Hrushovski's Amalgamation Construction

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Plan

Introduction

In 1986, Ehud Hrushovski modified Fraïssé's construction of a universial homogeneous countable relational structure from the class of its finite substructures, in order to obtain stable structures with particular properties. In particular, he constructed

- an $\aleph_0$-categorical stable pseudoplane,
- a strongly minimal set with an exotic geometry which is not disintegrated, but does not interpret any group,
- the fusion of two strongly minimal sets in disjoint languages in a third one,

obtaining counter-examples to conjectures by Lachlan and Zilber.

His method was taken up Baldwin, Baudisch, Evans, Hasson, Hils, Holland, Martín Pizarro, Poizat, Ziegler, Zilber and others, who constructed various other strange objects, most notably

- a one-based stable structure with a reduct which is not one-based,
- a new $\aleph_1$-categorical nilpotent group of class 2 and exponent p,
- a field of Morley rank 2 and an additive subgroup of Morley rank 1 in characteristic $p > 0$,
- a field of Morley rank 2 and a multiplicative subgroup of Morley rank 1 in characteristic 0,
- the fusion of two strongly minimal sets over a common $\aleph_0$-categorical reduct.

The original construction of Fraïssé

Let $\mathcal{C}$ be a class of finite structures in a finite relational language, closed under substructures, and with the amalgamation property AP (where we allow $A = \emptyset$):

> *For all injective $\sigma_i : A \to B_i$ in $\mathcal{C}$ for $i = 1, 2$ there are injective $\rho_i : B_i \to D \in \mathcal{C}$ with $\rho_1\sigma_1 = \rho_2\sigma_2$.*

Then there is a unique countable structure $\mathfrak{M}$ such that

> *for all finite $A \subset \mathfrak{M}$ and $A \subset B \in \mathcal{C}$ there is an embedding $B \to \mathfrak{M}$ which is the identity on A.*

The proof is by successive amalgamation over all possible situations, using AP.

We call $\mathfrak{M}$ the generic model; it is ultrahomogeneous, and hence $\aleph_0$-categorical (since the language is finite).

Strong embeddings

Rather than considering all inclusions $A \subset B \in \mathcal{C}$, we only consider certain inclusions $A \leq B$, which we call closed. We require $\leq$ to be transitive and preserved under intersections. We only demand AP for closed inclusions, and obtain a generic structure $\mathfrak{M}$ such that the richness condition holds:

> *For any finite $A \leq \mathfrak{M}$ and $A \leq B \in \mathcal{C}$ there is a closed embedding $B \hookrightarrow \mathfrak{M}$ which is the identity on A.*

Moreover, $\mathfrak{M}$ is ultrahomogeneous for closed subsets.

For a finite $A \subset \mathfrak{M}$ we define the closure $\mathrm{cl}_{\mathfrak{M}}(A)$ to be the smallest $B \leq \mathfrak{M}$ containing A.

Axiomatization

In order to axiomatize, we need to express $A \leq \mathfrak{M}$. If this can be done by a first-order formula, and if there is a bound on the number of possible closures (and hence types) of a finite subset of $\mathfrak{M}$, the generic model is $\aleph_0$-categorical.

However, in in a finite relational language, closedness need not be a definable property. If $A \leq \mathfrak{M}$ is only type-definable, or if closures can be infinite, we need approximate definability of richness:

> *If $A \leq B \in \mathcal{C}$ and A is sufficiently closed in $\mathfrak{M}$, then there is an embedding of B into $\mathfrak{M}$ over A whose image has a pre-described leved of closedness.*

This yields homogeneity for closures of finite subsets in an $\aleph_0$-saturated model, or for countable closed subsets in an $\aleph_1$-saturated model.

