Geometric Model Theory

Frank O Wagner
Institut Camille Jordan
Université Claude Bernard Lyon 1
France

17 July 2008

Partially supported by the European Research Network
MODNET MRTN-CT-2004-512234

Plan

1. Geometries
2. Transfinite Dimension
3. Structural Consequences
4. Combinatorics

Geometries

Geometric model theory studies geometric notions such as (combinatorial) geometries, independence, dimension/rank and measure in general structures, and tries to deduce structural properties from geometric data.

Geometries

Geometric model theory studies geometric notions such as (combinatorial) geometries, independence, dimension/rank and measure in general structures, and tries to deduce structural properties from geometric data.

Definition

A (finitary) *pre-geometry* is a set X together with a closure operator $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $Y \subseteq X$

1. $Y \subseteq \mathrm{cl}(Y)$
2. $\mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y)$
3. (Exchange) If $x \in \mathrm{cl}(Y, y) \setminus \mathrm{cl}(Y)$, then $y \in \mathrm{cl}(Y, x)$.
4. (Finitary) $\mathrm{cl}(Y) = \bigcup_{\bar{y} \in Y} \mathrm{cl}(\bar{y})$.

Geometries

Geometric model theory studies geometric notions such as (combinatorial) geometries, independence, dimension/rank and measure in general structures, and tries to deduce structural properties from geometric data.

Definition

A (finitary) *pre-geometry* is a set X together with a closure operator $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $Y \subseteq X$

1. $Y \subseteq \mathrm{cl}(Y)$
2. $\mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y)$
3. (Exchange) If $x \in \mathrm{cl}(Y, y) \setminus \mathrm{cl}(Y)$, then $y \in \mathrm{cl}(Y, x)$.
4. (Finitary) $\mathrm{cl}(Y) = \bigcup_{\bar{y} \in Y} \mathrm{cl}(\bar{y})$.
5. It is a *(combinatorial) geometry* if $\mathrm{cl}(x) = \{x\}$ for all $x \in X$.

Dimension

We call a subset $Y \subseteq X$ *independent* if $y \notin \mathrm{cl}(Y \setminus \{y\})$ for all $y \in Y$. A maximal independent subset of some $Y \subseteq X$ is called a *basis* for Y; it exists because the closure is finitary. By the exchange property, all bases for Y have the same cardinality, called the *dimension* $\dim(Y)$ of Y.

Dimension

We call a subset $Y \subseteq X$ *independent* if $y \notin \mathrm{cl}(Y \setminus \{y\})$ for all $y \in Y$. A maximal independent subset of some $Y \subseteq X$ is called a *basis* for Y; it exists because the closure is finitary. By the exchange property, all bases for Y have the same cardinality, called the *dimension* $\dim(Y)$ of Y.

Examples

1. A set X with $\mathrm{cl}(Y) = Y$ for all $Y \subseteq X$.
2. A vector space V over a division ring D with $\mathrm{cl}(Y)$ the linear span of Y, for all $Y \subseteq V$.
3. A field K with $\mathrm{cl}(Y)$ the relative algebraic closure of Y in K, for all $Y \subseteq K$.

Algebraic Closure

In order to generalize these notions, we consider a kind of universal domain: A big structure $\mathbb{U}$ satisfying a *compactness* condition:

> *Every intersection of sets definable with fewer than card*$(\mathbb{U})$ *parameters is non-empty, provided its finite subintersections are.*

Algebraic Closure

In order to generalize these notions, we consider a kind of universal domain: A big structure $\mathbb{U}$ satisfying a *compactness* condition:

> *Every intersection of sets definable with fewer than card*$(\mathbb{U})$ *parameters is non-empty, provided its finite subintersections are.*

Definition

The *algebraic closure* $\mathrm{acl}(Y)$ of a (small) subset $Y \subset \mathbb{U}$ is the set of elements of $\mathbb{U}$ having finite orbit under the action of the group of automorphisms of $\mathbb{U}$ fixing Y.

Clearly acl is a cosure operator; it is easy to see that compactness forces it to be finitary. We are thus looking for structures where acl has the exchange property.

