Théorie des modèles

Feuille 7 – Espaces de types.

Soit T une théorie. On rappelle que

$$S_n(T) = \{\mathrm{tp}^{\mathcal{M}}(\bar{a}) \colon \mathcal{M} \models T,\ \bar{a} \in M^n\}. \tag{1}$$

Ce sont tous des types en les variables libres $\bar{x} = (x_0, \ldots, x_{n-1})$.

Remark :

– A complete theory = a (complete) 0-type

– $S_0(T)$ is the set of all completions of T (complete theories containing T).

Exercise : Let $\bar{x} = (x_0, \ldots, x_{n-1})$ be variables, and $\pi(\bar{x})$ a set of formulas with free variables $\bar{x}$. Say that π is *consistent* is there exists a structure M and tuple $\bar{a} \in M$ (of the right length) such that $M \vDash \pi(\bar{a})$.

If every finite $\pi_0 \subseteq \pi$ is consistent, then π is consistent.

Add n new constants $\bar{c} = (c_0, \ldots, c_{n-1})$. We get a language $L(\bar{c}) = L \cup \bar{c}$. Now $\pi(\bar{c})$ is a set of $L(\bar{c})$-stentences.

An $L(\bar{c})$-structure M' = an L-structure M + a tuple $\bar{a} \in M^n$ (how ? take $\bar{a} = \bar{c}^{M'}$)

And : $M' \vDash \pi(\bar{c})$ iff $M \vDash \pi(\bar{a})$.

Conclusion : $\pi(\bar{x})$ is consistent iff $\pi(\bar{c})$ is consistent.

This allows us to reduce “compactness with free variables” to “compactness for sentences”.

Exercice 1 (Topologie sur l'espace des types)**.** Pour une formule $\varphi(\bar{x})$ on pose

$$[\varphi(\bar{x})] = \big\{p(\bar{x}) \in S_n(T) : \varphi(\bar{x}) \in p(\bar{x})\big\}.$$

Parfois on omet le $\bar{x}$:

$$[\varphi] = \big\{p \in S_n(T) : \varphi \in p\big\}.$$

1. Montrer que la famille des ensembles $[\varphi]$ forme une base d'ouvert pour une topologie sur $S_n(T)$.
   Enough to show that is is closed under (finite) intersection : Indeed, $[\varphi] \cap [\psi] = [\varphi \wedge \psi]$
   Dans la suite, on munit toujours $S_n(T)$ de cette topologie.
2. Montrer que chaque $[\varphi]$ est ouvert-fermé.
   Indeed, $S_n(T) \setminus [\varphi] = [\neg\varphi]$ is open so $[\varphi]$ is closed.
3. Montrer que $S_n(T)$ est totalement discontinu.
   I.e., show that any two disinct types are separated by a clopen set : If p and q are distinct, then there is some φ such that $\varphi \in p \setminus q$, say. Then $p \in [\varphi]$ and $q \in [\neg\varphi]$.
4. Montrer que $S_n(T)$ est compact.
   Indeed let C be an open cover of $S_n(T)$, and we need to show that it admits a finite sub-cover. For this we may assume that C consists of *basic* open sets. So : every member of C is of the form $[\varphi]$.

WTS : C admits a finite sub-cover. Assume that there is no finite sub-cover. This means that for every finite $C_0 \subseteq C$:

$$S_n(T) \neq \bigcup_{[\varphi] \in C_0} [\varphi].$$

Equivalently (passage to the complement) :

$$\bigcap_{[\varphi] \in C_0} [\neg\varphi] \neq \emptyset.$$

Let p belong to the intersection. By definition there exists $M \vDash T$ and $\bar{a} \in M^n$ such that $p = \mathrm{tp}^M(\bar{a})$. In particular for every $[\varphi] \in C_0$ we have $\mathrm{tp}^M(\bar{a}) \in [\neg\varphi]$, i.e. $\neg\varphi \in \mathrm{tp}^M(\bar{a})$ i.e. $M \vDash \neg\varphi(\bar{a})$.

