Die böse Farbe

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A bad field is a field of finite Morley rank with a predicate for a proper divisible non-trivial subgroup. The question of the existence of such fields arose naturally in the study of groups of finite Morley rank, where a Borel subgroup might have the form

$$K^+ \rtimes T$$

for some $T \leq K^x$; if one could show that $T = K^x$, this would imply the existence of involutions (i.e. a finite Morley rank version of the Feit-Thompson Theorem), and more generally the existence of elements of any finite order.
Positive characteristic

In characteristic $p > 0$ the existence of a bad field implies that there are only finitely many $p$-Mersenne primes, i.e. primes of the form

$$\frac{p^n - 1}{p - 1},$$

which is generally believed to be false.

Moreover, the absolutely algebraic numbers form an elementary substructure; it is thus impossible to construct a bad field of positive characteristic by generic (Hrushovski amalgamation style) methods, since they cannot tell us anything about $acl(\emptyset)$.

It may, however, still be possible to construct a non-saturated generic structure (not of finite Morley rank), or a simple bad field of finite SU-rank.
I shall sketch the recent construction of a bad field of characteristic 0
It is obtained by collapsing Poizat’s green field of characteristic zero and Morley rank $\omega \cdot 2$ with a multiplicative subgroup of Morley rank $\omega$, following the ideas of the collapse of Poizat’s red field of positive characteristic and Morley rank $\omega \cdot 2$ with an additive subgroup of Morley rank $\omega$ by Baudisch, Martín Pizarro and Ziegler.
This is joint work with Andreas Baudisch, Martin Hils and Amador Martín Pizarro.
Let $K$ be an algebraically closed field of characteristic 0. A variety $V$ will be a subvariety of some $(K^\times)^n$. A torus is a connected algebraic subgroup of $(K^\times)^n$. It is given by equations of the form $x_1^{r_1} \cdot \ldots \cdot x_n^{r_n} = 1$. For tori, linear dimension (of a generic point over $\mathbb{Q}$, modulo torsion) equals algebraic dimension.

Given an irreducible variety $V$, its minimal torus is the smallest torus $T$ such that $V$ lies in some coset $\bar{a} \cdot T$. The codimension of $V$ is then

$$\text{cd}(V) := \dim(T) - \dim(V) = \text{lin.dim}_\mathbb{Q}(V) - \dim(V).$$

A subvariety $W \subseteq V$ is $\text{cd}$-maximal if $\text{cd}(W') > \text{cd}(W)$ for every subvariety $W \subsetneq W' \subseteq V$. Clearly, irreducible components of $V$ and tori cosets maximally contained in $V$ are examples of $\text{cd}$-maximal subvarieties.
Poizat used Zilber’s weak CIT, a consequence of Ax’ differential Schanuel conjecture:

For any uniform family $\mathcal{V}$ of varieties there is a finite set $\{T_0, \ldots, T_n\}$ of associated tori, such that for any torus $T$, any $V \in \mathcal{V}$ and any irreducible component $W \ni \bar{a}$ of $V \cap \bar{a} \cdot T$ there is some $i$ with $W \subseteq \bar{a} \cdot T_i$ and

$$\dim(T_i) - \dim(V \cap \bar{a} \cdot T_i) = \dim T - \dim W.$$ 

Moreover, the minimal torus of every $\text{cd}$-maximal subvariety of $V$ belongs to the collection $\{T_0, \ldots, T_r\}$. We will assume that the above tori are all distinct, and $T_0 = (K^\times)^n$ and $T_1 = \{1\}^n$. 
Consequences

1. Given $T$, the set of irreducible $V \in \mathcal{V}$ with minimal torus $T$ is definable. In particular, for any irreducible $V \in \mathcal{V}$ there is a definable neighbourhood where $\text{lin.dim}_\mathbb{Q}$ and $\text{cd}$ remain constant.

2. Suppose $V \in \mathcal{V}$ decomposes into $m$ irreducible components $W_k$ with $d_k := \dim(W_k)$, $l_k := \text{lin.dim}_\mathbb{Q}(W_k)$ and $c_k := \text{cd}(W_k)$. Then this holds in some definable neighbourhood of $V$. 
The class $\mathcal{C}$

Let $\mathcal{C}$ be the class of divisible hulls of finitely generated multiplicative subgroups of a field of characteristic 0, augmented by 0, and with a predicate $\bar{U}$ (German: grün) for a torsion-free multiplicative subgroup, such that for all finitely generated subgroups $A$

$$\delta(A) = 2 \text{tr.deg}(A) - \text{lin.dim}_\mathbb{Q}(\bar{U}(A)) \geq 0.$$ 

We shall consider structures in $\mathcal{C}$ in a relational (apart from multiplication) Morleyization of $\text{ACF}_0$; embeddings will be with respect to this language (i.e. will extend to the fields generated).