Minimality

Definition

Let $\mathcal{L}$ be a collection of relations and functions on $\mathbb{U}$. Then $\mathbb{U}$ is *$\mathcal{L}$-minimal* if every definable subset of $\mathbb{U}$ is already definable using only relations and functions in $\mathcal{L}$ and arbitrary parameters in $\mathbb{U}$.

Minimality

Definition

Let $\mathcal{L}$ be a collection of relations and functions on $\mathbb{U}$. Then $\mathbb{U}$ is *$\mathcal{L}$-minimal* if every definable subset of $\mathbb{U}$ is already definable using only relations and functions in $\mathcal{L}$ and arbitrary parameters in $\mathbb{U}$.

Examples

A set without structure, a vector space over a division ring, an algebraically closed field are $=$-minimal.
A dense linear order, an ordered vector space over an ordered division ring, a real closed field are $\leq$-minimal.

Minimality

Definition

Let $\mathcal{L}$ be a collection of relations and functions on $\mathbb{U}$. Then $\mathbb{U}$ is *$\mathcal{L}$-minimal* if every definable subset of $\mathbb{U}$ is already definable using only relations and functions in $\mathcal{L}$ and arbitrary parameters in $\mathbb{U}$.

Examples

A set without structure, a vector space over a division ring, an algebraically closed field are $=$-minimal.
A dense linear order, an ordered vector space over an ordered division ring, a real closed field are $\leq$-minimal.

Trichotomy Conjecture (Zilber)
These are essentially the only ones.

False, but...

False, but...

In the ordered case, obvious counterexamples are a *group interval*, for instance $\mathbb{Q}$ with addition restricted to $[0,1]$, or a *field interval*, for instance $\mathbb{R}$ with addition and multiplication restricted to $[0,1]$. We thus have to consider the local structure.

Theorem (Peterzil-Starchenko)

The trichotomy conjecture holds locally for $\leq$-minimal structures. In the field interval case, the field may carry additionnal structure.

False, but...

In the ordered case, obvious counterexamples are a *group interval*, for instance $\mathbb{Q}$ with addition restricted to $[0, 1]$, or a *field interval*, for instance $\mathbb{R}$ with addition and multiplication restricted to $[0, 1]$. We thus have to consider the local structure.

Theorem (Peterzil-Starchenko)

The trichotomy conjecture holds locally for $\leq$-minimal structures. In the field interval case, the field may carry additionnal structure.

An important example of an $\leq$-minimal expansion of the reals is the real field with exponentiation. One of the main lines of research is to find further $\leq$-minimal expansions.

False, but...

Theorem (Hrushovski-Zilber)

The trichotomy conjecture is false for $=$-minimal structures. However it holds under an additionnal assumption on the dimension of the intersection of closed sets. In this case the resulting field will be pure (i.e. no additionnal structure).

False, but...

Theorem (Hrushovski-Zilber)

The trichotomy conjecture is false for $=$-minimal structures. However it holds under an additionnal assumption on the dimension of the intersection of closed sets. In this case the resulting field will be pure (i.e. no additionnal structure).

The counterexample is obtained by a modification of Fraïssé's construction of a countable homogeneous structure (dense linear order, random graph) from its finite substructures.

False, but...

Theorem (Hrushovski-Zilber)

The trichotomy conjecture is false for $=$-minimal structures. However it holds under an additionnal assumption on the dimension of the intersection of closed sets. In this case the resulting field will be pure (i.e. no additionnal structure).

The counterexample is obtained by a modification of Fraïssé's construction of a countable homogeneous structure (dense linear order, random graph) from its finite substructures.

Models of the additionnal assumption are called *Zariski structures*. In particular $=$-minimal sets in many algebraic contexts (algebraically, separably or differentially closed fields as well as existentially closed difference fields). These are the structures used in the applications.

Transfinite Dimension

Let us consider an algebraically closed field $\mathbb{U}$ with an algebraically closed subfield K and a transcendence basis $(x_0, x_1, x_2, \ldots)$ of $\mathbb{U}/K$. Then $\mathbb{U}$ contains arbitrary sums

$$\bigoplus_{i=0}^{n} x_i K$$

and should thus be considered as infinite dimensional over K.