Evans' Example

Consider the class $\mathcal{C}$ of finite directed graphs without directed cycles and out-valency 2. Define $A \leq B$ if all descendants of A in B are already in A. Note that the closure is disintegrated:

$$\mathrm{cl}_{\mathfrak{M}}(A) = \bigcup_{a \in A} \mathrm{cl}_{\mathfrak{M}}(a).$$

This class has AP (namely the free amalgam, i.e. the disjoint union over the intersection, with no edges added). Clearly, richness (for finite sets) is definable; let T be the theory of (finitely) rich directed graphs without directed cycles and out-valency 2. By compactness, an $\aleph_0$-saturated model of T is rich even for closures of finite sets, so T is complete, and isomorphic closed sets have the same type.

A non-disintegrated reduct

Now consider the undirected reduct of models of T. By induction, one sees that a finite undirected graph has an orientation without directed cycles and of out-valency 2 iff every subgraph has a vertex of valency at most 2. For these graphs we define $A \leq B$ if B has such an orientation in which A is closed in the directed sense. This class is closed under free amalgamation; its generic model is the reduct of a generic model of the directed class.

However, the closure of A in $\mathfrak{M}$ is the intersection of all directed closures of A in $\mathfrak{M}$, for all possible orientations. This closure is no longer disintegrated. Counting types, one can show that both theories are stable.

With a suitable notion of directed hypergraph, one can generalize this to arbitrary finite relational languages.

Ab initio

For every relation $R \in \mathcal{L}$ we choose a weight $\alpha_R > 0$ and define a predimension on finite $\mathcal{L}$-structures:

$$\delta(A) = |A| - \sum_{R \in \mathcal{L}} \alpha_R |R(A)|, \qquad \text{as well as}$$

$$\delta(A/B) = \delta(AB) - \delta(B) = |A \setminus B| - \text{weights of the new relations};$$

this makes sense even if B is infinite. Define

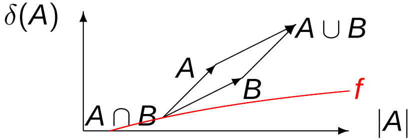
$$A \leq B \quad \Leftrightarrow \quad \delta(B'/A) \geq 0 \text{ for all } B' \subseteq B.$$

Let $\mathcal{C}$ be the universal class of all finite $\mathcal{L}$-structures whose substructures have non-negative predimension (ab initio).
It is closed under free amalgamation, and thus has AP.
Closedness is type-definable (this uses finiteness of the language), richness is approximately definable, and $\aleph_1$-saturated models are rich. The generic model is ω-stable if all the α_R are rational, and stable otherwise.

Since we have free amalgamation, cl is equal to algebraic closure (in the model-theoretic sense).
If A is a finite subset of the generic model, then $\mathrm{cl}_{\mathfrak{M}}(A)$ is the unique smallest superset of A with minimal predimension (for irrational α_R this may be a infinite, and we interpret its predimension as a suitable limit). Define the dimension of A in $\mathfrak{M}$ to be $d_{\mathfrak{M}}(A) = \delta(\mathrm{cl}_{\mathfrak{M}}(A))$.
Then two closed sets are independent in the forking sense iff they are freely amalgamated over their intersection, and the amalgam is closed in $\mathfrak{M}$. It is now easy to see that the generic model has weak elimination of imaginaries.
Finally, for rational α_R Evans has defined a notion of directed hypergraph such that $A \leq B$ in the predimension sense iff there is an orientation of B in which A is closed in the directed sense. So all ω-stable ab initio constructions arise as reducts of disintegrated geometries.

$\aleph_0$-categoricity

If we want the resulting structure to be $\aleph_0$-categorical, there must be a bound on the size of the closure of a finite set (as the closure is always contained in the algebraic closure).
Note that the free amalgam of A and B over $A \cap B$ forms the fourth point of a parallelogram:



We choose an unbounded increasing cut-off function f and consider the subclass $\mathcal{C}_f = \{A \in \mathcal{C} : \delta(A) \geq f(|A|)\}$.
If the slope of f at B is at most the minimal slope of an arrow at B, then $\mathcal{C}_f$ has free amalgamation; in particular $\delta(B/A)$ must be strictly positive, and the α_R linearly independent over $\mathbb{Q}$. Axiomatizability and $\aleph_0$-categoricity follow.