To sum up, for every finite $C_0 \subseteq C$ we found $M \vDash T$ and $a \in M^n$ such that $M \vDash \neg\varphi(\bar{a})$ for all $[\varphi] \in C_0$.

By compactness (with free variables) : there exist $M \vDash T$ and $a \in M^n$ such that $M \vDash \neg\varphi(\bar{a})$ for all $[\varphi] \in C$.

Let $p = \mathrm{tp}^M(\bar{a})$. Then $p \in S_n(T)$ but $p \in \bigcap_{[\varphi] \in C} [\neg\varphi]$ so $p \notin \bigcup C$, a contradiction.

5. Montrer que tout ouvert-fermé de $S_n(T)$ est de la forme $[\varphi]$ pour une formule $\varphi(\bar{x})$.

   Alternative statement : If X is any compact, totally disconnected top space, and B is a basis of clopen sets, closed under complement and intersection, then all clopen sets are in B.

   Proof : if $A \subseteq X$ is clopen, then it is compact (since closed), and a union of members of B (since open), so it is a finite union of members of B. But B is also closed under finite unions, so $A \in B$.

6. $[\varphi] \subseteq [\psi]$ iff ψ is a consequence of φ modulo T, i.e. :

$$T \vDash \forall \bar{x}(\varphi \to \psi)$$

   We may simply write $T \vDash \varphi \to \psi$

   Indeed, if $T \vDash \varphi \to \psi$ and $p \in [\varphi]$, then, say $p = \mathrm{tp}^M(\bar{a})$, where $M \vDash T$. Then $M \vDash \varphi(\bar{a})$ so $M \vDash \psi(\bar{a})$ so $\psi \in p$ and $p \in [\psi]$.

   If $T \nvDash \varphi \to \psi$ then there exists $M \vDash T$ and $\bar{a} \in M^n$ such that $M \vDash \varphi(\bar{a}) \wedge \neg\psi(\bar{a})$ so $p = \mathrm{tp}^M(\bar{a}) \in [\varphi] \setminus [\psi]$

7. If $p \in S_n(T)$ then $\{p\} = \bigcap_{\varphi \in p} [\varphi]$.

   Indeed $q \in \bigcap_{\varphi \in p} [\varphi]$ iff $p \subseteq q$. But if $p \neq q$ then there exists φ such that $\varphi \in p$ but $\neg\varphi \in q$ so p and q are incomparable. So $p \subseteq q$ implies $p = q$.

Homework : read about Boolean algebras and the Stone duality.
Then : the set of all formulas with free variables $\bar{x}$ MODULO the relation

$$T \vDash \forall \bar{x}(\varphi \leftrightarrow \psi)$$

(equivalence modulo T), is a Boolean algebra, and $S_n(T)$ is its Stone dual.

Exercice 2 (Type isolé)**.** Un type $p \in S_n(T)$ est dit *isolé* s'il existe une formule φ telle que $\varphi \in p$ et tout $\psi \in p$ est conséquence de φ modulo T (alors, φ *isole* p).

Montrer que $p \in S_n(T)$ est un type isolé si et seulement si c'est un point isolé de l'espace topologique $S_n(T)$.

Step I : p is isolated by φ iff $\{p\} = [\varphi]$. Proof : we know that $\{p\} = \bigcap_{\psi \in p}[\psi]$. So $\{p\} = [\varphi]$ iff $[\varphi] \subseteq [\psi]$ for all $\psi \in p$ iff every $\psi \in p$ is a consequence of φ modulo T.

Step II : p is topologically isolated iff $\{p\} = [\varphi]$ for some φ. Proof : right to left : the singleton $\{p\} = [\varphi]$ is open.

Left to right : If $\{p\}$ is open, then it is a union of basic open sets. So it is a basic open set. So it is equal to some $[\varphi]$.

