *Let* μ *be an entropy process solution to (1)-(2)-(3) and* κ *be in* $[A,B]$. *Then the following inequality holds: for all* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, $\varphi \ge 0$,

$$\iint_Q \int_0^1 \Big[|\mu(t,x,\alpha)-\kappa|\,\varphi_t(t,x) + \Phi(t,x,\mu(t,x,\alpha),\kappa)\cdot\nabla\varphi(t,x)\Big] \times d\alpha\,dx\,dt + \int_\Omega |u_0-\kappa|\,\varphi(0,x)\,dx + \iint_\Sigma \theta_\mu(t,\overline{x})\,\varphi(t,\overline{x},0) \times d\overline{x}\,dt + \iint_\Sigma \Phi_d(t,\overline{x},0,u^b(t,\overline{x}),\kappa)\,\varphi(t,\overline{x},0)\,d\overline{x}\,dt \ge 0\,. \tag{19}$$

Proof of Lemma 3 (see [12]): Let β be a function of $\mathcal{C}_c^\infty(]\,0,+\infty[\times\mathbb{R}^{d-1})$, $\beta \ge 0$, and define the functions $g^+_{\kappa,w,\beta}$, $h^+_{\kappa,w,\beta}$ (for $w \in [\,A,B]$) by:

$$g^+_{\kappa,w,\beta}(x_d) = -\iint_\Sigma \int_0^1 \Phi_d^+(t,x,\mu(t,x,\alpha),w\top\kappa)\,\beta(t,\overline{x})\,d\alpha\,d\overline{x}\,dt\,,$$

$$h^+_{\kappa,w,\beta}(x_d) = \iint_\Sigma \int_0^1 (\mu(t,x,\alpha)-\kappa\top w)^+\,\beta_t(t,\overline{x})\,d\alpha\,d\overline{x}\,dt + \sum_{i=1,d-1} \times \iint_\Sigma \int_0^1 \Phi_i^+(t,x,\mu(t,x,\alpha),\kappa\top w)\,\beta_{x_i}(t,\overline{x})\,d\alpha\,d\overline{x}\,dt\,.$$

Putting $\varphi = \beta\gamma$ in Inequation (17) where κ has been replaced by $\kappa\top w$ and $\gamma \in \mathcal{C}^\infty([\,0,+\infty[\,,\mathbb{R}_+)$, we get, if $\gamma \in \mathcal{C}_c^\infty(0,+\infty)$

$$h^+_{\kappa,w,\beta} - (g^+_{\kappa,w,\beta}(x_d))' \ge 0 \ \text{ in } \mathcal{D}'(]0,+\infty[) \tag{20}$$

and, if $\gamma(x_d) = \chi_{(0,\varepsilon)}(x_d)\left(1 - \dfrac{x_d}{\varepsilon}\right)$,

$$\frac{1}{\varepsilon}\int_0^\varepsilon g^+_{\kappa,w,\beta}(x_d)\,dx_d \geq -Lip(f)\iint_\Sigma (u^b(t,\overline{x}) - \kappa\top w)^+ \times\beta\,d\overline{x}\,dt + \mathcal{O}(\varepsilon)\,. \tag{21}$$

Considering that $h^+_{\kappa,w,\beta} \in L^1(0,+\infty)$, that $g^+_{\kappa,w,\beta} \in L^\infty(0,+\infty)$ and the inequality (20), we get $g^+_{\kappa,w,\beta} \in L^\infty \cap BV(0,1)$. Thus, $\operatorname*{ess\,lim}_{x_d\to 0^+} g^+_{\kappa,w,\beta}(x_d)$ exists and, by letting ε go to zero in the inequality (21), we get:

$$\operatorname*{ess\,lim}_{x_d\to 0^+} g^+_{\kappa,w,\beta}(x_d) \geq -Lip(f)\iint_\Sigma (u^b(t,\overline{x}) - \kappa\top w)^+\,\beta\,d\overline{x}\,dt\,. \tag{22}$$

Using the continuous dependency of $g^+_{\kappa,w,\beta}$ on $\beta \in L^1(\Sigma)$ and the density of $\mathcal{C}^\infty_c(\Sigma)$ in $L^1(\Sigma)$, we deduce: for all $\beta \in L^1(\Sigma)$, $\beta \geq 0$, $\operatorname*{ess\,lim}_{x_d\to 0^+} g^+_{\kappa,w,\beta}(x_d)$ exists and (22) still holds. Then, approaching u^b in $L^\infty(\Sigma)$ by simple functions u^b_ε, each of them taking a finite number of values w_i in $\mathbb{Q}$, say:

$$u^b_\varepsilon = \sum_{i=1}^p w_i\,\chi_{A_i}\,, \quad ((A_i)_i \text{ pairwise disjoints})$$

and taking $w = w_i$, $\chi_{A_i}\,\beta$ instead of β in (22), then summing with respect to $i \in \{1,\dots,p\}$ and, at last, letting ε go to zero yields the first point of Lemma 3. The same lines would be followed to prove the second point, or to prove the third point (by taking $\kappa = w$ at the beginning and by using the formula $(s-w)^+ + (s-w)^- = |s-w|$).

Proof of Lemma 4: for ε a positive number define the function ω_ε by

$$\omega_\varepsilon(x_d) = \begin{cases} x_d/\varepsilon \text{ if } 0 \leq x_d \leq \varepsilon \\ 1 \text{ if } \varepsilon \leq x_d \end{cases}.$$

Let $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d)$, $\varphi \geq 0$ and $\kappa \in [A,B]$. As the function μ is an entropy process solution to (1)-(2)-(3), it can easily be shown that it satisfies the inequality:

$$\iint_Q \int_0^1 \Big[|\mu(t,x,\alpha) - \kappa|\,\omega_\varepsilon(x_d)\,\varphi_t(t,x) + \Phi(t,x,\mu(t,x,\alpha),\kappa)$$
$$\cdot\nabla\varphi(t,x)\,\omega_\varepsilon(x_d)\Big]\,d\alpha\,dx\,dt + \int_\Omega |u_0 - \kappa|\,\varphi(0,x)\,\omega_\varepsilon(x_d)\,dx$$
$$+\frac{1}{\varepsilon}\int_0^\varepsilon \iint_\Sigma \int_0^1 \Phi_d(t,\overline{x},x_d,\mu(t,x,\alpha),\kappa)\,\varphi(t,\overline{x},x_d)\,d\alpha\,d\overline{x}\,dt\,dx_d \geq 0,$$

and, by letting ε go to zero:

$$\iint_Q \int_0^1 \Big[|\mu(t,x,\alpha)-\kappa|\,\varphi_t(t,x) \,+\, \Phi(t,x,\mu(t,x,\alpha),\kappa)\cdot\nabla\varphi(t,x)\Big]$$
$$\times d\alpha\,dx\,dt + \int_\Omega |u_0-\kappa|\,\varphi(0,x)\,dx + \lim_{x_d\to 0^+}\operatorname{ess\,sup}$$
$$\times \iint_\Sigma \int_0^1 \Phi_d(t,\overline{x},x_d,\mu(t,x,\alpha),\kappa)\,\varphi(t,\overline{x},x_d)\,d\alpha\,d\overline{x}\,dt \,\geq 0\,.$$

Moreover, using the formula

$$\Phi(t,x,s,\kappa) = 2\Big[\Phi^+(t,x,s,\kappa\top u^b(t,\overline{x})) + \Phi^-(t,x,s,\kappa\bot u^b(t,\overline{x}))\Big]$$
$$+\Phi(t,x,\kappa,u^b(t,\overline{x})) - \Phi(t,x,s,u^b(t,\overline{x}))\,,$$

we deduce from Lemma 3:

$$\lim_{x_d\to 0^+}\operatorname{ess\,sup}\iint_\Sigma\int_0^1 \Phi_d(t,\overline{x},x_d,\mu(t,x,\alpha),\kappa)\,\varphi(t,\overline{x},x_d)\,d\alpha\,d\overline{x}\,dt$$
$$\leq \iint_\Sigma \theta_\mu(t,\overline{x})\,\varphi(t,\overline{x},0)\,d\overline{x}\,dt$$
$$+\iint_\Sigma \Phi_d(t,\overline{x},0,u^b(t,\overline{x}),\kappa)\,\varphi(t,\overline{x},0)\,d\overline{x}\,dt\,,$$

which proves the inequality (19).