Using the weak CIT, Poizat has axiomatized $\mathcal{C}$:

1. $\bar{U}(M)$ is a torsion-free divisible multiplicative subgroup.
2. For every $\emptyset$-definable variety $V(\bar{x})$ of dimension $n$ with $|\bar{x}| = 2n + 1$, any $\bar{a} \in V \cap \bar{U}(M)$ lies in some associated torus. (This uses that $\bar{U}$ is torsion-free.)
Let $\tilde{C}$ be the class of structures whose finitely generated substructures are in $C$, and for $\langle \tilde{a}B \rangle \in \tilde{C}$ put

$$
\delta(\tilde{a}/B) = 2 \text{tr. deg}(\tilde{a}/B) - \text{lin. dim}_Q(\bar{U}(\langle \tilde{a}B \rangle)/\bar{U}(\langle B \rangle)).
$$

Properties:

1. $\delta(\tilde{a}\tilde{b}/C) = \delta(\tilde{b}/C) + \delta(\tilde{a}/\tilde{b}C)$.
2. Submodularity: $\delta(\tilde{a}/B) \leq \delta(\tilde{a}/B \cap \langle C\tilde{a} \rangle)$ for $C \subseteq B$.
3. Let $W$ be the locus of $\tilde{a}$ over $\text{acl}(B)$. Then
$$
\delta(\tilde{a}/\text{acl}(B)) = \dim(W) - \text{cd}(W).
$$
4. In general,
$$
\delta(\tilde{a}/B) = \dim(W) - \text{cd}(W) - \text{lin. dim}_Q(\langle \tilde{a}B \rangle \cap \text{acl}(B)/B).
$$
Strong embeddings

Given $A \subseteq B \in \tilde{C}$, we say that $A$ is strong in $B$, denoted $A \leq B$, if $\delta(\bar{b}/A) \geq 0$ for every $\bar{b} \in B$.

1. If $C \leq M$ and $C' \leq M$, then $C \cap C' \leq M$.

2. For every $A \subseteq M$ there exists a unique $A \subseteq C = \langle C \rangle \leq M$ minimal such. We call such a set the (strong) closure of $A$ (in $M$) and denote it by $\text{cl}_M(A)$.

3. If $(A_i)_{i < \alpha}$ is an increasing sequence with $A_i \leq K$ for all $i < \alpha$, then $\bigcup_i A_i \leq M$.

The class $C$ is countable up to isomorphism, and has AP and JEP with respect to strong embeddings. Let $M_\omega$ be its Fraïssé-Hrushovski limit. Using the weak CIT, Poizat has axiomatized its theory $T_\omega$, and shown that $M_\omega$ is $\omega$-saturated. It follows that $RM(M_\omega) = \omega \cdot 2$, and $RM(\bar{U}(M_\omega)) = \omega$. 
Pre-algebraicity

- Let $A \subseteq B \in \mathcal{C}$ with $\text{lin.dim}_Q(B/A) = n \geq 2$. The extension $B/A$ is minimal prealgebraic of length $n$ if $\delta(B/A) = 0$ and $\delta(B'/A) > 0$ for every $A \subsetneq B' \subsetneq B$ (or equivalently, if $\delta(B/B') < 0$).

- Let $B \subseteq \mathcal{C}$. A strong ACF$_0$-type $p(\bar{x}) \in S_n(B)$ is minimal prealgebraic, if the extension $\langle B\bar{a} \rangle /\langle B \rangle$ is minimal prealgebraic of length $n$ for some $\bar{a} |\!- p$ with $\bar{U}(\bar{a})$. In particular, $\bar{a}$ is multiplicatively independent over $B$. This is invariant under parallelism and multiplicative translation.

- An ACF$_0$-formula $\varphi(\bar{x})$ of Morley degree 1 is minimal prealgebraic if its generic type is minimal prealgebraic.