Pseudoplanes

Hrushovki has shown that for a single binary relation the set of suitable (irrational) alpha is co-meagre, thus constructing an $\aleph_0$-categorical stable complete pseudoplane.

An incidence system $I \subset P \times L$ is a pseudoplane if

- every point $p \in P$ lies on infinitely many lines $\ell \in L$,
- every line contains infinitely many points,
- every two points are incident with finitely many lines,
- every two lines intersect in finitely many points.

The pseudoplane is complete if I is a complete type.

A disintegrated stable theory is cannot type-define a complete pseudoplane. So if an ω-stable complete pseudoplane has a disintegrated reduct à la Evans, the pseudoplane obviously still exists in the expansion, but is no longer complete.

Geometry

Suppose all the α_R are integers. For a single point a and a set B (which we may assume to be closed) of parameters in a generic structure $\mathfrak{M}$, there are two possibilities:

- $d_{\mathfrak{M}}(a/B) = \delta(a/B) = 1$. Then $aB \leq \mathfrak{M}$, so this determines a unique type, the generic type.
- $d_{\mathfrak{M}}(a/B) = 0$. So a is in the geometric closure $\mathrm{gcl}(B)$.

Clearly gcl is increasing and idempotent, hence a closure operator, which in addition satisfies the exchange rule:

If $a \in \mathrm{gcl}(Bc) \setminus \mathrm{gcl}(B)$, then $c \in \mathrm{gcl}(Ba)$.

We should like to restrict the class $\mathcal{C}$ so that gcl becomes algebraic closure, thus yielding a strongly minimal set (every definable subset is uniformly finite or cofinite). For that, we have to bound the number of possible realisations of any $a \in \mathrm{gcl}(B)$ uniformly and definably.

Minimal and pre-algebraic extensions, codes

A proper closed extension $A \leq B \in \mathcal{C}$ is minimal if $A \leq A' \leq B$ implies $A' = A$ or $A' = B$. Equivalently, $\delta(B/A') < 0$ for all $A \subset A' \subset B$.

It is pre-algebraic if $\delta(B/A) = 0$.

For a minimal pre-algebraic extension $A \leq B$ let $A_0 \leq A$ be the closure of the points in A related to some points in $B \setminus A$. This is the unique minimal closed subset of A over which $B \setminus A$ is pre-algebraic (and in fact minimal).

We call $A_0 \leq B$ bi-minimal pre-algebraic; its code $\varphi(\bar{x}, \bar{y})$ is the quantifier-free diagram of $(B \setminus A, A_0)$.

Clearly it is sufficient to bound the number of realisations for each bi-minimal pre-algebraic code.

New strongly minimal sets

Let μ be a function from the set of codes to the integers, such that $\mu(\varphi(\bar{x},\bar{y})) \geq \delta(\bar{y})$. Let $\mathcal{C}^\mu$ be the class of $A \in \mathcal{C}$ such that for any code φ and any A_0 there are at most $\mu(\varphi)$ disjoint realizations of $\varphi(\bar{x}, A_0)$. Again, this is a universal class.

Hrushovski has shown that $\mathcal{C}^\mu$ has thrifty amalgamation:

> *If $A \leq B \in \mathcal{C}^\mu$ is minimal and $A \leq M \in \mathcal{C}^\mu$, then either the free amalgam of A and $\mathfrak{M}$ over B is still in $\mathcal{C}^\mu$, or B embeds closedly into M over A.*

So a generic model exists; since richness is still approximately definable, its theory is strongly minimal.

Poizat's red field of rank $\omega \cdot 2$

A red field is an ω-stable algebraically closed field K with a predicate R for a connected additive subgroup of comparable rank. Note that in characteristic 0 this gives rise to an infinite definable subfield $\{a \in K : aR \leq R\}$, so the structure has rank at least ω.