3.1 Proof of Lemma 2

Working on the entropy inequality (19), the doubling variable technique of Kruzkov (see [11]) is efficient. Let us detail it: let ρ be a function of $\mathcal{C}_c^\infty(]-1,0[,\mathbb{R}_+)$ such that $\displaystyle\int_{-1}^0 \rho(t)\,dt = 1$ (notice that ρ has a compact support located to the left of zero). Classically, a sequence of mollifiers (ρ_ε) on $\mathbb{R}$ can be defined by the formula

$$\rho_\varepsilon(t) \,=\, \frac{1}{\varepsilon}\rho\left(\frac{t}{\varepsilon}\right) \;,\varepsilon>0\,,$$

and a sequence of mollifiers $(\widetilde{\rho_\varepsilon})$ on $\mathbb{R}^q$ ($q\geq 1$) can be defined by the formula

$$\widetilde{\rho_\varepsilon}(x) \,=\, \rho_\varepsilon(x_1)\times\cdots\times\rho_\varepsilon(x_q)\,,\; x\in\mathbb{R}^q\,.$$

We also define R_ε by:

$$R_\varepsilon\,:\,t\longmapsto\int_{-\infty}^{-t}\rho_\varepsilon(s)\,ds\,.$$

Let ψ be in $\mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, $\psi \geq 0$ and define φ by

$$\varphi(t,x,s,y) = \psi(t,x)\rho_\varepsilon(t-s)\widetilde{\rho_\varepsilon}(x-y)\,.$$

We apply inequality (19) with $\kappa = \nu(s,y,\beta)$, $(t,x) \mapsto \varphi(t,x,s,y)$ as a test function and integrate w.r.t. (s,y,β). On the other hand, the function ν satisfies the inequation

$$\begin{aligned}
&\iint_Q \int_0^1 |\nu(s,y,\beta)-\kappa|\,\varphi_s(s,y) \;+\; \Phi(s,y,\nu(s,y,\beta),\kappa)\\
&\quad \cdot\nabla\varphi(s,y)\,d\beta\,dy\,ds + \int_\Omega \eta_\kappa^+(u_0)\,\varphi(0,y)\,dx\\
&\quad +Lip(f)\,\iint_\Sigma |u^b-\kappa|\,\varphi(s,\overline{y},0)\,d\overline{y}\,ds \geq 0\,;
\end{aligned} \tag{23}$$

In (23), we set $\kappa = \mu(t,x,\alpha)$, choose $(s,y) \mapsto \varphi(t,x,s,y)$ as a test function (notice that $\varphi(t,x,0,y) = \varphi(t,x,s,\overline{y},0) = 0$) and integrate w.r.t. (t,x,α). Summing the two inequalities thus obtained yields the following result:

$$A_{(\psi_t)} + A_{(\psi_x)} + A_{(\widetilde{\rho}_x)} + A_0 + A_1^b + A_2^b \geq 0, \tag{24}$$

where:

$$\begin{aligned}
A_{(\psi_t)} &= \iint_Q \int_0^1 \iint_Q \int_0^1 |\mu(t,x,\alpha)-\nu(s,y,\beta)|\,\psi_t(t,x)\\
&\quad \times\widetilde{\rho_\varepsilon}(x-y)\,\rho_\varepsilon(t-s)\,d\beta\,dy\,ds\,d\alpha\,dx\,dt,\\
A_{(\psi_x)} &= \iint_Q \int_0^1 \iint_Q \int_0^1 \Phi(t,x,\mu(t,x,\alpha),\nu(s,y,\beta))\cdot\nabla\psi(t,x)\\
&\quad \times\widetilde{\rho_\varepsilon}(x-y)\,\rho_\varepsilon(t-s)\,d\beta\,dy\,ds\,d\alpha\,dx\,dt,\\
A_{(\widetilde{\rho}_x)} &= \iint_Q \int_0^1 \iint_Q \int_0^1 [\Phi(t,x,\mu(t,x,\alpha),\nu(s,y,\beta))\\
&\quad -\Phi(s,y,\mu(t,x,\alpha),\nu(s,y,\beta))]\cdot\nabla\widetilde{\rho_\varepsilon}(x-y)\\
&\quad \times\psi(t,x)\,\rho_\varepsilon(t-s)\,d\beta\,dy\,ds\,d\alpha\,dx\,dt,\\
A_0 &= \iint_Q \int_0^1 \int_\Omega |u_0(x)-\nu(s,y,\beta)|\,\psi(0,x)\\
&\quad \times\widetilde{\rho_\varepsilon}(x-y)\,\rho_\varepsilon(-s)\,d\beta\,dy\,ds\,dx,\\
A_1^b &= \iint_Q \int_0^1 \iint_\Sigma \theta_\mu(t,\overline{x})\,\psi(t,\overline{x},0)\,\widetilde{\rho_\varepsilon}(\overline{x}-\overline{y})\\
&\quad \times\rho_\varepsilon(-y_d)\,\rho_\varepsilon(t-s)\,d\overline{x}\,dt\,d\beta\,dy\,ds\,,\\
A_2^b &= \iint_Q \int_0^1 \iint_\Sigma \Phi_d(t,\overline{x},0,u^b(t,\overline{x}),\nu(s,y,\beta))\,\psi(t,\overline{x},0)\\
&\quad \times\widetilde{\rho_\varepsilon}(\overline{x}-\overline{y})\,\rho_\varepsilon(-y_d)\,\rho_\varepsilon(t-s)\,d\overline{x}\,dt\,dy\,d\beta\,ds\,.
\end{aligned}$$

Now, we study the behaviour of each of those terms as ε goes to zero.

Terms $A_{(\psi_t)}$ *and* $A_{(\psi_x)}$. From the theorem of continuity in means is deduced the convergence of $(A_{(\psi_t)} + A_{(\psi_x)})$ to the right-hand side of the inequality (18) of Lemma 2, that is to say:

$$A_{(\psi_t)} \longrightarrow A^{\infty}_{(\psi_t)} \text{ and } A_{(\psi_x)} \longrightarrow A^{\infty}_{(\psi_x)} ,$$

where:

$$\begin{cases} A^{\infty}_{(\psi_t)} = \iint_Q \int_0^1 \int_0^1 |\mu(t,x,\alpha) - \nu(t,x,\beta)|\, \psi_t \, d\beta \, d\alpha \, dx \, dt \,, \\ A^{\infty}_{(\psi_x)} = \iint_Q \int_0^1 \int_0^1 \Phi(t,x,\mu(t,x,\alpha),\nu(t,x,\beta)) \cdot \nabla\psi \, d\beta \, d\alpha \, dx \, dt \,. \end{cases}$$

Term $A_{(\widetilde{\rho}_x)}$. Notice that, if f does not depend on (t,x), then $A_{(\widetilde{\rho}_x)} = 0$. Actually, using the fact that $\mathrm{div}_x f(t,x,s) = 0$ and the local Lipschitz continuity of $\dfrac{\partial f}{\partial s}$, we prove (see [6])

$$\limsup A_{(\widetilde{\rho}_x)} \leq 0 \,.$$

Term A_0. Let us consider Inequality (23) where $\kappa = u_0(x)$ and $(s,y) \mapsto \psi(0,x)\, R_\varepsilon(s)\, \widetilde{\rho}_\varepsilon(x-y)$ has been selected as a test function. Integrating the result w.r.t. $x \in \Omega$ yields an upper bound for A_0:

$$-A_0 + B_{(\widetilde{\rho}_x)} + B_0 \geq 0 \,,$$

where:

$$\begin{aligned} B_{(\widetilde{\rho}_x)} &= -\int_\Omega \iint_Q \int_0^1 \Phi(s,y,\nu(s,y,\beta),u_0(x)) \\ &\qquad \cdot \nabla \widetilde{\rho}_\varepsilon(x-y) \psi(0,x)\, R_\varepsilon(s)\, dy\, ds\, d\beta\, dx \,, \\ B_0 &= \int_\Omega \int_\Omega |u_0(x) - u_0(y)|\, \psi(0,x)\, \widetilde{\rho}_\varepsilon(x-y)\, dx\, dy \,. \end{aligned}$$

The theorem of continuity in means allows us to prove $B_0 \longrightarrow 0$. Let us denote by $C_{(\widetilde{\rho}_x)}$ the term defined by the expression of $B_{(\widetilde{\rho}_x)}$ where $u_0(x)$ has been replaced by $u_0(y)$. An integration by parts (w.r.t. the x variable) shows that $C_{(\widetilde{\rho}_x)} \longrightarrow 0$ and, from the theorem of continuity in means again and the fact that $||R_\varepsilon||_{L^1} \leq \varepsilon$ we deduce: $C_{(\widetilde{\rho}_x)} - B_{(\widetilde{\rho}_x)} \longrightarrow 0$ when $\varepsilon \to 0$.

Eventually, we have: $\limsup A_0 \leq 0$.

Terms A_1^b and A_2^b. Without any calculation, we have:

$$A_1^b = \iint_\Sigma \theta_\mu \, \psi_{|\Sigma} \, .$$

Moreover, in view of Lemma 3 (applied to ν), the following convergence holds:

$$A_2^b \longrightarrow - \iint_\Sigma \theta_\nu \, \psi_{|\Sigma} \, .$$

Eventually, by letting ε go to zero, is deduced from the inequality (24) the following inequation:

$$A^\infty_{(\psi_t)} + A^\infty_{(\psi_x)} + \iint_\Sigma \theta_\mu \, \psi_{|\Sigma} \; - \iint_\Sigma \theta_\nu \, \psi_{|\Sigma} \; \geq 0 \, .$$

Changing the roles of μ and ν and making the average of the two inequations obtained this way yields Inequality (18) of Lemma 2.