Note that if $B/A$ is minimal prealgebraic and $\bar{b}$ is a multiplicative green basis of $B$ over $A$, then $\text{stp}(\bar{b}/A)$ is minimal prealgebraic.
Minimal extensions

If $A \leq B$ in $\mathcal{C}$ with $\text{lin.dim}_\mathbb{Q}(B/A) < \infty$, we can find a decomposition $A = A_0 \leq A_1 \leq \ldots \leq A_{n-1} \leq A_n = B$ such that $A_{i+1}/A_i$ is minimal strong for all $i < n$.

Let $A \leq B$ be minimal strong. There are four possibilities:

1. **algebraic**: $\bar{U}(A) = \bar{U}(B)$ and $B = \langle Ab \rangle$ for some $b \in \text{acl}(A) \setminus A$. Then $\delta(B/A) = 0$.

2. **white generic**: $\bar{U}(A) = \bar{U}(B)$ and $B = \langle Ab \rangle$ for some element $b$ transcendental over $A$. Then $\delta(B/A) = 2$.

3. **green generic**: $B$ contains a basis consisting of a green singleton $b$ over $A$. Moreover, $b$ is transcendental over $A$ and $\delta(B/A) = 1$.

4. **minimal prealgebraic**: $A \leq B$ is minimal prealgebraic, i.e. $B$ contains a green basis $\bar{b}$ over $A$ such that $\text{stp}(\bar{b}/A)$ is minimal prealgebraic. Then $\delta(B/A) = 0$. 
A code is an $\text{ACF}_0$-formula $\varphi(\bar{x}, \bar{y})$ with $n_\varphi = |\bar{x}|$ such that

1. For all $\bar{b}$ either $\varphi(\bar{x}, \bar{b})$ is empty, or has Morley degree 1.
2. $RM(\bar{a}/\bar{b}) = n_\varphi/2$ and $\text{lin.dim}_Q(\bar{a}/\bar{b}) = n_\varphi$ for generic $\bar{a} \models \varphi(\bar{x}, \bar{b})$.
3. Let $T_0, T_1, \ldots$ be the tori associated to the Zariski closure $V$ of $\varphi(\bar{x}, \bar{b})$. For any $\bar{a} \models \varphi(\bar{x}, \bar{b})$, any $i = 2, \ldots, r$ and any irreducible component $W$ of $V \cap \bar{a} \cdot T_i$ of maximal dimension, $\dim(T_i) > 2 \cdot \dim(W)$ if $V \cap \bar{a} \cdot T_i$ is infinite.
4. If $RM(\varphi(\bar{x}, \bar{b}) \cap \varphi(\bar{x}, \bar{b}')) = n_\varphi/2$, then $b = b'$.
5. For any invertible $\tilde{m}$ and $\tilde{b}$ there is $\tilde{b}'$ with $\varphi(\bar{x} \cdot \tilde{m}, \bar{b}) \equiv \varphi(\bar{x}, \bar{b}')$. 
Let $\varphi(\bar{x}, \bar{y})$ be a code, and suppose $\varphi(\bar{x}, \bar{b})$ is non-empty.

1. If $\bar{a} \models \varphi(\bar{x}, \bar{b})$ is generic over $B \ni \bar{b}$ and green, then the extension $B \subseteq \langle B\bar{a} \rangle$ is minimal prealgebraic.

2. For all green $\bar{a} \models \varphi(\bar{x}, \bar{b})$ and $B \ni \bar{b}$
   - $\delta(\bar{a}/B) \leq 0$.
   - If $\delta(\bar{a}/B) = 0$, either $\bar{a} \in \langle B \rangle$ or $\bar{a}$ is generic in $\varphi(\bar{x}, \bar{b})$ over $B$.

3. $\bar{b}$ is the canonical parameter for $\varphi(\bar{x}, \bar{b})$.

Every minimal prealgebraic extension gives rise to some code.
GLₙ(ℚ) acts on the codes. Since this group is infinite, we cannot put invariance under GLₙ(ℚ) into the axioms, but have to deal with it externally. Using the weak CIT we obtain:

Let \( \varphi \) and \( \psi \) be codes. There is a finite set \( G(\varphi, \psi) \) of tori such that if \( \varphi(\bar{x}, \bar{b}) \neq \emptyset \) and \( T \cap (\varphi(\bar{x}, \bar{b}) \times \psi(\bar{x}, \bar{b}')) \) projects generically onto \( \varphi(\bar{x}, \bar{b}) \) and \( \psi(\bar{x}, \bar{b}') \) for some torus \( T \) with \( \dim(T) = |\bar{x}| \) (a toric correspondence), then \( T \in G(\varphi, \psi) \).