Let $\mathcal{C}$ be the class of finitely generated fields k of characteristic $p > 0$ with a predicate R for an additive subgroup, such that for all finitely generated subfields k'

$$\delta(k') = 2\,\mathrm{tr.deg}(k') - \mathrm{lin.dim}_{\mathbb{F}_p}(R(k')) \geq 0.$$

This condition is universal, since we have to say that $2n$ linearly independent red points do not lie in any variety of dimension $< n$.

For $k \leq k' \in \mathcal{C}$ put

$$\delta(k'/k) = 2\,\mathrm{tr.deg}(k'/k) - \mathrm{lin.dim}_{\mathbb{F}_p}(R(k')/R(k)).$$

This δ satisfies the two essential conditions for a predimension: $\delta(\emptyset) = 0$, and for closed A, B

$$\delta(A) + \delta(B) \geq \delta(A \cup B) + \delta(A \cap B) \qquad \text{Submodularity.}$$

Since $\mathcal{C}$ has free amalgamation, a generic model $\mathfrak{M}$ exists; as richness is approximately definable, $\aleph_0$-saturated models of $\mathrm{Th}(\mathfrak{M})$ are rich.

For a point a and a set B in $\mathfrak{M}$ there are three possibilities:

1. $d_{\mathfrak{M}}(a/B) = 2$. Then a is not red, and aB is closed. $RM(a/B) = \omega \cdot 2$.
2. $d_{\mathfrak{M}}(a/B) = 1$. There is a red point a' interalgebraic with a over B, and $a'B$ is closed. $\omega^2 > RM(a/B) \geq RM(a'/B) = \omega$.
3. $d_{\mathfrak{M}}(a/B) = 0$. Then either a is algebraic over B, or pre-algebraic.

The collaps (Baudisch, Martín Pizarro, Ziegler)

We have to restrict the number of bi-minimal pre-algebraic extensions. A code is a formula $\varphi(\bar{x}, \bar{y})$ with $n = |\bar{x}|$ such that

1. For all $\bar{b}$ either $\varphi(\bar{x}, \bar{b})$ is empty, or has Morley degree 1.
2. $RM(\bar{a}/\bar{b}) = n/2$ and $\mathrm{lin.dim}_{\mathbb{F}_p}(\bar{a}/\bar{b}) = n$ for generic $\bar{a} \models \varphi(\bar{x}, \bar{b})$, and for all non-trivial subspaces U of $\langle \bar{a} \rangle$ $2\,\mathrm{tr.deg}(\bar{a}/U\bar{b}) < n - \mathrm{lin.dim}_{\mathbb{F}_p}(U)$.
3. If $RM(\varphi(\bar{x}, \bar{b}) \cap \varphi(\bar{x}, \bar{b}')) = n/2$, then $b = b'$.
4. If $\varphi(\bar{x}, \bar{b})$ is disintegrated for some $\bar{b}$, it is disintegrated (or empty) for all $\bar{b}$.
5. For any $H \in \mathrm{GL}_n(\mathbb{F}_p)$, $\bar{m}$ and $\bar{b}$ there is $\bar{b}'$ with $\varphi(H\bar{x} + \bar{m}, \bar{b}) \equiv \varphi(\bar{x}, \bar{b}')$.

1. says that $\varphi(\bar{x}, \bar{b})$ determines a unique generic type $p_{\varphi(\bar{x},\bar{b})}$ (or is empty).
2. says that $\bar{b} \leq \bar{a}\bar{b}$ is minimally pre-algebraic. Moreover, $\delta(\bar{a}'/B) < 0$ for any $B \ni \bar{b}$ and non-generic $\bar{a}' \notin \mathrm{acl}(B)$ realizing $\varphi(\bar{x}, \bar{b})$.
3. says that $\bar{b}$ is the canonical base for $p_{\varphi(\bar{x},\bar{b})}$, so the extension $\bar{b} \leq \bar{a}\bar{b}$ is bi-minimal.
4. says that φ fixes the type of the extension: disintegrated, or generic in a group coset (minimal pre-algebraic types are locally modular).
5. says that affine transfomations preserve the code.