3.2 Proof of Theorem 2

Recall that, in the course of the proof of Lemma 2, the set Ω has been supposed to be the half-plane $\mathbb{R}^d_+$ but that Inequality (18) still holds when Ω is an open bounded polyhedral subset of $\mathbb{R}^d$: the details of the operations of localization and of transport have been eluded, but, for example, the dependance of the test-function ψ on the variable x has been carefully maintained. Indeed, we come back now to the case where Ω is any open bounded polyhedral subset of $\mathbb{R}^d$ and choose a test-function independant of x in (18), which is the function ψ_0 defined by:

$$\psi_0(t,x) = (T-t)\chi_{(0,T)}(t) \, ,$$

where $T > 0$. This yields:

$$\int_0^{\mathrm{T}} \int_\Omega \int_0^1 \int_0^1 |\mu(t,x,\alpha) - \nu(t,x,\beta)| \, dx \, dt \, d\alpha \, d\beta \, \leq 0 \, .$$

As T is any positive real number, the following equality holds:

for a.e. $(t,x,\alpha,\beta) \in Q \times (\,0,1) \times (\,0,1) \, , \; \mu(t,x,\alpha) = \nu(t,x,\beta) \, .$

Now, define the function u by the formula

$$u(t,x) = \int_0^1 \mu(t,x,\alpha) \, d\alpha \, .$$

Taking into account the product structure of the measurable space $Q \times (\,0,1) \times (\,0,1)$ we get:

$$\mu(t,x,\alpha) \; = \; u(t,x) \; = \; \nu(t,x,\beta) \;\; for \;\; a.e. \;\; (t,x,\alpha,\beta) \in Q \times (\,0,1)^2.$$

4 Convergence of the finite volumes scheme

4.1 Presentation of the scheme

Let $\mathcal{T}$ be a family of disjoint connected polygonal subsets of Ω (called control volumes) such that $\overline{\Omega}$ is the union of the closures of the elements of this family and such that the common interface of two control volumes is included in an hyperplane of $\mathbb{R}^d$. Let h be the size of the mesh: $h = \sup\{diam(K),\, K \in \mathcal{T}\}$. Notice that $h < +\infty$ (for the set Ω is bounded) and suppose: there exists $\alpha > 0$ such that

$$\begin{cases} \alpha h^d \leq m(K), \\ m(\partial K) \leq \frac{1}{\alpha} h^{d-1}\,,\ \forall K \in \mathcal{T}, \end{cases} \tag{25}$$

where $m(K)$ is the d-dimensionnal Lebesgue measure of K and $m(\partial K)$ is the $(d-1)$-dimensionnal Lebesgue measure of ∂K. If K and L are two control volumes having an edge σ in common we say that L is a neighbour of K and denote $L \in \mathcal{N}(K)$. We sometimes denote by $K|L$ the common edge σ between K and L and by $n_{K,\sigma}$ the unit normal to σ, oriented from K to L. Moreover, $\mathcal{E}$ denotes the set of all edges and $\mathcal{E}^b$ the set of boundary edges, that is: $\mathcal{E}^b = \{\sigma \in \mathcal{E}, m(\sigma \cap \partial\Omega) > 0\}$. If $K \in \mathcal{T}$, $\mathcal{E}_K$ is defined as the set of the edges determined by ∂K, i.e.: $\mathcal{E}_K = \{\sigma \in \mathcal{E}\,,\ m(\sigma \cap \partial K) > 0\}$.

Remark 4 Assumption (25) yields the following estimate on the number of control volumes:

$$|\mathcal{T}| \leq \frac{m(\Omega)}{\alpha} h^{-d}\,. \tag{26}$$

Let k be the time step. The numerical fluxes $F^n_{K,\sigma}$ (for $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$) are functions in $\mathcal{C}(\mathbb{R}^2, \mathbb{R})$ satisfying the following hypotheses of monotony, conservativity, regularity and consistency (recall that $B = \max(\operatorname*{ess\,sup}_{\Omega}(u_0)\,, \operatorname*{ess\,sup}_{\Sigma}(u^b))$ and $A = \min(\operatorname*{ess\,inf}_{\Omega}(u_0)\,, \operatorname*{ess\,inf}_{\Sigma}(u^b))\,)$:

$$\begin{cases} (i) & \text{on } [\,A, B]^2\,,\ (a,b) \mapsto F^n_{K,\sigma}(a,b) \text{ is nondecreasing w.r.t. } a \\ & \text{and nonincreasing w.r.t. } b\,, \\ (ii) & \text{for all } \sigma = K|L \in \mathcal{E} \setminus \mathcal{E}^b\,,\ \text{for all} \\ & a\,,b \in [\,A,B]\,, F^n_{K,\sigma}(a,b) = -F^n_{L,\sigma}(b,a)\,, \\ (iii) & \text{on } [\,A,B]^2\,,\ F^n_{K,\sigma} \text{ is Lipschitz continuous and admits} \\ & m(\sigma) Lip(f) \text{ as Lipschitz constant}\,, \\ (iv) & \text{for all } s \in [\,A,B]\,, \\ & F^n_{K,\sigma}(s,s) = \dfrac{1}{k}\displaystyle\int_{nk}^{(n+1)k}\int_{\sigma} f(t,x,s)\cdot n_{K,\sigma}\, d\gamma(x)\, dt\,. \end{cases} \tag{27}$$

Notation: If $\sigma = K|L \in \mathcal{E}$ we will denote $F^n_{K,K|L}$ by $F^n_{K,L}$.

Notice that the property of consistency (iv) above gives, owing to $\mathrm{div}_x f = 0$,

$$\forall s \in \, [A,B]\,,\ \forall K \in \mathcal{T}\,,\ \sum_{\sigma\in\mathcal{E}_K} F^n_{K,\sigma}(s,s) = 0\,. \tag{28}$$

The discrete unknowns u^n_K (for $n \in \mathbb{N}$ and $K \in \mathcal{T}$) are defined by the following set of equations:

$$u^0_K = \frac{1}{m(K)} \int_K u_0(x)\, dx\,,\ \forall K \in \mathcal{T}\,, \tag{29}$$

$$u^{b,n}_\sigma = \frac{1}{km(\sigma)} \int_{nk}^{(n+1)k} \int_\sigma u^b(t,x)\, d\gamma(x)\, dt\,,\ \forall \sigma \in \mathcal{E}^b\,,\ \forall n \in \mathbb{N}\,, \tag{30}$$

$$m(K)\frac{u^{n+1}_K - u^n_K}{k} + \sum_{\sigma\in\mathcal{E}_K} F^n_{K,\sigma}(u^n_K, u^n_{K,\sigma}) = 0\,,$$
$$\forall K \in \mathcal{T}\,,\ \forall n \in \mathbb{N}\,, \tag{31}$$

where

$$u^n_{K,\sigma} = \begin{cases} u^n_L & \text{if } \ \sigma = K|L\,, \\ u^{b,n}_\sigma & \text{if } \ \sigma \in \mathcal{E}^b\,. \end{cases}$$

The numerical solution is then defined by: for all $K \in \mathcal{T}$, for all $n \in \mathbb{N}$, $u_{\mathcal{T},k}(t,x) = u^n_K$ for all $(t,x) \in [nk,(n+1)k[\times K$.

Moreover, we will suppose that a CFL condition is fulfilled, that is to say:

$$\exists\, \xi \in]\,0,1[\ \text{ such that } k \le (1-\xi)\frac{\alpha^2 h}{2Lip(f)}\,. \tag{32}$$

Then, the monotony of the schemes is ensured. Indeed, we deduce from (31) and (28)

$$m(K)\frac{u^{n+1}_K - u^n_K}{k} + \sum_{\sigma\in\mathcal{E}_K} (F^n_{K,\sigma}(u^n_K, u^n_{K,\sigma})$$
$$-F^n_{K,\sigma}(u^n_K, u^n_K)) = 0\,. \tag{33}$$

Therefore

$$u^{n+1}_K = \Big(1 - \frac{k}{m(K)} \sum_{\sigma\in\mathcal{E}_K} \tau^n_{K,\sigma}\Big)\, u^n_K \ + \frac{k}{m(K)} \sum_{\sigma\in\mathcal{E}_K} \tau^n_{K,\sigma}\, u^n_{K,\sigma}\,,$$

where $\tau^n_{K,\sigma} = (F^n_{K,\sigma}(u^n_K, u^n_{K,\sigma}) - F^n_{K,\sigma}(u^n_K, u^n_K))/(u^n_K - u^n_{K,\sigma})$ if $u^n_K \neq u^n_{K,\sigma}$, or $\tau^n_{K,\sigma} = 0$ else. The monotony and the regularity of the fluxes (see (iii) of (27)) and the CFL condition ensures $\frac{k}{m(K)} \sum_{\sigma\in\mathcal{E}_K} \tau^n_{K,\sigma} \in [\,0,1]$. Thus, the following remark holds.

Remark 5 For all $K \in \mathcal{T}, \ n \in \mathbb{N}$, (31) can be rewritten in the following way:

$$u_K^{n+1} = H_K^n(u_K^n, (u_{K,\sigma}^n)_{\sigma \in \mathcal{E}_K}),$$

where the function H_K^n is nondecreasing with respect to each of its arguments and satisfies

$$H_K^n(u, (u)_{\sigma \in \mathcal{E}_K}) = u \ \text{ for all } u \in \mathbb{R}.$$

4.2 L^∞-stability and weak BV estimate

The scheme defined by (29)-(30)-(31) is L^∞-stable and a weak BV estimate – that is a weak estimate on the time and space derivatives of $u_{\mathcal{T},k}$ – holds.