There exists a collection \( S \) of codes such that for every minimal prealgebraic definable set \( X \) there is a unique code \( \varphi \in S \) and finitely many tori \( T \) such that \( T \) induces a toric correspondence between \( X \) and some instance of \( \varphi \).
Proof

Suppose $T$ induces a toric correspondence between $\varphi(\bar{x}, \bar{b})$ and $\psi(\bar{x}, \bar{b}')$. Let $\mathcal{T}$ be the finite family of tori associated to the Zariski closure $V$ of $\varphi(\bar{x}, \bar{b}) \times \psi(\bar{x}, \bar{b}')$, and put $B = \text{acl}(\bar{b}\bar{b}')$. Choose some $B$-generic point $(\bar{a}, \bar{a}') \in V \cap \mathcal{T}$. Let $W \subseteq V \cap \mathcal{T}$ be the locus of $(\bar{a}, \bar{a}')$ over $B$. Then $T$ is the minimal torus of $W$, and $\dim(W) = \text{cd}(W) = n_\varphi/2$.

Suppose $W \subsetneq W' \subseteq V$ and $(\bar{g}, \bar{g}')$ is $B$-generic in $W'$.

\[
\text{cd}(W') = \text{lin.dim}_\mathbb{Q}(\bar{g}, \bar{g}'/B) - \text{tr.deg}(\bar{g}, \bar{g}'/B) \\
= [\text{lin.dim}_\mathbb{Q}(\bar{g}/B) - \text{tr.deg}(\bar{g}/B)] \\
+ [\text{lin.dim}_\mathbb{Q}(\bar{g}'/B\bar{g}) - \text{tr.deg}(\bar{g}'/B\bar{g})] \\
= \text{cd}(W) + \text{lin.dim}_\mathbb{Q}(\bar{g}'/B\bar{g}) - \text{tr.deg}(\bar{g}'/B\bar{g}) \\
> \text{cd}(W) - \delta(\bar{g}'/B\bar{g}) \geq \text{cd}(W),
\]

since $W \subsetneq W'$ implies $\text{tr.deg}(\bar{g}'/B\bar{g}) > 0$, and $\bar{g}'$ realizes the code $\psi(\bar{x}, \bar{b}')$. So $W \subseteq V$ is $\text{cd}$-maximal, and $G(\varphi, \psi) \subseteq \mathcal{T}$. 
The collection $S$ can now be constructed recursively. List all minimal prealgebraic subsets $(X_i : i < \omega)$ up to isomorphism. Suppose that $S_i$ has been already defined encoding all $X_j$ for $j < i$. If $X_i$ can be encoded by some element in $S_i$ and some torus $T$, then set $S_{i+1} = S_i$. Otherwise $X_i$ is equivalent to some code instance $\varphi(\bar{x}, \bar{b})$. Put

$$
\rho(\bar{z}) := \forall \bar{y} \left( \bigwedge_{\psi \in S_i} \bigwedge_{T \in G(\psi, \varphi)} \neg \chi_{\psi, \varphi}^T(\bar{y}, \bar{z}) \right),
$$

where $\chi_{\psi, \varphi}^T(\bar{b}, \bar{b}')$ expresses that $T$ induces a toric correspondence between $\psi(\bar{x}, \bar{b})$ and $\varphi(\bar{x}, \bar{b}')$. Then $S_{i+1} := S_i \cup \{ \varphi(\bar{x}, \bar{z}) \land \neg \rho(\bar{z}) \}$ will do.
For a code $\varphi$ and some $\bar{b}$ consider a generic Morley sequence $(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_k, f)$ for $\varphi(\bar{x}, \bar{b})$, and put $\bar{e}_i = \bar{a}_i \cdot f^{-1}$. We can then find a formula $\psi^k_\varphi \in \text{tp}(\bar{e}_0, \ldots, \bar{e}_k)$ such that