There is a set $\mathcal{S}$ of codes such that every minimal pre-algebraic extension is coded by a unique $\varphi \in \mathcal{S}$.

Difference sequences

For a code φ and some $\bar b$ consider a Morley sequence $(\bar a_0, \bar a_1, \ldots, \bar a_k, f)$ for $p_{\varphi(\bar x, \bar b)}$, and put $\bar e_i = \bar a_i - \bar f$.
We can then find a formula $\psi_\varphi \in \mathrm{tp}(\bar e_0, \ldots, \bar e_k)$ such that

- Any realization $(\bar e'_0, \ldots, \bar e'_k)$ of ψ_φ is $\mathbb{F}_p$-linearly independent, and $\models \varphi(\bar e'_i, \bar b')$ for some unique $\bar b'$ definable over sufficiently large finite subsets of the $\bar e'_i$.
- ψ_φ is invariant under the finite group of derivations generated by $\partial_i : \bar x_j \mapsto \begin{cases} \bar x_j - \bar x_i & \text{if } j \neq i \\ -\bar x_i & \text{if } j = i \end{cases}$ for $0 \leq i \leq k$.
- Some condition ensuring dependence of affine combinations, and invariance under the stabiliser of the group for coset codes.

A difference sequence for a code φ is any realization of ψ_φ.

A counting Lemma

Given a code φ and natural numbers m, n, there is some λ such that for every $M \leq N \in \mathcal{C}$ and difference sequence $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ in N with canonical parameter $\bar{b}$, either

- the canonical parameter for some derived sequence lies in M, or
- for every $A \subset N$ of size m the sequence $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ contains a Morley subsequence in $p_{\varphi(\bar{x}, \bar{b})}$ over MA of length n.

Let μ be a sufficiently fast-growing finite-to-one function from $\mathcal{S}$ to ω, and $\mathcal{C}^\mu$ the class of $A \in \mathcal{C}$ which do not contain a difference sequence for φ of length $\mu(\varphi)$ for any $\varphi \in \mathcal{S}$.

The above lemma allows us to characterize when a minimal pre-algebraic extension of some $M \in \mathcal{C}^\mu$ is no longer in $\mathcal{C}^\mu$, and to prove thrifty amalgamation for $\mathcal{C}^\mu$.

Axiomatization

Hence there is a generic model $\mathfrak{M}$; since richness remains approximately definable, $\aleph_0$-saturated models of $\mathrm{Th}(\mathfrak{M})$ are rich, $\mathfrak{M}$ has Morley rank 2, and $R^{\mathfrak{M}}$ has Morley rank 1.

An alternative axiomatization:

Universal axioms

- Finitely generated subfields are in $\mathcal{C}^{\mu}$.

Inductive axioms

- ACF_p.
- The extension of the model generated by a red generic realization of some code instance $\varphi(\bar{x}, \bar{b})$ is not in $\mathcal{C}^{\mu}$.

Since any complete theory of fields of finite Morley rank is $\aleph_1$-categorical, Lindström's theorem implies that $\mathrm{Th}(\mathfrak{M})$ is model-complete.

Poizat's green field of rank $\omega \cdot 2$

A green field is an ω-stable algebraically closed field K with a predicate $\ddot{U}$ for a connected multiplicative subgroup of comparable rank. Note that in characteristic $p > 0$ this gives rise to only finitely many p-Mersenne primes $\frac{p^n-1}{p-1}$, and has $\tilde{\mathbb{F}}_p$ as prime model. Its existence is thus improbable; in any case it cannot be constructed as generic model by amalgamation methods.