Lemma 5 *Assume that (27) and (32) hold. Then the approximate solution $u_{\mathcal{T},k}$ of (1)-(2)-(3) defined by (29), (30) and (31) satisfies:*

$$A \le u_{\mathcal{T},k}(t,x) \le B \ \textit{ for a.e. } (t,x) \in Q.$$

Proof. We prove by induction on $n \in \mathbb{N}$:

$$\forall K \in \mathcal{T}, \ A \le u_K^n \le B.$$

It is true for $n = 0$ and the heredity of the proposition is a consequence of the monotony of the scheme (see Remark 5).

Let us now detail the weak BV estimate.

Lemma 6 *Assume that (25), (27) and (32) hold. Let $u_{\mathcal{T},k}$ be the approximate solution of the problem (1)-(2)-(3) defined by (29), (30) and (31). Let T be positive and set $N = \max\{n \in \mathbb{N}, n < T/k\}$ and $\mathcal{E}_{\text{int}}^n = \{(K,L) \in \mathcal{T}^2, L \in \mathcal{N}(K)$ and $u_K^n > u_L^n\}$. Then there exists $C \ge 0$ only depending on Ω, u_0, u^b, $Lip(f)$, T, α and ξ such that, if $k < T$,*

$$\sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\text{int}}^n} \left[\begin{array}{l} \max\limits_{u_L^n \le c \le d \le u_K^n} (F_{K,\sigma}^n(d,c) - F_{K,\sigma}^n(d,d)) \\ + \max\limits_{u_L^n \le c \le d \le u_K^n} (F_{K,\sigma}^n(d,c) - F_{K,\sigma}^n(c,c)) \end{array} \right] \le \frac{C}{\sqrt{h}}, \tag{34}$$

and

$$\sum_{n=0}^{N} \sum_{K\in\mathcal{T}} m(K)|u_K^{n+1} - u_K^n| \le \frac{C}{\sqrt{h}}. \tag{35}$$

Proof. In the following, we denote by C various quantities that only depend on Ω, u_0, u^b, $Lip(f)$, T, α and ξ.

Multiplying (33) by $k\, u_K^n$, then summing over $K \in \mathcal{T}$ and $n \in \{0, \cdots, N\}$ yields the following equality:

$$B_1 + B_2 = 0, \tag{36}$$

where

$$B_1 = \sum_{n=0}^{N} \sum_{K \in \mathcal{T}} m(K)(u_K^{n+1} - u_K^n) u_K^n \,,$$

$$B_2 = \sum_{n=0}^{N} k \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} u_K^n (F_{K,\sigma}^n(u_K^n, u_{K,\sigma}^n) - F_{K,\sigma}^n(u_K^n, u_K^n)) \,.$$

The last two summations in the expression of B_2 can be gathered by edges, according to the following lemma:

Lemma 7 *Let $n \in \mathbb{N}$. Let ρ be an application $\mathcal{T} \times \mathcal{E} \to \mathbb{R}$ such that $\rho_{K,K|L}^n = -\rho_{L,K|L}^n$ if $u_K^n = u_L^n$. Then we have*

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \rho_{K,\sigma}^n = \sum_{(K,L) \in \mathcal{E}_{\text{int}}^n} (\rho_{K,K|L}^n + \rho_{L,K|L}^n) + \sum_{\sigma \in \mathcal{E}^b} \rho_{K,\sigma}^n \,.$$

(Notice that if $\sigma \in \mathcal{E}^b$ then there exists a unique $K \in \mathcal{T}$ such that $\sigma = \partial K \cap \partial \Omega$; therefore the $\rho_{K,\sigma}^n$ in the last sum are well defined.)

The proof of this lemma is left to the reader.

From this result is deduced $B_2 = B_3 + b_{2,3}$, where

$$B_3 = \sum_{n=0}^{N} k \sum_{(K,L) \in \mathcal{E}_{\text{int}}^n} \begin{bmatrix} u_K^n (F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n)) \\ -u_L^n (F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_L^n, u_L^n)) \end{bmatrix} ,$$

$$b_{2,3} = \sum_{n=0}^{N} k \sum_{\sigma \in \mathcal{E}^b} u_K^n (F_{K,\sigma}^n(u_K^n, u_\sigma^{b,n}) - F_{K,\sigma}^n(u_K^n, u_K^n)) \,.$$

An estimate on the quantity $b_{2,3}$ of the kind:

$$|b_{2,3}| \leq C \tag{37}$$

is available since:

$$\begin{aligned} |b_{2,3}| &\leq 2NkLip(f) \max(|A|, |B|)^2 \sum_{\sigma \in \mathcal{E}^b} m(\sigma) \\ &= 2TLip(f) \max(|A|, |B|)^2 m(\partial \Omega) \,. \end{aligned}$$

Now, define the function $\Psi^n_{K,L}$, primitive of the function $s \longmapsto s\dfrac{d}{ds} F^n_{K,L}(s,s)$, by

$$\Psi^n_{K,L}(s) = \int_A^s \tau \, \frac{d}{ds} F^n_{K,L}(\tau,\tau)\, d\tau \, .$$

From an integration by parts is deduced the formula: $\forall \; (a,b) \in \mathbb{R}^2$,

$$\Psi^n_{K,L}(b) - \Psi^n_{K,L}(a) = \begin{bmatrix} b(F^n_{K,L}(b,b) - F^n_{K,L}(a,b)) \\ \\ -a(F^n_{K,L}(a,a) - F^n_{K,L}(a,b)) \end{bmatrix} - \int_a^b (F^n_{K,L}(s,s) - F^n_{K,L}(a,b)) ds \, ,$$

so that

$$B_3 = B_4 + b_{3,4} \, , \tag{38}$$

with

$$b_{3,4} = -\sum_{n=0}^{N} \sum_{(K,L)\in \mathcal{E}^n_{\text{int}}} k(\Psi^n_{K,L}(u^n_K) - \Psi^n_{K,L}(u^n_L)) \, ,$$

$$B_4 = \sum_{n=0}^{N} \sum_{(K,L)\in \mathcal{E}^n_{\text{int}}} k \int_{u^n_L}^{u^n_K} (F^n_{K,L}(u^n_K, u^n_L) - F^n_{K,L}(s,s)) ds \, .$$

The relation (28) ensures $\sum_{L\in\mathcal{N}(K)} \Psi^n_{K,L} = 0$; from this and Lemma 7 (summation over the edges) it appears that $b_{3,4}$ reduces to a sum over the edges of the boundary and, as $b_{2,3}$, satisfies

$$|b_{3,4}| \le C \, . \tag{39}$$

Now, fix $a\,, b\,, c\,, d \in \mathbb{R}$ s.t. $a \le c \le d \le b$. Taking into account the monotony of $F^n_{K,L}$, the following inequality holds:

$$\int_a^b (F^n_{K,L}(b,a) - F^n_{K,L}(s,s)) ds \ge \int_c^d (F^n_{K,L}(d,c) - F^n_{K,L}(d,s)) ds.$$

Moreover $F^n_{K,L}(d,.)$ is Lipschitz continuous and nonincreasing so that (see [9])

$$\int_c^d (F^n_{K,L}(d,c) - F^n_{K,L}(d,s))\, ds \ge \frac{1}{2m(K|L)Lip(f)} (F^n_{K,L}(d,c) - F^n_{K,L}(d,d))^2 \, .$$

Therefore, we get:

$$\int_{u_L^n}^{u_K^n} (F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(s,s))ds \\ \geq \frac{1}{2m(K|L)Lip(f)} \max_{u_L^n \leq c \leq d \leq u_K^n} (F_{K,L}^n(d,c) - F_{K,L}^n(d,d))^2\,,$$

and

$$\int_{u_L^n}^{u_K^n} (F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(s,s))ds \\ \geq \frac{1}{2m(K|L)Lip(f)} \max_{u_L^n \leq c \leq d \leq u_K^n} (F_{K,L}^n(d,c) - F_{K,L}^n(c,c))^2\,,$$

so that

$$B_4 \geq \mathcal{B}\,, \tag{40}$$

where $\mathcal{B}$ is defined by:

$$\mathcal{B} = \frac{1}{4Lip(f)} \sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\text{int}}^n} \\ \times \left[\begin{array}{l} \frac{1}{m(K|L)} \max_{u_L^n \leq c \leq d \leq u_K^n} (F_{K,L}^n(d,c) - F_{K,L}^n(d,d))^2 \\ + \frac{1}{m(K|L)} \max_{u_L^n \leq c \leq d \leq u_K^n} (F_{K,L}^n(d,c) - F_{K,L}^n(c,c))^2 \end{array} \right]. \tag{41}$$

Recalling the equality $B_2 = B_4 + b_{2,3} + b_{3,4}$ and the estimates on the terms $b_{2,3}$ and $b_{3,4}$ described in (37) and (39), it appears that

$$B_2 \geq \mathcal{B} - C\,. \tag{42}$$

On the other hand, the quantity B_1 reads:

$$B_1 = -\frac{1}{2} \sum_{n=0}^{N} \sum_{K\in\mathcal{T}} m(K)(u_K^{n+1} - u_K^n)^2 + \frac{1}{2} \sum_{K\in\mathcal{T}} m(K)(u_K^{N+1})^2 \\ -\frac{1}{2} \sum_{K\in\mathcal{T}} m(K)(u_K^0)^2.$$