Transfinite Dimension

Let us consider an algebraically closed field $\mathbb{U}$ with an algebraically closed subfield K and a transcendence basis $(x_0, x_1, x_2, \ldots)$ of $\mathbb{U}/K$. Then $\mathbb{U}$ contains arbitrary sums

$$\bigoplus_{i=0}^{n} x_i K$$

and should thus be considered as infinite dimensional over K.

A similar phenomenon arises in differentially closed fields, where the field $\mathbb{U}$ is infinite dimensional over the subfield C of constants. A rough measure for the dimension of a point P is the transcendence degree (in the sense of C) of the differential subfield generated by P. A more precise measure is the *Lascar rank* which is combinatorially defined.

Dividing and Independence

Definition

Let X_B be a subset of $\mathbb{U}$ defined with parameters in B. Then X_B *divides* over parameters $A \subset \mathbb{U}$ if there is an *indiscernible sequence* $B = B_0, B_1, B_2, \ldots$ over A such that $\bigcap_i X_{B_i} = \emptyset$.

A sequence is *indiscernible* over A if for all n all increasing n-tuples are in the same orbit under the group of automorphisms of $\mathbb{U}$ fixing A.

Dividing and Independence

Definition

Let X_B be a subset of $\mathbb{U}$ defined with parameters in B. Then X_B *divides* over parameters $A \subset \mathbb{U}$ if there is an *indiscernible sequence* $B = B_0, B_1, B_2, \ldots$ over A such that $\bigcap_i X_{B_i} = \emptyset$.

A sequence is *indiscernible* over A if for all n all increasing n-tuples are in the same orbit under the group of automorphisms of $\mathbb{U}$ fixing A.

Definition

Let $\bar{a}$ be a tuple in $\mathbb{U}$, and $A, B \subset \mathbb{U}$ be sets of parameters. Then $\bar{a}$ is *independent* of B over A if no $(A \cup B)$-definable set containing $\bar{a}$ divides over A.

In this case we write $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A B$.

Lascar rank

Definition

Lascar rank is the smallest ordinal-valued function U s.t.: $U(\bar{a}/A) \geq \alpha + 1$ if there is $B \supseteq A$ with $\bar{a} \not\perp_A B$ and $U(\bar{a}/B) \geq \alpha$.

In an $=$-minimal structure $a \not\perp_A B$ iff $a \in \mathrm{acl}(A \cup B) \setminus \mathrm{acl}(A)$. Hence dimension is equal to Lascar rank.

Lascar rank

Definition

Lascar rank is the smallest ordinal-valued function U s.t.: $U(\bar{a}/A) \geq \alpha + 1$ if there is $B \supseteq A$ with $\bar{a} \not\mathrel{\raise0.2em\hbox{$\smile$}\kern-0.9em\lower0.1em\hbox{$\mid$}}_A B$ and $U(\bar{a}/B) \geq \alpha$.

In an $=$-minimal structure $a \not\mathrel{\raise0.2em\hbox{$\smile$}\kern-0.9em\lower0.1em\hbox{$\mid$}}_A B$ iff $a \in \mathrm{acl}(A \cup B) \setminus \mathrm{acl}(A)$. Hence dimension is equal to Lascar rank.

Theorem (Lascar Inequalities)

Lascar rank is well-behaved with respect to fibers:

$$U(\bar{a}/A, f(\bar{a})) + U(f(\bar{a})/A) \leq U(\bar{a}/A) \leq U(\bar{a}/A, f(\bar{a})) \oplus U(f(\bar{a})/A)$$

Here $\oplus$ is the Cantor symmetric sum of ordinals.

Simplicity

Definition

$\mathbb{U}$ is *simple* if independence is symmetric.
$\mathbb{U}$ is *supersimple* if it is simple and Lascar rank exists.