1. $\psi^k_\varphi$ implies $\psi^{k'}_\varphi$ for all $k' < k$.
2. $\psi^k_\varphi$ is invariant under the finite group of derivations generated by $\partial_i : \bar{x}_j \mapsto \begin{cases} \bar{x}_j \cdot \bar{x}_i^{-1} & \text{if } j \neq i \\ \bar{x}_i^{-1} & \text{if } j = i \end{cases}$.
3. Any realization $(\bar{e}'_0, \ldots, \bar{e}'_k)$ of $\psi^k_\varphi$ is disjoint, and $\models \varphi(\bar{e}'_i, \bar{b}')$ for some unique canonical parameter $\bar{b}'$ definable over any $m_\varphi$ elements among the $\bar{e}'_i$.
4. If $\bar{e}'_i$ is generic and there is some toric correspondence $T$ on $\varphi$ with $(\bar{e}_j, \bar{e}') \in T$ for some $i \neq j$ and $\bar{e}'$, then $\bar{e}_i \not\models_{\bar{b}} \bar{e}' \cdot \bar{e}_i^{-1}$.

A difference sequence for $\varphi$ of length $k$ is a realization of $\psi^k_\varphi$.
Proof

Consider the following type-definable property $\Sigma(\bar{e}_0, \ldots, \bar{e}_\lambda)$: there exist $\bar{b}'$ and a Morley sequence $\bar{e}'_0, \ldots, \bar{e}'_\lambda, \bar{f}$ in $\varphi(\bar{x}, \bar{b}')$ with $\bar{e}_i = \bar{e}'_i \cdot \bar{f}^{-1}$.

$\Sigma$ satisfies 1.–3. Now $(\bar{e}_i : i \leq \lambda)$ is a Morley sequence over $\bar{b}'\bar{f}$. If $\bar{e}_i$, $\bar{e}_j$, $\bar{e}'$ and $T$ are as in 4., then $\bar{e}' \in \text{acl}(\bar{e}_j)$, so $\bar{e}' \perp_{\bar{b}'\bar{f}} \bar{e}_i$. If $\bar{e}_i \perp_{\bar{b}'\bar{f}} \bar{e}' \cdot \bar{e}_i^{-1}$, then $\bar{e}_i^{-1}$, $\bar{e}'$ and $\bar{e}' \cdot \bar{e}_i^{-1}$ will determine a pairwise $\bar{b}'\bar{f}$-independent triple.

By a Lemma of Ziegler all three are generic types for cosets of some torus $T$. A contradiction, since by code property 2.

$$0 \geq \delta(\bar{e}_i/\bar{b}'\bar{f}) = \delta(T) = \text{dim}(T).$$

Let $\psi_0 \in \Sigma$ imply properties 1.,3. and 4., and put

$$\psi_\varphi^k(\bar{x}_0, \ldots, \bar{x}_k) := \bigwedge_{\partial \text{ derivation}} \psi_0(\partial(\bar{x}_0, \ldots, \bar{x}_\lambda)).$$
A counting Lemma

Given a code $\varphi$ and natural number $n$, there is some $\lambda = \lambda_\varphi(n)$ such that for every $M \leq N \in C$ and difference sequence $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ in $N$ with canonical parameter $\bar{b}$, either

- the canonical parameter for some derived sequence lies in $M$, or
- the sequence $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ contains a generic subsequence over $M\bar{b}$ of length $n$.

Suppose the first part does not hold. Put

$$X_1 = \{ i \in [m_\varphi, \lambda] : \bar{e}_i \text{ generic over } M \cup \{ \bar{e}_0, \ldots, \bar{e}_{i-1} \} \},$$
$$X_2 = \{ i \in [m_\varphi, \lambda] : \bar{e}_i \subseteq \langle M \cup \{ \bar{e}_0, \ldots, \bar{e}_{i-1} \} \rangle \},$$
$$X_3 = [m_\varphi, \lambda] \setminus (X_1 \cup X_2).$$

We may assume that $X_1 < X_3 < X_2$. 
Bounding $|X_3|$

Note that

\[
\delta(\bar{e}_i/M, \bar{e}_0, \ldots, \bar{e}_{i-1}) \leq -1 \quad \text{for } i \in X_3 \quad \text{and}
\]

\[
\delta(\bar{e}_i/M, \bar{e}_0, \ldots, \bar{e}_{i-1}) = 0 \quad \text{for } i \in X_1 \cup X_2.
\]

Since $M \leq N$

\[
0 \leq \delta(\bar{e}_0, \ldots, \bar{e}_\lambda/M)
\]

\[
\leq \delta(\bar{e}_0, \ldots, \bar{e}_{m_\varphi-1}/M) + \sum_{i=m_\varphi}^{\lambda} \delta(\bar{e}_i/M, \bar{e}_0, \ldots, \bar{e}_{i-1})
\]

\[
\leq m_\varphi n_\varphi + (-1)|X_3|,
\]

whence $|X_3| \leq m_\varphi n_\varphi$.  