There exists $C \geq 0$ such that $-C$ is a lower bound for the last two terms of the previous equality. Moreover, the Cauchy-Schwarz inequality and (31) lead to:

$$(u_K^{n+1}-u_K^n)^2 \le \frac{k^2}{m(K)^2}\Big(\sum_{\sigma\in\mathcal{E}_K} m(\sigma)\Big) \times\Big(\sum_{\sigma\in\mathcal{E}_K}\frac{1}{m(\sigma)}(F^n_{K,\sigma}(u^n_K,u^n_{K,\sigma})-F^n_{K,\sigma}(u^n_K,u^n_K))^2\Big)\,.$$

From this last inequality, from Lemma 7 (summation over the edges), from the assumption (25) on the mesh and from the CFL condition (32) are deduced the following inequalities:

$$\frac{1}{2}\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)(u_K^{n+1}-u_K^n)^2 \le (1-\xi)\mathcal{B}+C$$

and:

$$B_1 \ge -(1-\xi)\mathcal{B}-C\,. \tag{43}$$

Now, as the equality $B_1+B_2=0$ holds and as (43) and (42) are satisfied, we have $\xi\mathcal{B}\le C$, that is to say (recall that C may depend on ξ):

$$\mathcal{B}\le C\,. \tag{44}$$

Moreover, from the Cauchy-Schwarz inequality is deduced:

$$\sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}^n_{\rm int}} \left[\begin{array}{l}\max\limits_{u^n_L\le c\le d\le u^n_K}(F^n_{K,\sigma}(d,c)-F^n_{K,\sigma}(d,d)) \\ +\max\limits_{u^n_L\le c\le d\le u^n_K}(F^n_{K,\sigma}(d,c)-F^n_{K,\sigma}(c,c))\end{array}\right]$$

$$\le C\Big(\sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}^n_{\rm int}} m(K|L)\Big)^{1/2}\mathcal{B}^{1/2}\,; \tag{45}$$

and, taking into account the following estimate (deduced from (25) and (26))

$$\sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}^n_{\rm int}} m(K|L) \le C/h,$$

it appears that the weak BV estimate on space derivatives (34) holds. To get the weak BV estimate on time derivatives (35), use (33) to get the following estimate:

$$m(K)|u_K^{n+1}-u_K^n| \le k\sum_{\sigma\in\mathcal{E}_K}|F^n_{K,\sigma}(u^n_K,u^n_{K,\sigma}))-F^n_{K,\sigma}(u^n_K,u^n_K)|\,.$$

Summing the result over $K\in\mathcal{T}$ and $n\in\{0,\cdots,N\}$, then using Lemma 7 and (34), yields Inequation (35).

4.3 Entropy inequalities

We wish to prove that the approximate solution $u_{\mathcal{T},k}$ satisfies approximate entropy inequalities that have already been discussed in the introduction of the entropy process solution (see Sect. 2, relation (16)). To this purpose, we will work with the semi Kruzkov entropies; that is one of the keys of the following results (the other key being the weak BV estimate).

We recall some notations about it.

Notations: η_κ^+ denotes the function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\eta_\kappa^+(s) = (s-\kappa)^+ \,, \tag{46}$$

and Φ_κ^+ the associated flux-function from $Q \times \mathbb{R}$ to $\mathbb{R}$ defined by

$$\Phi_\kappa^+(t,x,s) = \mathrm{sgn}^+(s-\kappa)(f(t,x,s) - f(t,x,\kappa)) \,. \tag{47}$$

Notice that, if $a \top b = \max(a,b)$ and $a \perp b = \min(a,b)$, then we have

$$\eta_\kappa^+(s) = s \top \kappa - \kappa \,,$$

and

$$\Phi_\kappa^+(t,x,s) = f(t,x,s\top\kappa) - f(t,x,\kappa) \,.$$

Therefore, the associated entropy numerical flux function is defined by the formula

$$\Phi_{K,\sigma,\kappa}^{+,n}(a,b) = F_{K,\sigma}^n(a\top\kappa, b\top\kappa) - F_{K,\sigma}^n(\kappa,\kappa) \,. \tag{48}$$

If $\sigma = K|L$, $(K,L) \in \mathcal{T}^2$, then $\Phi_{K,K|L,\kappa}^{+,n}$ is denoted by $\Phi_{K,L,\kappa}^{+,n}$.

4.3.1 Discrete entropy inequalities

Lemma 8 *Assume that (25), (27) and (32) hold. Let $u_{\mathcal{T},k}$ be the approximate solution of the problem (1)-(2)-(3) defined by (29), (30) and (31). Then, for all $\kappa \in [A,B]$, for all $K \in \mathcal{T}$, $n \in \mathbb{N}$, the following local discrete entropy inequality holds:*

$$\frac{\eta_\kappa^+(u_K^{n+1}) - \eta_\kappa^+(u_K^n)}{k} + \frac{1}{m(K)} \sum_{\sigma\in\mathcal{E}_K} (\Phi_{K,\sigma,\kappa}^{+,n}(u_K^n, u_{K,\sigma}^n) - \Phi_{K,\sigma,\kappa}^{+,n}(u_K^n, u_K^n)) \leq 0. \tag{49}$$

Proof. From the monotony of the scheme (Remark 5) is deduced the fact that $H_K^n(u_K^n\top\kappa, (u_{K,\sigma}^n\top\kappa)_{\sigma\in\mathcal{E}_K})$ is an upper bound for u_K^{n+1} and κ, thus for $u_K^{n+1}\top\kappa$ too, that is to say:

$$u_K^{n+1}\top\kappa \leq u_K^n\top\kappa - \frac{k}{m(K)} \times \sum_{\sigma\in\mathcal{E}_K} (F_{K,\sigma}^n(u_K^n\top\kappa, u_{K,\sigma}^n\top\kappa) - F_{K,\sigma}^n(u_K^n\top\kappa, u_K^n\top\kappa)) \,. \tag{50}$$

Substracting from this inequality the equality

$$\kappa = \kappa - \frac{k}{m(K)} \sum_{\sigma \in \mathcal{E}_K} \left(F^n_{K,\sigma}(\kappa,\kappa) - F^n_{K,\sigma}(\kappa,\kappa) \right),$$

yields the result.

4.3.2 Continuous entropy inequality

Theorem 3 *Assume that (25), (27) and (32) hold. Let $u_{\mathcal{T},k}$ be the approximate solution of the problem (1)-(2)-(3) defined by (29), (30) and (31). Then the following approximate continuous entropy inequalities hold:*

$$\text{(51)} \qquad \begin{cases} \forall \kappa \in [\,A,B]\,,\ \forall \varphi \in \mathcal{C}^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)\,, \\ \displaystyle\iint_Q \Big(\eta^+_\kappa(u_{\mathcal{T},k})\varphi_t + \Phi^+_\kappa(t,x,u_{\mathcal{T},k}) \cdot \nabla\varphi \Big) dx\,dt \\ \displaystyle + \int_\Omega \eta^+_\kappa(u_0)\varphi(0)dx \\ \displaystyle + Lip(f) \int_\Sigma \eta^+_\kappa(u^b)\varphi(t,x)d\gamma(x)\,dt \geq -\varepsilon_{\mathcal{T},k}(\varphi)\,, \end{cases}$$

where:

$$\forall \varphi \in \mathcal{C}^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)\,,\ \varepsilon_{\mathcal{T},k}(\varphi) \longrightarrow 0 \text{ when } h \to 0\,.$$

The same result holds when the negative semi-Kruzkov entropies are considered.

Proof. Let φ be in $\mathcal{C}^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)$ and κ be in $[\,A,B]$. We fix $T \geq 0$ such that $\varphi \equiv 0$ on $[\,T,\infty[\times\Omega$ and set $N = [T/k+1]$. We also denote by $u^0_{\mathcal{T}}$ the application defined by $u^0_{\mathcal{T}}(x) = u^0_K$ for a.e. $x \in K$, and by $u^b_{\mathcal{T},k}$ the application defined by $u^b_{\mathcal{T},k}(x) = u^{b,n}_\sigma$ for a.e. $(t,x) \in [\,nk,(n+1)k[\times\sigma$.

