The Model Theory of Stable Skew Braces

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Introduction

Skew braces are one of the main algebraic tools controlling the structure of a non-degenerate bijective set-theoretic solution of the Yang–Baxter equation from statistical mechanics. The aim of this talk is to present the basic model theory of skew braces under the additionnal hypothesis of stability.

In particular, we shall characterise ω-categorical stable and ω-stable skew braces, prove analogues of the Berline-Lascar and Hrushovski decompositions, and show that an arbitrary nilpotent skew sub-brace is contained in a definable one of the same nilpotency class.

The Yang-Baxter equation

Given a vector space V and a linear map $R: V \otimes V \to V \otimes V$, the Yang–Baxter equation states that

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R).$$

Named after Chen-Ning Yang and Rodney Baxter, it plays an important role in many areas of mathematics such as knot theory, braid theory, operator theory, Hopf algebras, quantum groups, 3-manifolds and the monodromy of differential equations.

In 1990, Drinfel'd posed the question of finding all *set-theoretic* solutions of this equation, where V is a set and $R: V \times V \to V \times V$.

Let $R(a, b) = (\lambda_a(b), \rho_b(a))$, with $\lambda_a, \rho_b: V \to V$, for all $a, b \in V$.

We say that R is *left/right non-degenerate* if λ_a/ρ_a is bijective for every $a \in V$, non-degenerate if it is both left and right non-degenerate, and degenerate otherwise.

Skew braces

Definition

A set B with two binary operations $+$ and $\circ$ and a constant 0 is a *skew brace* if

- $(B, 0, +)$ and $(B, 0, \circ)$ are groups (not necessarily abelian),
- a left quasi-distributive law holds:

$$a \circ (b + c) = a \circ b - a + a \circ c.$$

If P is a group-theoretic property, a skew brace is *of type* P if its additive group satisfies P. Skew braces of abelian type are *braces*.

When we use group-theoretic notation, we shall indicate by sub- or superscripts whether we consider them additively or multiplicatively, e.g. $[a, b]_+$, $Z^\circ(B)$, $C_B^+(X)$.

A skew brace is *trivial* if $a + b = a \circ b$ for all $a, b \in B$; it is *almost trivial* if $a + b = b \circ a$ for all $a, b \in B$.

Examples

- Every group can be considered as a trivial brace, or as an almost trivial brace.
- Skew braces form a variety, and hence are closed under direct products, ultraproducts, substructures and quotients.
- There is an algorithm to calculate all finite skew braces of a given size.
- Let $(B, +, \cdot)$ be a radical ring, i.e. for all $b \in B$ there is $c \in B$ with $c + b - c \cdot b = 0$. Put $a \circ b = a \cdot b + a + b$. Then $(B, +, \circ)$ is a brace.

Skew braces and the Yang-Baxter equation

Theorem (Guarnieri, Vendramin)

Let B be a skew brace. The map $r_B : B \times B \to B \times B$ given by

$$r_B(a, b) = (-a + a \circ b, (-a + a \circ b)^{-1} \circ a \circ b)$$

is a non-degenerate set-theoretic solution of the Yang-Baxter equation.

The *structure group* of a set-theoretic solution (X, r) is the group

$$G(X, r) = \langle X \mid xy(uv)^{-1} : r(x, y) = (u, v) \rangle.$$

Theorem (Smoktunowicz, Vendramin)

Let (X, r) be a non-degenerate solution. There is a unique skew brace structure on $G(X, r)$ such that

$$(\iota \times \iota)r = r_{G(X,r)}(\iota \times \iota),$$

where $\iota : X \to G(X, r)$ is the canonical map. It is universal.

Algebraic properties of skew braces

It follows from quasi-distributivity that for all $a \in B$ the map

$$\lambda_a : x \mapsto -a + a \circ x$$

is an additive automorphism, and the map

$$\lambda : a \mapsto \lambda_a$$

is a homomorphism from $(B, \circ)$ to $\mathrm{Aut}(B, +)$. We put

$$G(B) = (B, +) \rtimes (B, \circ).$$

In analogy with ring theory, a third binary operation (non-necessarily associative) is defined as follows:

$$a * b = \lambda_a(b) - b = -a + a \circ b - b.$$

In particular, B is trivial iff $B * B = 0$.

An easy computation shows that in $G(B)$ we have

$$\big[(0, a), (b, 0)\big] = (a * b, 0).$$

Definition

An additive subgroup I of a skew brace B is a *left ideal* if $\lambda_a(I) \subseteq I$ for all $a \in B$.
A left ideal is an *ideal* if it is additively and multiplicatively normal in B.