[Might be on exam :]

Exercice 3 (Type algébrique)**.** Soit $\mathcal{M}$ une L-structure, $T = \mathrm{Th}(\mathcal{M})$ et $p \in S_n(T)$. Supposons que pour chaque extension élémentaire $\mathcal{N}$ de $\mathcal{M}$, il y a au plus un nombre fini de réalisations de p dans $\mathcal{N}$ (un tel type est dit algébrique).

1. Montrer qu'il existe une formule $\phi(\bar{x})$ dans $p(\bar{x})$ qui n'est satisfaite que par un nombre fini d'éléments dans $\mathcal{M}$.
2. Montrer que toute réalisation de p dans une extension élémentaire de $\mathcal{M}$ est déjà dans $\mathcal{M}$.
3. Soit $\phi(\bar{x}) \in p(\bar{x})$ ayant un nombre fini m de solutions dans $\mathcal{M}$, et telle que m est minimal. Montrer que ϕ isole p, c'est-à-dire que p est l'unique type de $S_n(T)$ contenant ϕ.
4. Et si au lieu d'un type complet $p(\bar{x})$ on avait un type *partiel* $\pi(\bar{x})$?

On rappelle qu'étant donné une structure $\mathcal{M}$ et $A \subseteq M$:

$$S_n(A) = \{\mathrm{tp}^{\mathcal{N}}(\bar{a}/A) \colon \mathcal{N} \succeq \mathcal{M},\ \bar{a} \in N^n\}. \tag{2}$$

Ceci dépend de $\mathcal{M}$ (et non seulement de A).

Nous avons un souci : la définition de $S_n(T)$ prend en compte tous les modèles de T, alors que pour des types au-dessus de A on ne considère que les extension élémentaires de $\mathcal{M}$.

Exercice 4 (Types avec et sans paramètres)**.** On rappelle que $L(A) = L \cup A$ (ou plus précisément $L \cup \{c_a : a \in A\}$, mais parfois on oublie cette distinction) et $T(A) = \mathrm{Th}_{L(A)}(\mathcal{M})$. Montrer que

$$S_n(A) = S_n\big(T(A)\big),$$

où $S_n\big(T(A)\big)$ est au sens de (1).

Easy direction : say $p \in S_n(A)$. Then there exists $N \succeq M$, and $\bar{b} \in N^n$ such that $p = \mathrm{tp}^N(\bar{b}/A)$.

We may view M as an L-structure, or as an $L(A)$-structure. As the latter : $M \vDash T(A)$. We have $N \succeq M$ as L-structures, and therefore also as $L(A)$-strctures. (Indeed : $N \succeq M$ as L-structures iff for all $\bar{m} \in M$:

$$M \vDash \varphi(\bar{m}) \iff N \vDash \varphi(\bar{m})$$

iff for all $\bar{m} \in M \cup A$:

$$M \vDash \varphi(\bar{m}) \iff N \vDash \varphi(\bar{m})$$

iff $N \succeq M$ as $L(A)$-structures.)

So $N \models T(A)$. Also $\operatorname{tp}_L^N(\bar{b}/A) = \operatorname{tp}_{L(A)}^N(\bar{b})$ (= all formulas with parameters in A which are true of $\bar{b}$ in N)

Conclusion $p = \operatorname{tp}_{L(A)}^N(\bar{b}) \in S_n\big(T(A)\big)$.

Conversely : assume $p \in S_n\big(T(A)\big)$. Then there exists an $L(A)$-structure N such that $N \models T(A)$ and $\bar{b} \in N^n$ such that $p = \operatorname{tp}_{L(A)}^N(\bar{b})$. No reason that $N \succeq M$!

We know that $N \equiv_{L(A)} M$ (same complete $L(A)$-theory $T(A)$). By a previous exercise there exists an ultrapower M^U and a map $f\colon N \to M^U$ which is an $L(A)$-elementary embedding.

Then $p = \operatorname{tp}_{L(A)}^N(\bar{b}) = \operatorname{tp}_{L(A)}^{M^U}(f(\bar{b})) = \operatorname{tp}_L^{M^U}(f(\bar{b})/A) \in S_n(A)$. (Keeping in mind that $M \preceq M^U$.)