Let $\mathcal{C}$ be the class of finitely generated fields k of characteristic 0 with a predicate $\ddot{U}$ for a torsion-free multiplicative subgroup, such that for all finitely generated subfields k'

$$\delta(k') = 2\,\mathrm{tr.deg}(k') - \mathrm{lin.dim}_{\mathbb{Q}}(\ddot{U}(k')) \geq 0,$$

where the linear dimension is taken multiplicatively. Put

$$\delta(k'/k) = 2\,\mathrm{tr.deg}(k'/k) - \mathrm{lin.dim}_{\mathbb{Q}}(\ddot{U}(k')/\ddot{U}(k)).$$

The weak CIT

While linear dimension over a finite field is definable, this is no longer true for dimension over $\mathbb{Q}$, as there are infinitely many scalars (exponents). Poizat used Zilber's weak CIT, a consequence of Ax' differential Schanuel conjecture:

For any uniform family $V_{\bar{z}}$ of varieties there is a finite set $T_0, \ldots, T_r$ of tori, such that for any torus T, any $V_{\bar{b}}$ and any irreducible component $W \ni \bar{a}$ of $V_{\bar{b}} \cap \bar{a} \cdot T$, for some $i \in [0, r]$ $W \subseteq \bar{a} \cdot T_i$ and $\dim(T_i) - \dim(V \cap \bar{a} \cdot T_i) = \dim T - \dim W$.

This specifies finitely many possibilities for $\mathbb{Q}$-linear relations on a family of varieties which could render δ negative. Hence $\mathcal{C}$ is again universal, richness approximately axiomatizable, the generic model exists, and $\aleph_0$-saturated models of its theory are rich. It has Morley rank $\omega \cdot 2$, and a generic green point has rank ω.

The collaps (Baudisch, Hils, Martín Pizarro, W.)

A code is a formula $\varphi(\bar{x}, \bar{y})$ with $n = |\bar{x}|$ such that

1. For all $\bar{b}$ either $\varphi(\bar{x}, \bar{b})$ is empty, or has Morley degree 1.
2. $RM(\bar{a}/\bar{b}) = n/2$ and $\mathrm{lin.dim}_{\mathbb{Q}}(\bar{a}/\bar{b}) = n$ for generic $\bar{a} \models \varphi(\bar{x}, \bar{b})$, and for $i = 2, \ldots, r$ and any W irreducible component of $V \cap \bar{a}T_i$ of maximal dimension, $\dim(T_i) > 2 \cdot \dim(W)$ if $V \cap \bar{a}T_i$ is infinite.
3. If $RM(\varphi(\bar{x}, \bar{b}) \cap \varphi(\bar{x}, \bar{b}')) = n/2$, then $b = b'$.
4. For any invertible $\bar{m}$ and $\bar{b}$ there is $\bar{b}'$ with $\varphi(\bar{x} \cdot \bar{m}, \bar{b}) \equiv \varphi(\bar{x}, \bar{b}')$.

Toric correspondences

This time $\mathrm{GL}_n(\mathbb{Q})$ acts on the codes, which is infinite. Hence we cannot put invariance under $\mathrm{GL}_n(\mathbb{Q})$ into the axioms, but have to deal with it outside the codes. Using weak CIT we obtain:

There exists a collection $\mathcal{S}$ of codes such that for every minimal prealgebraic definable set X there is a unique code $\varphi \in \mathcal{S}$ and finitely many tori T such that $T \cap (X \times \varphi(\bar{x}, \bar{b}))$ projects generically onto X and $\varphi(\bar{x}, \bar{b})$ for some $\bar{b}$. We call such a T a toric correspondence. In particular, for any code φ only finitely many tori can induce a toric correspondence between instances of φ.