It is easy to see that an additive subgroup I is a left ideal iff $B * I \subseteq I$. Thus a left ideal is also a multiplicative subgroup, and $a + I = a \circ I$ for all $a \in B$. (This need *not* hold for right cosets.)

An additively normal left ideal I is an ideal iff $I * B \subseteq I$.

Ideals are precisely the kernels of skew brace homomorphisms, and analogues of the homomorphism theorems hold.

We shall call an ideal *trivial* if it is trivial as a skew brace.

The *socle* of B is defined as $\mathrm{Soc}(B) = \mathrm{Ker}(\lambda) \cap Z^+(B)$.

The *annihilator* of B is defined as $\mathrm{Ann}(B) = \mathrm{Soc}(B) \cap Z^\circ(B)$.

Both the socle and the annihilator are ideals.

Generic types

Let B be a stable skew brace, and p an additive generic type over B, realized by some element a. Then for any $b \in B$ the map λ_b is an additive automorphism, and maps p to another additive generic type. Therefore

$$b \circ p = \mathrm{tp}(b \circ a/G) = \mathrm{tp}(b + \lambda_b(a)) = b + \lambda_b(p)$$

is again an additive generic type, and p has only boundedly many multiplicative translates. Hence an additive generic type is also a multiplicative generic type.

Conversely, if p is a multiplicatively generic type, let $b \in B$ be such that $b \circ p$ is an additive generic type. So $b^{-1} + \lambda_{b^{-1}}(b \circ p)$ is an additive generic type. But if a realizes p, then

$$b^{-1} + \lambda_{b^{-1}}(b \circ p) = \mathrm{tp}(b^{-1} - b^{-1} + b^{-1} \circ (b \circ a)/G) = \mathrm{tp}(a/G) = p.$$

It follows that additive and multiplicative generic types coincide.

Connected components

Recall that in a stable group G, there is a unique minimal type-definable group of bounded index, the connected component G^0. It is the intersection of all definable subgroups of finite index, and definably characteristic. Moreover, a stable group is connected iff it has a unique generic type.

Theorem

Let I be a type-definable ideal in a stable skew brace B. Then I^0_+ is an ideal in B, and multiplicatively connected.

Proof.

I^0_+ is definably characteristic, whence additively normal and invariant under λ_b for all $b \in B$. So it is a left ideal in B, whence a skew subbrace. It has a unique additive generic type, which is also its unique multiplicative generic type. So I^0_+ is multiplicatively connected, and $I^0_+ = I^0_\circ =: I^0$. Since $I^0_\circ$ is multiplicatively definably characteristic, it is normal in B, and an ideal. $\square$

Definable hulls

As customary, given a subgroup H of a stable group G, the *definable hull* $\bar{H}$ of H is the intersection of all definable supergroups of H. Obviously, it is a type-definable group. In the skew brace context, if H is a subgroup (additive or multiplicative) of B, we indicate this by a superscript: $\bar{H}^+$ and $\bar{H}^\circ$.

If C is a skew sub-brace of B, it seems very difficult to obtain a definable hull which is a brace, since the lack of a right (quasi-) distributive law means that there is only little connection between the additive and the multiplicative hull. The same holds for ideals instead of skew subbraces.

On the other hand, if I is a *left ideal*, $\bar{I}^+$ is again a left ideal, since for any $b \in B$ we have $\lambda_b(I) = I$, so $\lambda_b(\bar{I}^+) = \bar{I}^+$.

However, there are two cases where we actually do get definable skew sub-braces / ideals, namely when C is a *component*, or when C is *trivial* (or more generally *nilpotent*).

Components

Let Φ be an invariant collection of partial types.

Fact

Let G be a stable group, g realize its principal generic type, and

$$g_\Phi = (g_0 \in \mathrm{dcl}(g) : \mathrm{tp}(g_0) \textit{ is } \Phi\textit{-internal}).$$

Then there is a unique normal type-definable subgroup N of G such that $\mathrm{dcl}(g_\Phi) = \mathrm{dcl}(gN)$. It does not depend on the choice of g, and is definably characteristic.

N is the intersection of all definable subgroups H such that G/H is Φ-internal.

Note that a priori N need not be connected.

Definition

We call N the Φ-*component* of G, denoted G^Φ.

Theorem

Let B be a connected stable skew brace, and Φ an invariant collection of partial types. Then the additive Φ-component B_+^Φ is an ideal in B.

Proof.