$\delta(\bar{e}_i/M, \bar{e}_0, \ldots, \bar{e}_{i-1})$
Bounding $|X_2|$ 

Put $r = m_\varphi + |X_1| + |X_3|$ and $s = r(n_\varphi + 1)$. For simplicity, assume that there are varieties $V, W$ with $\psi_\varphi = V \setminus W$, and let $\mathcal{I}$ be the family of tori associated to $V$. Let $I \subset [r, \lambda]$ of cardinality $rn_\varphi + 1$; for simplicity assume $I = [r, s]$. Let $W'$ be the locus of $(\bar{e}_0, \ldots, \bar{e}_s)$ over $\text{acl}(M)$, and choose $W' \subseteq W'' \subseteq V$ maximal with $\text{cd}(W'') \leq \text{cd}(W')$.

By construction $W''$ is $\text{cd}$-maximal, so its minimal torus is some $T \in \mathcal{I}$. Fix some $\bar{m} \in \text{acl}(M)$ with $W'' \subseteq \bar{m}T \cap V$. Choose $(\bar{a}_0, \ldots, \bar{a}_s)$ a generic point of $W''$ over $\text{acl}(M)$ and paint it green. It lies in $V \setminus W$, since $(\bar{e}_0, \ldots, \bar{e}_s)$ is a specialization of $(\bar{a}_0, \ldots, \bar{a}_s)$, so $\models \psi_\alpha(\bar{a}_0, \ldots, \bar{a}_s)$. 


Bounding $|X_2|$

\[
\begin{align*}
\text{rn}_\varphi & \geq \text{lin. dim}_Q(\bar{e}_{<r}/M) = \text{lin. dim}_Q(\bar{e}_{\leq s}/M) \geq \text{cd}(W') \geq \text{cd}(W'') \\
& = \sum_{i \leq s} \text{lin. dim}_Q(\bar{a}_i/\tilde{M}, \bar{a}_{<i}) - \text{tr.deg}(\bar{a}_i/\tilde{M}, \bar{a}_{<i}) \\
& \geq \sum_{r \leq i \leq s} \text{lin. dim}_Q(\bar{a}_i/\tilde{M}, \bar{a}_{<i}) - \text{tr.deg}(\bar{a}_i/\tilde{M}, \bar{a}_{<i}).
\end{align*}
\]

By property 2. of codes $\delta(\bar{a}_i/\tilde{M}, \bar{a}_{<i}) \leq 0$ for $i \geq r \geq m_\varphi$, so

\[
2 \text{ tr.deg}(\bar{a}_i/\tilde{M}, \bar{a}_{<i}) \leq \text{lin. dim}_Q(\bar{a}_i/\tilde{M}, \bar{a}_{<i}).
\]

Hence, if $\bar{a}_i \not\in \langle \tilde{M}, \bar{a}_{<i} \rangle$ then

\[
\text{lin. dim}_Q(\bar{a}_i/\tilde{M}, \bar{a}_{<i}) - \text{tr.deg}(\bar{a}_i/\tilde{M}, \bar{a}_{<i}) \geq 1.
\]

Therefore, there is some $t \in \{r, \ldots, s\}$ with $\bar{a}_t \in \langle \tilde{M}, \bar{a}_{<t} \rangle$. 

Bounding $|X_2|$ 

The linear dependence will be determined by the coset $\bar{m}T$. So $\bar{m}T$ also determines that $\bar{e}_t \in \langle \tilde{M}, \bar{e}_{<t} \rangle$. Consider now all possible pairs $(t, T)$. This determines a $(rn_\varphi + 1)|T|$-coloring of all $(rn_\varphi + 1)$-subsets of $\{r, \ldots, \lambda\}$. By the (finite) Ramsey theorem, there is some number $\lambda_0$, such that for $\lambda \geq \lambda_0$ there is a monochromatic subset $I \subseteq \{r, \ldots, \lambda\}$ of cardinality $|I| \geq m_\alpha + rn_\alpha + 1$.

Thus for some $t \in \{r, \ldots, s\}$ and some $T \in \mathcal{T}$

$$\bar{e}_{i_t} \in \langle \tilde{M}, \bar{e}_{<r}, \bar{e}_{i_r}, \ldots, \bar{e}_{i_{t-1}} \rangle,$$

for all $i_r < \cdots < i_s$ in $I$, and the linear dependence comes from some $\bar{m}T_j$ with $\bar{m} \in \tilde{M}$.