Multiplying (49) by $k\,m(K)\,\varphi^n_K = \displaystyle\int_{nk}^{(n+1)k} \int_K \varphi\,dx\,dt$, and summing over $K \in \mathcal{T}\,,\ n \in \mathbb{N}$, yields the inequality:

$$\text{(52)} \qquad T_1 + T_2 \leq 0\,,$$

where

$$\text{(53)} \qquad T_1 = \sum_{n=0}^{N} \sum_{K \in \mathcal{T}} m(K)\,(\eta(u^{n+1}_K) - \eta(u^n_K))\varphi^n_K\,,$$

and, by summing over the edges, $T_2 = T_2^{\text{int}} + T_2^b$, with

$$T_2^{\text{int}} = \sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}^n_{\text{int}}} \begin{bmatrix} \varphi^n_K(\Phi^{+,n}_{K,L,\kappa}(u^n_K,u^n_L) - \Phi^{+,n}_{K,L,\kappa}(u^n_K,u^n_K)) \\ \\ -\varphi^n_L(\Phi^{+,n}_{K,L,\kappa}(u^n_K,u^n_L) - \Phi^{+,n}_{K,L,\kappa}(u^n_L,u^n_L)\Big) \end{bmatrix}, \qquad \text{(54)}$$

$$T_2^b = \sum_{n=0}^{N} k \sum_{\sigma\in\mathcal{E}^b} \varphi_K^n(\Phi^{+,n}_{K,\sigma,\kappa}(u_K^n, u_\sigma^{n,b}) - \Phi^{+,n}_{K,\sigma,\kappa}(u_K^n, u_K^n))\,. \tag{55}$$

Proving the approximate continuous entropy inequalities comes back to prove

$$T_{10} + T_{20} \le \varepsilon_{\mathcal{T},k}(\varphi) \tag{56}$$

where T_{10} and T_{20} are defined by

$$\begin{aligned} T_{10} &= -\iint_Q \eta_\kappa^+(u_{\mathcal{T},k})\,\varphi_t\,dx\,dt - \int_\Omega \eta_\kappa^+(u_0)\,\varphi(0)\,dx\,,\\ T_{20} &= -\iint_Q \Phi_\kappa^+(t,x,u_{\mathcal{T},k})\cdot\nabla\varphi\,dx\,dt\\ &\qquad -Lip(f)\int_\Sigma \eta_\kappa^+(u^b)\varphi(t,x)\,d\gamma(x)\,dt\,. \end{aligned} \tag{57}$$

To this purpose, we compare T_{10} to T_1 and T_{20} to T_2.

1 Estimate on $T_{10} - T_1$

Using the definitions of $u^0_{\mathcal{T}}$ and $u_{\mathcal{T},k}$, the quantity T_{10} reads:

$$\begin{aligned} T_{10} = \sum_{n=0}^{N}\sum_{K\in\mathcal{T}} \frac{\eta_\kappa^+(u_K^{n+1}) - \eta_\kappa^+(u_K^n)}{k} \int_{nk}^{(n+1)k}\int_K \varphi(x,(n+1)k)\,dx\,dt\\ + \int_\Omega (\eta_\kappa^+(u^0_{\mathcal{T}}) - \eta_\kappa^+(u_0))\,\varphi(0)\,dx\,. \end{aligned}$$

From the fact that η_κ^+ is 1-Lipschitz continuous is deduced:

$$|T_{10} - T_1| \le \varepsilon^0_{\mathcal{T}}(\varphi) + \varepsilon^1_{\mathcal{T},k}(\varphi)\,, \tag{58}$$

where

$$\left\{\begin{aligned} \varepsilon^0_{\mathcal{T}}(\varphi) &= \int_\Omega |u^0_{\mathcal{T},k} - u_0|\,\varphi(0)\,dx\,,\\ \varepsilon^1_{\mathcal{T},k}(\varphi) &= \sum_{n=0}^{N}\sum_{K\in\mathcal{T}} \frac{|u_K^{n+1} - u_K^n|}{k}\\ &\quad\times \int_{nk}^{(n+1)k}\int_K |\varphi(x,(n+1)k) - \varphi(x,t)|\,dx\,dt\,. \end{aligned}\right. \tag{59}$$

Before giving precise estimates on these quantities, we study the difference $T_{20} - T_2$.

2 *Comparison of* T_{20} *and* T_2 We divide the study into two steps. Indeed, we have to take care to what happens inside and on the boundary of Ω. From the definition of the function $u_{\mathcal{T},k}$, from the fact that $\operatorname{div}_x f = 0$ and from Lemma 7 is deduced the equality:

$$T_{20} = T_{20}^{\mathrm{int}} + T_{20}^{b}\,,$$

where

$$\begin{aligned} T_{20}^{\mathrm{int}} = -\sum_{n=0}^{N} \sum_{(K,L)\in\mathcal{E}_{\mathrm{int}}^n} &\times \left[\begin{array}{l} \displaystyle\int_{nk}^{(n+1)k} \int_{K|L} \Phi_\kappa^+(t,x,u_K^n)\cdot n_{K|L}\,\varphi\, d\gamma(x)\, dt \\ \displaystyle -\int_{nk}^{(n+1)k} \int_{K|L} \Phi_\kappa^+(t,x,u_L^n)\cdot n_{K|L}\,\varphi\, d\gamma(x)\, dt \end{array} \right], \\ T_{20}^{b} = -\sum_{n=0}^{N} \sum_{\sigma\in\mathcal{E}^b} &\int_{nk}^{(n+1)k} \int_{\sigma} \Phi_\kappa^+(t,x,u_K^n)\cdot n_{K,\sigma}\varphi\, d\gamma(x)\, dt \\ &-Lip(f) \int_{\Sigma} \eta_\kappa^+(u^b)\,\varphi\, d\gamma(x)\, dt\,. \end{aligned}$$

2.1 Estimate on $|T_{20}^{\mathrm{int}} - T_2^{\mathrm{int}}|$

In order to compare T_{20}^{int} to T_2^{int}, let us introduce the average value of φ on an edge, defined by

$$\tilde{\varphi}_\sigma^n = \frac{1}{k m(\sigma)} \int_{nk}^{(n+1)k} \int_{\sigma} \varphi\, d\gamma(x)\, dt\,.$$

Notice that, φ being a regular function, its average values on K, L and $K|L$, with $(K,L)\in\mathcal{E}_{\mathrm{int}}^n$ are "close" each to other. That is why we rewrite:

$$\begin{aligned} T_{20}^{\mathrm{int}} = -\sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\mathrm{int}}^n} &\Big[\Big(\frac{1}{k}\int_{nk}^{(n+1)k} \int_{K|L} \Phi_\kappa^+(t,x,u_K^n) \\ &\cdot n_{K|L}\,\varphi\, d\gamma(x)\, dt - \Phi_{K,L,\kappa}^{+,n}(u_K^n,u_L^n)\tilde{\varphi}_{K|L}^n\Big) \\ &-\Big(\frac{1}{k}\int_{nk}^{(n+1)k} \int_{K|L} \Phi_\kappa^+(t,x,u_L^n) \\ &\cdot n_{K|L}\,\varphi\, d\gamma(x)\, dt - \Phi_{K,L,\kappa}^{+,n}(u_K^n,u_L^n)\tilde{\varphi}_{K|L}^n\Big)\Big]\,, \end{aligned}$$

then

$$T_{20}^{\text{int}} = -\sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\text{int}}^n} \Big[\Big(\frac{1}{k}\int_{nk}^{(n+1)k}\int_{K|L} \varPhi_\kappa^+(t,x,u_K^n)$$
$$\cdot n_{K|L}\,\varphi\,d\gamma(x)\,dt - \varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_K^n)\tilde{\varphi}_{K|L}^n\Big)$$
$$+\Big(\varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_K^n) - \varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_L^n)\Big)\tilde{\varphi}_{K|L}^n$$
$$-\Big(\frac{1}{k}\int_{nk}^{(n+1)k}\int_{K|L} \varPhi_\kappa^+(t,x,u_L^n)\cdot n_{K|L}\,\varphi\,d\gamma(x)\,dt$$
$$-\varPhi_{K,L,\kappa}^{+,n}(u_l^n,u_L^n)\tilde{\varphi}_{K|L}^n\Big)$$
$$-\Big(\varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_K^n) - \varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_L^n)\Big)\tilde{\varphi}_{K|L}^n\Big].$$

Now, let $\varepsilon_{\mathcal{T},k}^{c,int}$, $\varepsilon_{\mathcal{T},k}^{int,+}$, $\varepsilon_{\mathcal{T},k}^{int,-}$ be defined by

(60)

$$\left\{\begin{aligned}
\varepsilon_{\mathcal{T},k}^{c,int}(\varphi) &= \sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\text{int}}^n} \Big|\frac{1}{k}\int_{nk}^{(n+1)k}\int_{K|L} \Big(\varPhi_\kappa^+(t,x,u_K^n)\cdot n_{K|L} \\
&\quad -\frac{1}{m(K|L)}\varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_K^n)\Big)\,\varphi\,d\gamma(x)\,dt \\
&\quad -\frac{1}{k}\int_{nk}^{(n+1)k}\int_{K|L} \Big(\varPhi_\kappa^+(t,x,u_L^n)\cdot n_{K|L} \\
&\quad -\frac{1}{m(K|L)}\varPhi_{K,L,\kappa}^{+,n}(u_L^n,u_L^n)\Big)\,\varphi\,d\gamma(x)\,dt\Big|, \\
\varepsilon_{\mathcal{T},k}^{int,+}(\varphi) &= \sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\text{int}}^n} \Big|\Big(\varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_K^n) - \varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_L^n)\Big) \\
&\quad \times\Big(\varphi_K^n - \tilde{\varphi}_{K|L}^n\Big)\Big|, \\
\varepsilon_{\mathcal{T},k}^{int,-}(\varphi) &= \sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\text{int}}^n} \Big|\Big(\varPhi_{K,L,\kappa}^{+,n}(u_L^n,u_L^n) - \varPhi_{K,L,\kappa}^{+,n}(u_K^n,u_L^n)\Big) \\
&\quad \times\Big(\varphi_L^n - \tilde{\varphi}_{K|L}^n\Big)\Big|.
\end{aligned}\right.$$

The following estimate holds:

(61) $$|T_{20}^{\text{int}} - T_2^{\text{int}}| \le \varepsilon_{\mathcal{T},k}^{int,c}(\varphi) + \varepsilon_{\mathcal{T},k}^{int,+}(\varphi) + \varepsilon_{\mathcal{T},k}^{int,-}(\varphi)\,.$$

Notice that $\varepsilon_{\mathcal{T},k}^{c,int}$ is a consistency error (it tends to zero with h thanks to the property (iv) of assumption (27) satisfied by the numerical fluxes). The quantities $\varepsilon_{\mathcal{T},k}^{\text{int},+}$ and $\varepsilon_{\mathcal{T},k}^{\text{int},-}$ are those which measure the difference between T_{20}^{int} and T_2^{int}. We will use the weak BV estimate (34) to prove that they tend to zero as h does. But we first study the quantity $T_2^b - T_{20}^b$.

