Cours du 31 mars 2020
Exercise 3.3 : D given by $\varphi(\bar{x})$ and $\psi(\bar{x})$ in $\mathfrak{M}$. That is,

$$D = \varphi(\mathfrak{M}) = \{\bar{d} \in M^{|\bar{x}|} : \mathfrak{M} \models \varphi(\bar{d})\} = \psi(\mathfrak{M}) = \{\bar{d} \in M^{|\bar{x}|} : \mathfrak{M} \models \psi(\bar{d})\}.$$

Show that for $\mathfrak{M} \prec \mathfrak{N}$ we have

$$D' = \varphi(\mathfrak{N}) = \{\bar{d} \in N^{|\bar{x}|} : \mathfrak{N} \models \varphi(\bar{d})\} = \psi(\mathfrak{N}).$$

This is because

$$\mathfrak{M} \models \forall \bar{x}\, (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Since $\mathfrak{M} \prec \mathfrak{N}$, we have

$$\mathfrak{N} \models \forall \bar{x}\, (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Thus,

$$D' = \varphi(\mathfrak{M}) = \{\bar{d} \in N^{|\bar{x}|} : \mathfrak{N} \models \varphi(\bar{d})\} = \psi(\mathfrak{N}) = \{\bar{d} \in N^{|\bar{x}|} : \mathfrak{N} \models \psi(\bar{d})\}.$$

Exercise 3.14. $\pi(\bar{x})$ partial type over A, where $A \subset M$ for some model $\mathfrak{M}$. We want to complete it to $p(\bar{x}) \in S_n(A)$.
Enumerate all $\mathcal{L}(A)$-formulæ $(\varphi_i(\bar{x}) : i < \alpha)$ for some ordinal α. Put $\pi_0 = \pi$, and for all $i < \alpha$, put

$$\pi_{i+1} = \pi_i \cup \{\varphi_i\}$$

if this is (finitely) consistent, and otherwise

$$\pi_{i+1} = \pi_i \cup \{\neg\varphi_i\}.$$

At limits, take unions. These always remain (finitely) consistent.
Suppose $\pi_i \cup \{\varphi_i\}$ is not finitely consistent, say a finite bit $\pi_i^0(\bar{x}) \cup \{\varphi_i(\bar{x})\}$ is not consistent. Since π_i is consistent, there is $\bar{m}$ in M such that $\mathfrak{M} \models \pi_i(\bar{m})$. Then $\mathfrak{M} \not\models \varphi_i(\bar{m})$, so $\mathfrak{M} \models \neg\varphi_i(\bar{m})$. If there is a finite bit π_i^1 of π_i such that $\pi_i^1(\bar{x}) \cup \{\neg\varphi_i(\bar{x})\}$ is inconsistent, since π_i is consistent, there is $\bar{m} \in M$ such that

$$\mathfrak{M} \models (\pi_i^0 \cup \pi_i^1)(\bar{m}).$$

But then either $\mathfrak{M} \models \varphi_i(\bar{m})$ or $\mathfrak{M} \models \neg\varphi_i(\mathfrak{M})$, a contradiction in either case.
So we can always add either φ_i or $\neg\varphi_i$, and π_α is the completion we are looking for.
Proposition 3.17 : Every type can be realized in some elementary extension.
Just write down what that means :
Elementary extension = model $\mathfrak{N}$ of $\mathrm{Th}(\mathfrak{M}, M)$, where we identify $m \in M$ with $m^{\mathfrak{N}} \in N$. We want an element $\bar{n} = c^{\mathfrak{N}}$ in N realizing p, so we want $\mathfrak{N}$ to satisfy $p(\bar{c})$.
Example : $p(x) = \{\exists y\, p \cdot y = x :\ p \text{ a (standard) prime}\}$.
This is finitely consistent in $\mathbb{N}$. Hence there is $\mathbb{N} \prec \mathbb{N}^*$ and some element $n^* \in \mathbb{N}^*$ which realises this (partial) type.