Simplicity

Definition

$\mathbb{U}$ is *simple* if independence is symmetric.
$\mathbb{U}$ is *supersimple* if it is simple and Lascar rank exists.

Separably closed fields are simple; differentially closed fields and existentially closed difference fields are supersimple of Lascar rank ω.

Simplicity

Definition

$\mathbb{U}$ is *simple* if independence is symmetric.
$\mathbb{U}$ is *supersimple* if it is simple and Lascar rank exists.

Separably closed fields are simple; differentially closed fields and existentially closed difference fields are supersimple of Lascar rank ω.

In a supersimple theory, we can define a variety of closure operators:

Definition

$\mathrm{cl}_\alpha(A) = \{a \in \mathbb{U} : U(a/A) < \omega^\alpha\}$.

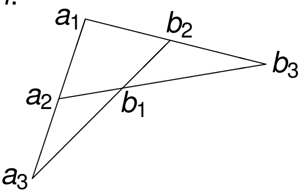
On a set of Lascar rank ω^α this α-closure induces a pre-geometry.

Structural Consequences

In the $=$- or $\leq$-minimal context, the most sophisticated structural consequences of geometrical properties come from the use of the trichotomy. However, these have only been proven for the finite rank context, and even there the proof is not well-understood.

The Group Configuration

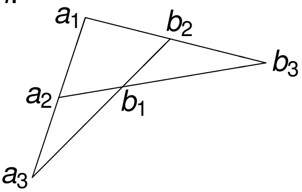
In general, the most important geometric configuration is the *group configuration*:



- $a_i \in \mathrm{cl}(a_j, a_k)$ for $\{i, j, k\} = \{1, 2, 3\}$.
- $b_i \in \mathrm{cl}(a_j, b_k)$ for $\{i, j, k\} = \{1, 2, 3\}$.
- all other triples and all pairs are independent.

The Group Configuration

In general, the most important geometric configuration is the *group configuration*:



- $a_i \in \mathrm{cl}(a_j, a_k)$ for $\{i, j, k\} = \{1, 2, 3\}$.
- $b_i \in \mathrm{cl}(a_j, b_k)$ for $\{i, j, k\} = \{1, 2, 3\}$.
- all other triples and all pairs are independent.

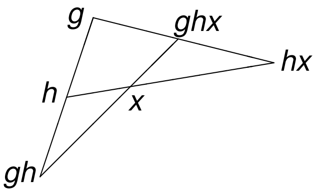
In a group take $a_3 = a_1 a_2$, $b_3 = b_2 a_1$ and $b_1 = b_2 a_1 a_2$.

The Group Configuration Theorem

Theorem (Weil, Zilber, Hrushovski, Ben Yaacov, Tomašić, W)

Given a group configuration in a simple theory, we can find a group G acting transitively on a set X.

If the configuration is non-reducible, the original configuration is closely related to the configuration given by the action:



The Field Configuration

Theorem (Hrushovski)

If in addition X is $=$-minimal, then

1. *G is $=$-minimal and abelian, or*
2. *X is the affine line and G the group of affine transformations, or*
3. *X is the projective line and G the group of projective transformations,*

over some definable algebraically closed field.

The Field Configuration

Theorem (Hrushovski)

If in addition X is $=$-minimal, then

1. *G is $=$-minimal and abelian, or*
2. *X is the affine line and G the group of affine transformations, or*
3. *X is the projective line and G the group of projective transformations,*

over some definable algebraically closed field.

Note that $=$-minimality of X follows from a suitable minimality condition on the b_i.

...and the order ?

But what about the ordered context ?

...and the order ?

But what about the ordered context ?

There is no higher-dimensional (even no two-dimensional!) generalisation of $\leq$-minimality.

...and the order ?

But what about the ordered context ?

There is no higher-dimensional (even no two-dimensional!) generalisation of $\leq$-minimality.

And what about other interesting structures, such as *p*-adically closed fields or algebraically closed valued fields ?

...and the order ?

But what about the ordered context ?

There is no higher-dimensional (even no two-dimensional!) generalisation of $\leq$-minimality.

And what about other interesting structures, such as *p*-adically closed fields or algebraically closed valued fields ?