2.2 Comparison of T_{20}^b and T_2^b

Recall that the quantity T_{20}^b is defined by:

$$
\begin{aligned}
T_{20}^b = &-\sum_{n=0}^{N} k \sum_{\sigma\in\mathcal{E}^b} \frac{1}{k}\int_{nk}^{(n+1)k}\int_\sigma \Phi_\kappa^+(t,x,u_K^n)\cdot n_{K,\sigma}\varphi\, d\gamma(x)\, dt \\
&-Lip(f)\sum_{n=0}^{N}\sum_{\sigma\in\mathcal{E}^b}\int_{nk}^{(n+1)k}\int_\sigma \eta_\kappa^+(u^b)\,\varphi\, d\gamma(x)\, dt\,.
\end{aligned}
\tag{62}
$$

Now, let us denote by $\tilde{T}_{20}^b$ the following quantity

$$
\begin{aligned}
\tilde{T}_{20}^b = &-\sum_{n=0}^{N} k \sum_{\sigma\in\mathcal{E}^b} \Phi_{K,\sigma,\kappa}^{+,n}(u_K^n, u_K^n)\,\varphi_K^n \\
&-\sum_{n=0}^{N} k \sum_{\sigma\in\mathcal{E}^b} Lip(f)\, m(\sigma)\,\eta_\kappa^+(u_\sigma^{b,n})\,\varphi_K^n\,.
\end{aligned}
\tag{63}
$$

Then T_2^b can be compared to $\tilde{T}_{20}^b$:

$$
T_2^b - \tilde{T}_{20}^b = \sum_{n=0}^{N} k \sum_{\sigma\in\mathcal{E}^b} \varphi_K^n(\Phi_{K,\sigma,\kappa}^{+,n}(u_K^n, u_\sigma^{b,n}) + Lip(f)\, m(\sigma)\,\eta_\kappa^+(u_\sigma^{b,n}))\,,
$$

and this quantity is nonnegative:

$$
T_2^b - \tilde{T}_{20}^b \geq 0\,. \tag{64}
$$

Indeed, the following lemma holds:

Lemma 9 *Assume that (27) holds. Let σ be in $\mathcal{E}^b$ and K be in $\mathcal{T}$ s.t. $\sigma = \partial K\cap\partial\Omega$. Then:* $\forall\kappa\in[\,A,B]\,,\ \forall a\,,\ b\in[\,A,B]$,

$$
F_{K,\sigma}^n(a\top\kappa, b\top\kappa) - F_{K,\sigma}^n(\kappa,\kappa) + Lip(f)\, m(\sigma)\,(b-\kappa)^+ \geq 0\,.
$$

Proof. The numerical fluxes being non-decreasing functions with respect to their first variables, the following inequality holds:

$$
F_{K,\sigma}^n(a\top\kappa, b\top\kappa) \geq F_{K,\sigma}^n(\kappa, b\top\kappa)\,.
$$

Moreover, from the fact that $F^n_{K,\sigma}$ is a $m(\sigma)\,Lip(f)$-Lipschitz continuous function, is deduced the inequality

$$F^n_{K,\sigma}(\kappa, b\top\kappa) - F^n_{K,\sigma}(\kappa,\kappa) + Lip(f)\,m(\sigma)\,(b-\kappa)^+ \geq 0\,,$$

which yields the result. □

Now, let us estimate the quantity $T^b_{20} - \tilde{T}^b_{20}$. To compare the expressions (62) and (63), we write

$$\begin{aligned}\eta^+_\kappa(u^{b,n}_\sigma)\varphi^n_K = \eta^+_\kappa(u^{b,n}_\sigma)\Big(\varphi^n_K - \frac{1}{k\,m(\sigma)}\int_{nk}^{(n+1)k}\int_\sigma \varphi\Big) \\ +\frac{1}{k\,m(\sigma)}\int_{nk}^{(n+1)k}\int_\sigma (\eta^+_\kappa(u^b_{\mathcal{T},k}) - \eta^+_\kappa(u^b))\varphi \\ +\frac{1}{k\,m(\sigma)}\int_{nk}^{(n+1)k}\int_\sigma \eta^+_\kappa(u^b)\varphi\,,\end{aligned}$$

and get

$$|T^b_{20} - \tilde{T}^b_{20}| \leq \varepsilon^{c,b}_{\mathcal{T},k}(\varphi) + \varepsilon^b_{\mathcal{T},k}(\varphi) + \tilde{\varepsilon}^{c,b}_{\mathcal{T},k}(\varphi)\,, \tag{65}$$

where

$$\begin{cases} \varepsilon^{c,b}_{\mathcal{T},k}(\varphi) = \displaystyle\sum_{n=0}^N k \sum_{\sigma\in\mathcal{E}^b} \Big|\frac{1}{k}\int_{nk}^{(n+1)k}\int_\sigma \Phi^+_\kappa(t,x,u^n_K) \\ \qquad \cdot n_{K,\sigma}\,\varphi - \Phi^{+,n}_{K,\sigma,\kappa}(u^n_K,u^n_K)\,\varphi^n_K\Big|\,, \\ \varepsilon^b_{\mathcal{T},k}(\varphi) = Lip(f)\displaystyle\int_\Sigma |u^b_{\mathcal{T},k} - u^b|\varphi\,d\gamma(x)\,dt\,, \\ \tilde{\varepsilon}^{c,b}_{\mathcal{T},k}(\varphi) = \displaystyle\sum_{n=0}^N\sum_{\sigma\in\mathcal{E}^b} Lip(f)\,k\,m(\sigma)\,\eta^+_\kappa(u^{b,n}_\sigma)\,|\varphi^n_K - \tilde{\varphi}^n_\sigma|\,. \end{cases} \tag{66}$$

Eventually, from (52), (58), (61), (64) and (65) is deduced the approximate continuous entropy inequality (16) with

$$\varepsilon_{\mathcal{T},k} = \varepsilon^c_{\mathcal{T},k} + \varepsilon^0_{\mathcal{T}} + \varepsilon^1_{\mathcal{T},k} + \varepsilon^{int,+}_{\mathcal{T},k} + \varepsilon^{int,-}_{\mathcal{T},k} + \varepsilon^b_{\mathcal{T},k}\,, \tag{67}$$

$\varepsilon^c_{\mathcal{T},k}$ being a consistency error defined by

$$\varepsilon^c_{\mathcal{T},k} = \varepsilon^{c,int}_{\mathcal{T},k} + \varepsilon^{c,b}_{\mathcal{T},k} + \tilde{\varepsilon}^{c,b}_{\mathcal{T},k} \tag{68}$$

(see (60) and (66)).

Let us now turn to the study of $\varepsilon_{\mathcal{T},k}$.

3 Estimate on $\varepsilon_{\mathcal{T},k}$

We first turn to the consistency errors, for example to $\varepsilon_{\mathcal{T},k}^{c,int}$ which, we recall, is defined by

$$\varepsilon_{\mathcal{T},k}^{c,int}(\varphi) = \Bigg| \sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\rm int}^n} \frac{1}{k} \int_{nk}^{(n+1)k} \int_{K|L} \Big(\Phi_\kappa^+(t,x,u_K^n) \cdot n_{K|L} - \frac{1}{m(K|L)} \Phi_{K,L,\kappa}^{+,n}(u_K^n,u_K^n) \Big) \varphi \, d\gamma(x)\, dt - \frac{1}{k} \int_{nk}^{(n+1)k} \int_{K|L} \Big(\Phi_\kappa^+(t,x,u_L^n) \cdot n_{K|L} - \frac{1}{m(K|L)} \times \Phi_{K,L,\kappa}^{+,n}(u_L^n,u_L^n) \Big) \varphi \, d\gamma(x)\, dt \Bigg| .$$

As the numerical fluxes, the numerical entropy fluxes are consistents: for all $s \in [\,A,B]$,

$$\Phi_{K,\sigma,\kappa}^{+,n}(s,s) = \frac{1}{k} \int_{nk}^{(n+1)k} \int_\sigma \Phi_\kappa^+(t,x,s) \cdot n_{K,\sigma}\, d\gamma(x)\, dt \, .$$