Difference sequences

For every code φ there is some formula $\psi(\bar{x}_0, \ldots, \bar{x}_k) \in \mathrm{tp}(\bar{e}_0 \cdot \bar{f}^{-1}, \ldots, \bar{e}_k \cdot \bar{f}^{-1})$ for some Morley sequence $(\bar{e}_0, \ldots, \bar{e}_k, \bar{f})$ in $p_{\varphi(\bar{x},\bar{b})}$ such that:

1. Any realization $(\bar{e}'_0, \ldots, \bar{e}'_k)$ of ψ is disjoint, and $\models \varphi(\bar{e}'_i, \bar{b}')$ for some unique $\bar{b}'$ definable over sufficiently large finite subsets of the $\bar{e}'_i$.
2. If $\models \psi(\bar{e}_0, \ldots, \bar{e}_k)$, then $\models \psi(\bar{e}_0, \ldots, \bar{e}_{k'})$ for each $k' \leq k$, and ψ is invariant under derivations.
3. Let $i \neq j$ and $(\bar{e}_0, \ldots, \bar{e}_k)$ realize ψ with canonical parameter $\bar{b}$. If there is some toric correspondence T on φ and $\bar{e}'_j$ with $(\bar{e}_j, \bar{e}'_j) \in T$, then $\bar{e}_i \not\perp_{\bar{b}} \bar{e}'_j \cdot \bar{e}_i^{-1}$ in case $\bar{e}_i$ is a generic realization of $\phi(\bar{x}, \bar{b})$.

Counting

Miraculously, this is enough to obtain the same counting lemma as before.

Since $\mathbb{Q}$-linear dependence need not be uniform, we have to use the weak CIT in order to uniformize dependencies in a non-generic difference sequence, and then the finite Ramsey theorem to obtain a long derived sequence inside the original model.

Again, we choose a fast-growing finite-to-one function μ from $\mathcal{S}$ to ω, and define $\mathcal{C}^\mu$ to be the class of all $A \in \mathcal{C}$ who do not have a difference sequence for φ of length $\mu(\varphi)$. We obtain the same characterisation when a structure in $\mathcal{C}^\mu$ has a minimal pre-algebraic extension not in $\mathcal{C}^\mu$, and can prove thrifty amalgamation. Hence there is a generic model $\mathfrak{M}$.

Axiomatization

When we want to approximately axiomatize richness, we have to say that for all $\bar{a} \in A \leq \mathfrak{M}$, a code instance $\varphi(\bar{x}, \bar{a})$ has an A-generic realization in $\mathfrak{M}$, unless for a generic realization B we would have $AB \notin \mathcal{C}^\mu$. The weak CIT allows us to limit the possible $\mathbb{Q}$-linear dependencies we have to consider, with an extra twist: We may first have to extend by finitely many green generic points.

It follows that $\aleph_0$-saturated models of $\mathrm{Th}(\mathfrak{M})$ are rich, $\mathfrak{M}$ has Morley rank 2, and $\ddot{U}^{\mathfrak{M}}$ has Morley rank 1.

Moreover, there also is an alternative axiomatization analogous to the red case, which yields model-completeness.

Fusion

1. Two strongly minimal sets with definable multiplicity property (DMP) in disjoint languages can be amalgamated freely with predimension $\delta(A/B) = RM_1(A/B) + RM_2(A/B) - |A \setminus B|$, and collapsed to a strongly minimal set (Hrushovski).
2. Two strongly minimal sets with DMP with a common $\aleph_0$-categorical reduct, one preserving multiplicities, can be amalgamated freely with predimension $\delta(A/B) = RM_1(A/B) + RM_2(A/B) - RM_0(A/B)$, and collapsed to a strongly minimal set (Baudisch, Martín Pizarro, Ziegler; partial results by Hasson and Hils).
3. Two theories of finite and definable Morley rank with DMP can be amalgamated freely with predimension $\delta(A/B) = n_1 \cdot RM_1(A/B) + n_2 \cdot RM_2(A/B) - n \cdot |A \setminus B|$, where $n_1 \cdot RM(T_1) = n_2 \cdot RM(T_2) = n$, and collapsed to a structure of Morley rank n (Ziegler).

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