Let $\gamma_i$ be the $(t + i)^{th}$ element in $I$. For $i > 0$ we have that $\bar{e}_{\gamma_i} \bar{e}_{\gamma_0}^{-1} \in \tilde{M}$, so the canonical parameter of the derived sequence lies in $\tilde{M}$, a contradiction.
Let $\mu^*$, $\mu$ be finite-to-one functions from $S$ to $\omega$ with

$$\mu^*(\varphi) \geq \max\left\{ \frac{n_{\varphi}^2}{2} + 1, \lambda_{\varphi}(m_{\varphi} + 1) \right\} \quad \text{and} \quad \mu(\varphi) \geq \lambda_{\varphi}(\mu^*(\varphi)).$$

Let $\mathcal{C}_\mu$ be the class of $M \in \mathcal{C}$ such that no $\varphi \in S$ has a (green) difference sequence in $M$ of length $> \mu(\varphi)$. The class $\mathcal{C}_\mu$ is universally axiomatizable relative to $ACF_0$. 
Let $M \leq M'$ be minimal prealgebraic, $M \in C_\mu$ but $M' \notin C \setminus C_\mu$, as witnessed by some difference sequence $(\bar{e}_0, \ldots, \bar{e}_\mu(\varphi))$ for some code instance $\varphi(\bar{x}, \bar{b})$. If $\bar{b} \in \text{acl}(M)$, there is an $M$-generic $\bar{e}_i$ generating $M'$ over $M$, and $\bar{e}_j \in M$ for $j \neq i$.

We may assume that $M$ is algebraically closed. As $M \leq M'$, any $\bar{e}_j \notin M$ is $M$-generic. Since $M \in C_\mu$ there must be some generic $\bar{e}_i$, and $M' = \langle M\bar{e}_i \rangle$ by minimality.

Suppose $\bar{e}_j$ is also $M$-generic. Since $M' = \langle M\bar{e}_i \rangle = \langle M\bar{e}_j \rangle$ there is $\bar{m} \in M$ and a toric correspondence $T \ni (\bar{e}_i \cdot \bar{m}, \bar{e}_j)$.

Let $\bar{e}_j' := \bar{e}_i \cdot \bar{m}$, so $\bar{e}_j' \cdot \bar{e}_i^{-1} \in M$. Since $\bar{e}_i$ is $M$-generic,

$$\bar{e}_i \downarrow \bar{e}_j' \cdot \bar{e}_i^{-1},$$

contradicting property 4. of a difference sequence.
Let $M \leq M' \in \mathcal{C}$ be minimal, $M \in C_\mu$. If $\text{lin.dim}_\mathbb{Q}(M'/M) = 1$, then $M' \in C_\mu$. Otherwise $M'/M$ is minimal prealgebraic, and $M' \notin C_\mu$ iff there is $\varphi \in S$ and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu}(\varphi))$ for $\varphi$ in $M'$ with canonical parameter $\bar{b}$, s.t.

1. $\varphi$ is unique, $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1} \in M$ and $\langle M, \bar{e}_{\mu}(\varphi) \rangle = M'$, or
2. there is a subsequence of length $\mu^*(\varphi)$ which is a Morley sequence for $\varphi(\bar{x}, \bar{b})$ over $M\bar{b}$.

$M' \notin C_\mu$ yields a generic realisation $\bar{e}$ of some code $\varphi(\bar{x}, \bar{b})$ over $M\bar{b}$, and $\text{lin.dim}_\mathbb{Q}(M'/M) \geq \text{lin.dim}_\mathbb{Q}(\bar{e}/M\bar{b}) \geq 2$.

1. or 2. imply $M' \notin C_\mu$. Conversely, if $M' \notin C_\mu$ we obtain a long difference sequence for a code $\varphi \in S$; if 1. does not hold, the counting Lemma yields 2. If $\varphi'$ is a second such code, $\langle M\bar{e} \rangle = M' = \langle M\bar{e}' \rangle$ yields a toric correspondence in $G(\varphi, \varphi')$, whence $\varphi = \varphi'$ by construction of $S$. 
Suppose \( B \leq A \) and \( B \leq C \) are in \( C_\mu \) and minimal, and the free amalgam \( M' \) is not in \( C_\mu \). Then both extensions are prealgebraic, and there is \( \varphi \in S \) and a difference sequence \((\bar{e}_0, \ldots, \bar{e}_\mu(\varphi)) \subset M'\) for \( \varphi \) in \( M' \) with canonical parameter \( \bar{b} \).