Since B_+^Φ is definably characteristic, it is additively normal and invariant under λ_b for all $b \in B$, whence a left ideal.
Thus B_+^Φ is a multiplicative subgroup, and the multiplicative quotient B/B_+^Φ is Φ-internal (since additive and multiplicative cosets are the same). But then $B_\circ^\Phi \leq B_+^\Phi$, and for a generic g we have

$$\mathrm{dcl}(gB_+^\Phi) = \mathrm{dcl}(g_\Phi) = \mathrm{dcl}(gB_\circ^\Phi).$$

As B is connected, this can only happen if $B_+^\Phi = B_\circ^\Phi =: B^\Phi$. It follows that B^Φ is multiplicatively normal, whence an ideal. $\square$

Note that if I is a connected type-definable ideal of B, then I^ϕ is additively and multiplicatively definably characteristic, whence an ideal in B.

Corollary (Berline-Lascar analysis)

If $U(B) = \sum_{i \leq k} \omega^{\alpha_i} \cdot n_i$ and $\Phi = \{p : U(p) < \omega^{\alpha_j}\}$, then $(B^0)^\Phi$ is the unique type-definable connected ideal of Lascar-rank $\sum_{i \leq j} \omega^{\alpha_i} \cdot n_i$.

Corollary (Hrushovski analysis)

Let π be a partial type such that a generic type of B is not foreign to π. Then there is a unique minimal type-definable ideal I of infinite index such that B/I is π-internal.

Again, I need not be connected.

Nilpotency

While nilpotency in a group is about how far the group law is from being commutative, and nilpotency in a ring is about how far multiplication is from being null, nilpotency in a skew brace is about how far multiplication is from addition, i.e. how far the brace is from being trivial.

Due to the lack of associativity of the $*$-product, and depending on whether we also require additive or multiplicative centrality, there are various kinds of nilpotency in skew braces:

1. Annihilator-nilpotency,
2. Socle-nilpotency,
3. Right nilpotency, and
4. Left nilpotency.

We have 1. $\Rightarrow$ 2. $\Rightarrow$ 3. and 1. $\Rightarrow$ 4.

Annihilator-nilpotency

We define the *upper annihilator series* as follows:

$$\begin{aligned} \mathrm{Ann}_0(B) &= (0), \\ \mathrm{Ann}_{n+1}(B) &= \{b \in B : b * B, [b, B]_+, [b, B]_\circ \subseteq \mathrm{Ann}_n(B)\}, \end{aligned}$$

and the *lower annihilator series* as:

$$\begin{aligned} \Gamma_0(B) &= B, \\ \Gamma_{n+1}(B) &= \langle \Gamma_n(B) * B, B * \Gamma_n(B), [B, \Gamma_n(B)]_+, [B, \Gamma_n(B)]_\circ \rangle_+. \end{aligned}$$

All of the above are ideals in B, and B is *annihilator-nilpotent of class n* if $\mathrm{Ann}_n(B) = B$ iff $\Gamma_n(B) = (0)$.

Socle-nilpotency

We define the *(upper) socle series* as follows:

$$\begin{aligned}\mathrm{Soc}_0(B) &= (0),\\ \mathrm{Soc}_{n+1}(B) &= \{b \in B : b * B, [b, B]_+ \subseteq \mathrm{Soc}_n(B)\},\end{aligned}$$

and the *lower socle series* as:

$$\begin{aligned}\Delta_0(B) &= B,\\ \Delta_{n+1}(B) &= \langle \Delta_n(B) * B, B * \Delta_n(B), [B, \Delta_n(B]_+\rangle_+.\end{aligned}$$

All of the above are ideals in B, and B is *socle-nilpotent of class* n if $\mathrm{Soc}_n(B) = B$ iff $\Delta_n(B) = (0)$.

Right and left nilpotency

We define two descending series $B^{(n)}$ and B^n as follows:

$$\begin{aligned} B^{(1)} &= B & B^1 &= B, \\ B^{(n+1)} &= \langle B^{(n)} * B \rangle_+ & B^{n+1} &= \langle B * B^n \rangle_+. \end{aligned}$$

While the $B^{(n)}$ are ideals, in general the B^n are only left ideals in B, and B^{n+1} is an ideal in B^n.

B is *right nilpotent of class n* if $B^{(n+1)} = (0)$, and *left nilpotent of class n* if $B^{n+1} = (0)$.

Theorem (Cedo, Smoktunowicz, Vendramin)

B is socle-nilpotent iff B is right nilpotent of nilpotent type.

ω-categorical stable skew braces

Theorem

Let B be an ω-categorical stable skew brace. Then B has an ideal I of finite index which is left nilpotent of nilpotent type; if B is ω-stable, then I is trivial.

Proof.