If you don't see a solution, generalize the problem until the solution becomes obvious.

...and the order ?

But what about the ordered context ?

There is no higher-dimensional (even no two-dimensional!) generalisation of $\leq$-minimality.

And what about other interesting structures, such as p**-adically closed fields or algebraically closed valued fields ?**

If you don't see a solution, generalize the problem until the solution becomes obvious.

We shall have to go back to combinatorics.

þ-independence

We shall modify the definition of dividing and independence.

Definition

Let X_B be a subset of $\mathbb{U}$ defined with parameters in B. Then X_B *þ-divides* over parameters $A \subset \mathbb{U}$ if there is finite n and $A' \supseteq A$ such that B lies in an infinite $\mathrm{Aut}(\mathbb{U}/A')$-orbit $\mathcal{O}$ and $\bigcap_i X_{B_i} = \emptyset$ for all distinct $B_0, \ldots, B_n \in \mathcal{O}$.

þ-independence

We shall modify the definition of dividing and independence.

Definition

Let X_B be a subset of $\mathbb{U}$ defined with parameters in B. Then X_B *þ-divides* over parameters $A \subset \mathbb{U}$ if there is finite n and $A' \supseteq A$ such that B lies in an infinite $\mathrm{Aut}(\mathbb{U}/A')$-orbit $\mathcal{O}$ and $\bigcap_i X_{B_i} = \emptyset$ for all distinct $B_0, \ldots, B_n \in \mathcal{O}$.

Definition

Let $\bar{a}$ be a tuple in $\mathbb{U}$, and $A, B \subset \mathbb{U}$ be sets of parameters. Then $\bar{a}$ is *þ-independent* of B over A if no $(A \cup B)$-definable set containing $\bar{a}$ þ-divides over A.

In this case we write $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{þ}_{A} B$.

Rosiness

The Lascar þ-rank $U^{þ}$ is defined as the Lascar rank, but with þ-independence instead of independence.

Definition

We call $\mathbb{U}$ *rosy* if þ-independence is symmetric.
It is *superrosy* if it is rosy and Lascar þ-rank exists.

(Super-)rosy theories are a commong generalization of (super-)simple theories and $\leq$-minimal theories. In particular, they allow a higher (even ordinal) generalization of $\leq$-minimality for ordered structures. The Lascar Inequalities are still valid for Lascar þ-rank.

Rosiness

However, many definability results fail; in particular type-definability seems much weaker than definability. We do not know whether there is a group configuration theorem.

Rosiness

However, many definability results fail; in particular type-definability seems much weaker than definability. We do not know whether there is a group configuration theorem.

Unfortunately, important algebraic examples are not rosy.

We have to do even better.

The Tree Property of the Second Kind...

Definition

$\mathbb{U}$ has the tree property of the second kind if there is finite n and uniformly definable sets $X_{i,j}$ such that the families $\{X_{i,j} : i < \omega\}$ are n-inconsistent, but for all $\sigma : \omega \to \omega$ the set $\bigcap_j X_{\sigma(j),j}$ is non-empty.

We are interested in $\mathbb{U}$ *without* the tree property of the second kind. These include simple and $\leq$-minimal structures (but not all rosy ones), as well as algebraically closed valued fields.

...and the Independence Property

Moreover, structures without the tree property of the second kind include those *lacking* the following

Definition (Independence Property)

A uniformly definable family of sets $\{X_i : i \in I\}$ has the *independence property* if there is infinite $A \subset \mathbb{U}$ such that for all $B \subseteq A$ there is i_B with $B = A \cap X_{i_B}$.

(The lack of the Independence Property was introduced as having VC-dimension by Vapnik and Chervonenkis in statistical learning theory in the 1960's.)

Where to go ?

In structures without the tree property of the second kind, dividing still has some good properties, but the derived notion of independence is no longer symmetric.

Nevertheless, there is hope to recover some of the dimension theory at least *generically* (i.e. relativized to a single orbit), or via *simple domination* (i.e. every orbit is in some sense conrolled by some simple orbit).