Therefore, the quantity $\varepsilon_{\mathcal{T},k}^{c,int}(\varphi)$ can be rewritten as:

$$\varepsilon_{\mathcal{T},k}^{c,int}(\varphi) = \Bigg| \sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\rm int}^n} \frac{1}{k} \int_{nk}^{(n+1)k} \int_{K|L} \Big(\Phi_\kappa^+(t,x,u_K^n) \cdot n_{K|L} - \frac{1}{m(K|L)} \Phi_{K,L,\kappa}^{+,n}(u_K^n,u_K^n) \Big) (\varphi - \tilde{\varphi}_{K|L}^n)\, d\gamma(x)\, dt - \frac{1}{k} \int_{nk}^{(n+1)k} \int_{K|L} \Big(\Phi_\kappa^+(t,x,u_L^n) \cdot n_{K|L} - \frac{1}{m(K|L)} \Phi_{K,L,\kappa}^{+,n}(u_L^n,u_L^n) \Big) (\varphi - \tilde{\varphi}_{K|L}^n)\, d\gamma(x)\, dt \Bigg| .$$

Now, writing

$$\varphi(t,x) - \tilde{\varphi}_{K|L}^n = \frac{1}{k\, m(K|L)} \int_{nk}^{(n+1)k} \int_{K|L} \times (\varphi(t,x) - \varphi(s,y))\, d\gamma(y)\, ds \, , \tag{69}$$

the following majoration holds: for all $(t,s,x,y) \in [\,nk,(n+1)k[^2 \times (K|L)^2$

$$|\varphi(t,x) - \varphi(s,y)| \le (k+h) \Big|\Big| (|\nabla\varphi| + |\varphi_t|) \Big|\Big|_{L^\infty} . \tag{70}$$

Besides, the function f is Lipschitz continuous so that there exists $C \geq 0$, only depending on Ω, f, A, B, such that for all $s \in [A, B]$, for all $(K, L) \in \mathcal{E}_{int}$, for all $t \in [nk, (n+1)k[$ and all $x \in K|L$,

$$\left| f(t, x, s) \cdot n_{K|L} - \frac{1}{m(K|L)} F^n_{K,L}(s, s) \right| \leq C(h + k)\,.$$

Discussing the respective positions of u^n_K, u^n_L and κ, we deduce from this result the following estimate

$$\left| \Phi^+_\kappa(t, x, u^n_K) \cdot n_{K|L} - \frac{1}{m(K|L)} \Phi^{+,n}_{K,L,\kappa}(u^n_K, u^n_K) \right| \leq 2C(h + k)\,, \tag{71}$$

which is still true when u^n_K is replaced by u^n_L.

From (69), (70) and (71), is deduced the estimate

$$\begin{aligned} \varepsilon^{c,int}_{\mathcal{T},k}(\varphi) &\leq 2\,C \left|\left| (|\nabla \varphi| + |\varphi_t|) \right|\right|_{L^\infty} (h + k)^2 \sum_{n=0}^{N} \sum_{(K,L) \in \mathcal{E}^n_{\text{int}}} k\, m(K|L) \\ &\leq 2\,C \left|\left| (|\nabla \varphi| + |\varphi_t|) \right|\right|_{L^\infty} \frac{m(\Omega)}{\alpha^2}\, (T + 1)\, (h + k)^2 h^{-1}\,, \end{aligned}$$

the second inequality being a consequence of assumption (25) on the mesh. Eventually, the CFL condition (32) holding, we get

$$\varepsilon^{c,int}_{\mathcal{T},k}(\varphi) \overset{h \to 0}{\longrightarrow} 0\,.$$

We would do the same to get an estimate on $\varepsilon^{b,c}_{\mathcal{T},k}$ and $\tilde{\varepsilon}^{b,c}_{\mathcal{T},k}$, in order to prove that they are shrinking to zero when h does.

We now study the errors $\varepsilon^{\text{int},+}_{\mathcal{T},k}$ and $\varepsilon^{\text{int},-}_{\mathcal{T},k}$ defined by (60). Here, the weak BV estimate on space derivatives (34) is required. Indeed, we have

$$\begin{aligned} &\left| \Phi^{+,n}_{K,L,\kappa}(u^n_K, u^n_K) - \Phi^{+,n}_{K,L,\kappa}(u^n_K, u^n_L) \right| \\ &\qquad \leq \max_{u^n_L \leq c \leq d \leq u^n_K} \left(F^n_{K,\sigma}(d, c) - F^n_{K,\sigma}(d, d) \right) \end{aligned}$$

and, thanks to the integral Taylor formula, we get an estimate on the difference between the average value of φ on a controle volume and on one of its edge: there exists C_φ, depending only upon φ, such that

$$\forall (K, L) \in \mathcal{E}^n_{\text{int}}\,,\ |\varphi^n_K - \tilde{\varphi}^n_{K|L}| \leq\ C_\varphi\, (h + k)\,.$$

Therefore, the following estimate on $\varepsilon_{\mathcal{T},k}^{\text{int},+}(\varphi)$ holds

$$\begin{aligned}\varepsilon_{\mathcal{T},k}^{\text{int},+}(\varphi) \leq C_\varphi\,(h+k)\sum_{n=0}^{N} k \sum_{(K,L)\in\mathcal{E}_{\text{int}}^n} \\ \times \left[\begin{array}{l}\max\limits_{u_L^n\leq c\leq d\leq u_K^n}\left(F_{K,\sigma}^n(d,c)-F_{K,\sigma}^n(d,d)\right) \\ +\max\limits_{u_L^n\leq c\leq d\leq u_K^n}\left(F_{K,\sigma}^n(d,c)-F_{K,\sigma}^n(c,c)\right)\end{array}\right] \\ \leq C_\varphi\, C\,\frac{h+k}{\sqrt{h}}\,,\end{aligned}$$

where the constant C is given by (34). We would follow the same lines to get a similar estimate on $\varepsilon_{\mathcal{T},k}^{\text{int},-}(\varphi)$. Moreover, the continuity of the translations in L_{loc}^1 ensures that the quantities $\varepsilon_{\mathcal{T}}^0(\varphi)$ and $\varepsilon_{\mathcal{T},k}^b(\varphi)$ tend to zero when h does.

Convergence of the scheme

We come back to the discussion preceding the introduction of the entropy process solution (see Sect. 2): we know that $u_{\mathcal{T},k}$ is bounded in L^∞ (Lemma 5); the compacity result given in Theorem 1 proves that there exists $\mu \in L^\infty(Q\times(0,1))$ such that, up to a subsequence, for all $g\in\mathcal{C}(\mathbb{R})$,

$$g(u_{\mathcal{T},k}) \longrightarrow \int_0^1 g(\mu(.,\alpha))\,d\alpha \;\text{ in } L^\infty(Q)\; \text{ weak} - \star \;\text{ when } h\to 0\,.$$

Then, taking Theorem 3 into account, it is clear that μ is an entropy process solution to problem (1)-(2)-(3). Thus, the function μ does not depend on its third variable (Theorem 2): there exists $u\in L^\infty(Q)$ such that $\mu(t,x,\alpha) = u(t,x)$ for a.e. $(t,x,\alpha)\in Q\times(0,1)$. Consequently, the function u is a weak entropy solution to problem (1)-(2)-(3) and the whole sequence $(u_{\mathcal{T},k})$ converges to u in $L_{loc}^p(Q)$ for every $p\in[1,\infty[$, as proved by the following lemma.

Lemma 10 *Let $\mathcal{O}$ be a bounded borelian subset of $\mathbb{R}^m$ and let (v^n) be a bounded sequence of $L^\infty(\mathcal{O})$ such that there exists $v\in L^\infty(\mathcal{O})$ satisfying: for all $g\in\mathcal{C}(\mathbb{R})$,*

$$g(v^n)\longrightarrow g(v)\;\textit{ in } L^\infty(\mathcal{O})\;\textit{ weak} - \star\,.$$

Then, for all p such that $1\leq p<+\infty$, $v^n\longrightarrow v$ in $L^p(\mathcal{O})$.

Proof. The space $L^\infty(\mathcal{O})$ is continuously imbedded in the space $L^2(\mathcal{O})$; taking $g(x) = x$ we therefore have: $v^n \longrightarrow v$ in $L^2(\mathcal{O})$ weak-$\star$. Taking $g(x) = x^2$, we also prove $||v^n||_{L^2(\mathcal{O})} \to ||v||_{L^2(\mathcal{O})}$ and the Hilbert structure of the space $L^2(\mathcal{O})$ allows us to get the result when $p = 2$, then, using the fact that (v^n) is bounded in $L^\infty(\mathcal{O})$, for every p.

Eventually, we have proven the following results.

Theorem 4 *There exists a unique weak entropy solution to problem (1)-(2)-(3).*

Theorem 5 *Let $\alpha \in \mathbb{R}_+$ and $\xi \in (\,0, 1)$ be fixed. Assume that assumptions (25), (27) and (32) hold. Let $u_{\mathcal{T},k}$ be the numerical approximate solution of the problem (1)-(2)-(3) defined by (29), (30), (31). Then, for every p in $[\,1, +\infty[$, $u_{\mathcal{T},k}$ converges to the weak entropy solution to (1)-(2)-(3) in $L^p_{loc}(Q)$.*

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