Put $I = B^0$, a connected ideal of finite index in B. We consider $G(I)$, a connected stable ω-categorical group. It must be nilpotent; if B (and I) are ω-stable, it is abelian.
In the latter case it follows immediately that I is trivial.
In the stable case, it suffices to note that in $G(I)$ for all n

$$I^n \times (0) = [(0) \times \gamma_n(I, \circ), (I, +) \times (0)] \leq \gamma_n G(I).$$

Since $\gamma_n G(I) = (1)$ for n sufficiently large, $I^n = (0)$. $\square$

Definable hulls of nilpotent skew sub-braces

Theorem

Let B be a stable skew brace, and C a skew sub-brace. Then C is contained in a definable skew sub-brace D such that:

1. *If C is annihilator-nilpotent of class n, so is D.*
2. *If C is socle-nilpotent of class n, so is D.*
3. *If C is right nilpotent of class n, so is D.*
4. *If C is left nilpotent of class n, so is D.*

Sketch of proof

Recall that if G is a group, A a subgroup and X a set, then $C_G(X/A) = \{g \in N_G(A) : [g, X] \subseteq A\}$, and $Z(G/A) = C_G(G/A)$.

We define a similar notion with respect to triviality rather than commutativity. (If A is trivial, it is omitted.)

Let B be a brace, A and H additive subgroups, G a multiplicative subgroup, and $X \subseteq B$ and $Y \subseteq \mathrm{Stab}_B(A)$ subsets.

$$\mathrm{Stab}_G(A) = \{h \in G : \lambda_h(A) = A\},$$
$$\mathrm{Fix}^l_G(X/A) = \{y \in \mathrm{Stab}_G(A) \ : \ \lambda_y(x) \in x + A \text{ for all } x \in X\},$$
$$\mathrm{Fix}^r_H(Y/A) = \{x \in N^+_H(A) \ : \ \lambda_y(x) \in x + A \text{ for all } y \in Y\}.$$

- $g \in \mathrm{Stab}_G(A)$ iff $g \in G$ and $g \circ A = g + A$.
- $\mathrm{Stab}_G(A)$ and $\mathrm{Fix}^l_G(X/A)$ are multiplicative subgroups of G.
- $\mathrm{Fix}^r_H(Y/A)$ is an additive subgroup of H.
- $y \in \mathrm{Fix}^l_B(x/A)$ iff $x \in \mathrm{Fix}^r_B(y/A)$, for $y \in \mathrm{Stab}_B(A)$, $x \in N^+_B(A)$.

Recall that

$$\operatorname{Ann}_{n+1}(C) = Z^{+}(C/\operatorname{Ann}_n(C)) \cap Z^{\circ}(C/\operatorname{Ann}_n(C)) \cap \operatorname{Fix}^{l}_{C}(C/\operatorname{Ann}_n(C)).$$

We define inductively $G_0 = H_0 = B$ and $A_0^* = A_0 = \{0\}$, and put

$$\begin{aligned}
G_{n+1} &= C^{\circ}_{G_n}(\operatorname{Ann}_{n+1}(C)/A_n) \cap \operatorname{Stab}_{G_n}(A_n),\\
H_{n+1} &= \operatorname{Fix}^{r}_{H_n}(\operatorname{Ann}_{n+1}(C)/A_n) \cap C^{+}_{H_n}(\operatorname{Ann}_{n+1}(C)/A_n),\\
A^*_{n+1} &= \operatorname{Fix}^{l}_{G_{n+1}}(H_{n+1}/A_n) \cap Z^{\circ}(G_{n+1}/A_n) \cap Z^{+}(H_{n+1}/A_n), \quad \text{and}\\
A_{n+1} &= \bigcap_{c \in C} \lambda_c(A'_{n+1}).
\end{aligned}$$

1. G_n is a multiplicative and H_n an additive subgroup.
2. $A_n \subseteq G_n \cap H_n$ and $C \subseteq G_n \cap H_n$.
3. A_n^* and A_n are skew sub-braces, annihilator-nilpotent of class n.
4. $A_n \cap C = \operatorname{Ann}_n(C)$.

In particular, if C is annihilator-nilpotent of class c, then $C \leq A_c$.

The proofs of the other cases are similar. □

Concluding remarks

We define inductively the *derived series* as

$$B_{(1)} = B, \qquad B_{(n+1)} = B^2_{(n)} = \langle B_{(n)} * B_{(n)} \rangle_+.$$

The skew brace B is *soluble of derived length* ℓ if $B_{(\ell+1)} = (0)$.

Questions

- Is every soluble skew sub-brace of a stable brace contained in a definable on of the same derived length?
- Is a type-definable skew sub-brace intersection of definable skew sub-braces?
- Is a type-definable skew brace contained in a definable skew brace?
- Can the results be generalized to other neostability hypotheses (simplicity, dependence, *o*-minimality, NSOP, pseudofiniteness,...)?

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Thank you

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Happy Birthday, Katrin !