Exercice 3.21. Suppose $D = \varphi(\mathfrak{M}, \bar{m})$, and for some $A \subseteq M$ and all $\mathfrak{M} \prec \mathfrak{N}$ and all automorphisms σ of $\mathfrak{N}$ fixing A, we have $\sigma[D'] = D' = \varphi(\mathfrak{N}, \bar{m})$. Then there is an $\mathcal{L}(A)$-formula $\psi(\bar{x}, \bar{a})$ defining D (in $\mathfrak{M}$ and hence D' in $\mathfrak{N}$).
Proof : Consider $p(\bar{y}) = \mathrm{tp}(\bar{m}/A)$.
Claim : $\mathrm{Th}(\mathfrak{M}, M) \cup p(\bar{y}) \models \forall \bar{x}(\varphi(\bar{x}, \bar{m}) \leftrightarrow \varphi(\bar{x}, \bar{y}))$.
Let $\mathfrak{M} \prec \mathfrak{N}$, and $\bar{n}$ in N realize p. Then $\mathrm{tp}_{\mathfrak{N}}(\bar{n}/A) = \mathrm{tp}_{\mathfrak{N}}(\bar{m}/A) = \mathrm{tp}_{\mathfrak{M}}(\bar{m}/A) = p$. Hence there is an elementary extenison $\mathfrak{N} \prec \mathfrak{N}'$ and an automorphism σ of $\mathfrak{N}'$ fixing A with $\sigma(\bar{m}) = \bar{n}$. Let $D' = \varphi(\mathfrak{N}', \bar{m})$. By assumption

$$\varphi(\mathfrak{N}', \bar{m}) = D' = \sigma[D'] = \varphi(\mathfrak{N}', \sigma(\bar{m})) = \varphi(\mathfrak{N}', \bar{n}).$$

By compactness, there is a finite bit of $p(\bar{y})$, say $\theta(\bar{y})$, such that

$$\mathrm{Th}(\mathfrak{M}, M) \cup \theta(\bar{y}) \models \forall \bar{x}(\varphi(\bar{x}, \bar{m}) \leftrightarrow \varphi(\bar{x}, \bar{y})).$$

Hence $\exists \bar{y}\, (\theta(\bar{y}) \wedge \varphi(\bar{x}, \bar{y}))$ defines D with parameters in A. QED

Section 3.3 : Quantifier Elimination
Note : $\vdash$ is the same as $\models$.
Why QE ? Because quantifier-free definable sets are easier to understand than general sets ; QE allows us to get a better hold on general definable sets.
Often, we do not have full QE, but just QE down to some set of "nice" formulas.
For instance, in the (noncommutative) free groups, every formula is equivalent to a boolean combination of $\forall\exists$-formulas.
QE for formulas is "equivalent" to qe for types, in the sense of Proposition 3.23.
(Note that the converse is obvious : if T has qe and $\mathrm{tpsq}_{\mathfrak{M}}(\bar{a}) = \mathrm{tpsq}_{\mathfrak{N}}(\bar{b})$ for models $\mathfrak{M}, \mathfrak{N}$ of T, then $\mathrm{tp}_{\mathfrak{M}}(\bar{a}) = \mathrm{tp}_{\mathfrak{N}}(\bar{b})$.)
Remark : What about types over parameters ? If every formula is equivalent to a b.c. of formulas from Φ, then a formula $\varphi(\bar{x}, \bar{y})$ is equivalent to a b.c. $\theta(\bar{x}, \bar{y})$ of formulas from Φ. So $\varphi(\bar{x}, \bar{a})$ is equivalent to $\theta(\bar{x}, \bar{a})$, and it is in some type $p \in S(A)$ iff $\theta(\bar{x}, \bar{a})$ is.