If (after derivation) \( \bar{b} \notin \text{acl}(A) \cup \text{acl}(C) \) we obtain a pairwise independent triple, a contradiction.

So wlog \( \bar{b} \subseteq A \), \( \bar{e}_i \subseteq A \) for \( i < \mu(\varphi) \), \( \bar{e}_\mu(\varphi) \) is \( A \)-generic and \( M' = \langle A\bar{e}_\mu(\varphi) \rangle \); write \( \bar{e}_\mu(\varphi) = \bar{a} \cdot \bar{c} \) for some \( \bar{a} \subseteq A \) and \( \bar{c} \subseteq C \).

Suppose (after derivation) \( \bar{b} \subseteq C \). As \( \bar{e}_\mu(\varphi) \notin C \) would imply \( \bar{a}, \bar{c}, \bar{e}_\mu(\varphi) \) pairwise \( B \)-independent, we have \( C = \langle B\bar{e}_\mu(\varphi) \rangle \); \( C \in C_\mu \) yields \( \bar{e}_i \subseteq A \setminus B \), and \( \bar{e}_i \mapsto \bar{e}_\mu(\varphi) \) induces \( A \cong_B C \).

Ow there is \( \bar{e}_i \downarrow_{B\bar{b}} \bar{a} \). Then \( \bar{e}_\mu(\varphi) \) and \( \bar{e}_i \) have the same type over \( B\bar{b}\bar{a} \), as do \( \bar{c} = \bar{e}_\mu(\varphi) \cdot \bar{a}^{-1} \) and \( \bar{e}_i \cdot \bar{a}^{-1} \). So \( \bar{c} \mapsto \bar{e}_i \cdot \bar{a}^{-1} \) is the required isomorphism.
Axiomatization

When we want to axiomatize richness for the Fraïssé-Hrushovski limit $\mathcal{M}_\mu$, we have to say that for all $\bar{a} \in A \leq \mathcal{M}$, a code instance $\varphi(\bar{x}, \bar{a})$ has an $A$-generic realization in $\mathcal{M}_\mu$, unless for a generic realization $\bar{b}$ we would have $\langle A\bar{b} \rangle \notin C_\mu$.

The weak CIT allows us to limit the possible $\mathbb{Q}$-linear dependencies we have to consider, with an extra twist: We may first have to extend by finitely many green generic points.

It follows that $\aleph_0$-saturated models of $T_\mu = \text{Th}(\mathcal{M}_\mu)$ are rich, $\mathcal{M}_\mu$ has Morley rank 2, and $\bar{U}(\mathcal{M}_\mu)$ has Morley rank 1.
Model-completeness

We show $M \leq N$ for any two models $M \subseteq N$ of $T_\mu$; since then $\text{cl}_M = \text{cl}_N$, homogeneity for closed sets yields $M \prec N$.

**Claim.** If $M \models T_\mu$ and $M \subseteq N \in C_\mu$, then $M \leq N$.

If not, assume $\text{lin.dim}_Q(N/M) = d$ is minimal. Then $d \geq 2$, as $M = \text{acl}(M)$. Choose $M \subset N' \subset N$ with $\text{lin.dim}_Q(N'/M) = d - 1$. By minimality $M \leq N'$. Now

$$-1 \geq \delta(N/M) = \delta(N/N') + \delta(N'/M),$$

and $\delta(N/N') \geq -1$ implies that $\delta(N'/M) \leq 0$. So $N'/M$ is prealgebraic.

Hence there is $M < N'' \leq N'$ with $N''/M$ minimal prealgebraic. But there are only finitely many extensions of this type, which must all lie already in $M$, a contradiction.
An alternative axiomatization

Universal axioms

- Finitely generated subfields are in $C_\mu$.

Inductive axioms

- $ACF_0$.
- The extension of the model generated by a green generic realization of some code instance $\varphi(\bar{x}, \bar{b})$ is not in $C_\mu$.

Since any complete theory of fields of finite Morley rank is $\aleph_1$-categorical, Lindström’s theorem implies that $\text{Th}(\mathcal{M}_\mu)$ is model-complete.
References