Proof of 3.23. Start with $\psi(\bar{x})$. If it is inconsistent with T, it is equivalent to any inconsistent formula. (Here we need Φ non-empty.)
Otherwise consider a model $\mathfrak{M} \models T$ and $\bar{a}$ in M realizing ψ. Put

$$\Sigma = T \cup \{\varphi(\bar{c}) : \varphi \in \Phi, \mathfrak{M} \models \varphi(\bar{a})\} \cup \{\neg\varphi(\bar{c}) : \varphi \in \Phi, \mathfrak{M} \models \neg\varphi(\bar{a})\} \cup \{\neg\psi(\bar{c})\}.$$

By hypothesis, this is inconsistent : If it were realized by $\mathfrak{N}$ and a tuple $\bar{b}$, then $\bar{a}$ in $\mathfrak{M}$ and $\bar{b}$ in $\mathfrak{N}$ realize the same formulas in Φ, and hence have the same type. As $\mathfrak{M} \models \psi(\bar{a})$ we must have $\mathfrak{N} \models \psi(\bar{b})$, a contradiction to Σ.
Hence there is a finite bit which is inconsistent. We get a finite b.c. $\varphi_{\bar{a}}(\bar{x})$ of Φ-formulas which implies ψ modulo T. Note that $\varphi_{\bar{a}}(\bar{a})$ holds.
We obtain such a $\varphi_{\bar{a}_i}$ for every $\bar{a}_i$ realising ψ in every model of T. This only depends on the type p_i of $\bar{a}_i$. Suppose these are $\{p_i : i \in I\}$. Then $\varphi_{\bar{a}_i} \in p_i$. Hence

$$T \cup \{\neg\varphi_{\bar{a}_i}(\bar{c}) : i \in I\} \cup \{\psi(\bar{c})\}$$

is inconsistent : if in some model $\mathfrak{M}'$ of T we have $\psi(\bar{c})$ for some tuple $\bar{c}$ in M', then $\bar{c} \models p_i$ for some $i \in I$, whence $\varphi_{\bar{a}_i}(\bar{c})$ is true.
Hence a finite bis is inconsistent. If this uses $\{\neg\varphi_{\bar{a}_i} : i \in I_0\}$, then

$$\bigvee_{i \in I_0} \varphi_{\bar{a}_i}(\bar{x})$$

is equivalent to $\psi(\bar{x})$. QED

Remark 3.28 Morleyization : QE depends on the language. We can always expand the language, in a definable way without increasing the collection of definable sets, in order to have qe.

Section 3.4. Algebraically closed fields.
Theorem (Tarski 1948, Chevalley in algebraic geometry). Algebraically closed fields have qe in the language of rings.
Take two tuples $\bar{a}$ and $\bar{b}$ which satisfy the same atomic formulas in algebraically closed fields K and L. Then $\bar{a}$ and $\bar{b}$ generate isomorphic fields : $P(\bar{a}) = Q(\bar{a})$ iff $P(\bar{b}) = Q(\bar{b})$. Replace K and L be elementary extension of infinite transcendence degree. Then partial isomorphisms of subfields form a back-an-forth system. Hence they are elementary, and $\bar{a} \mapsto \bar{b}$ is elementary. Thus $\mathrm{tp}_K(\bar{a}) = \mathrm{tp}_L(\bar{b})$. QED

Note : If T has qe and $\mathfrak{M}, \mathfrak{N} \models T$ with $\mathfrak{M} \subseteq \mathfrak{N}$, then $\mathfrak{M} \prec \mathfrak{N}$: If $\varphi(\bar{m}) \in \mathcal{L}(M)$ and $\mathfrak{M} \models \varphi(\bar{m})$, then there is quantifier-free $\psi(\bar{x})$ equivalent to $\varphi(\bar{x})$, and $\mathfrak{M} \models \psi(\bar{m})$. As quantifier-free formulas are preserved by sub-/superstructures, we get $\mathfrak{N} \models \psi(\bar{m})$, whence $\mathfrak{N} \models \varphi(\bar{m})$.
This yields lemma 3.36.

Theorem 3.37. (Hilbert's Nullstellensatz)
Note : The ideal generated by Q and J are all polynomials of the the form

$$TQ + S$$

for T some polynomial and $S \in J$.
Since J is prime, $K[X_1, \ldots, X_n]/J$ is a domain, and has a field of fractions L. Clearly $\alpha \mapsto \alpha + J$ embeds K into L.