Groups definable in orthogonal sets

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Plan

Structures

A *structure* $\mathfrak{M}$ is just a set M, its *domain*, together with some functions $(f_i : i \in I)$ and some relations $(R_j : j \in J)$ of arbitrary finite arity.
The relations are supposed to include equality, although this will not be mentioned explicitly.
We can also name some particular constants, although we shall be allowed to use any element of M as parameter.

Examples

- A group $\langle G, 1, \cdot, ^{-1}\rangle$
- A K-vector space $\langle V, 0, +, -, \lambda_k : k \in K\rangle$
- A ring $\langle R, 0, 1, +, -, \cdot\rangle$
- An ordered field $\langle K, 0, 1, +, -, \cdot, \leq\rangle$.

Formulas

Using parameters, variables, the functions and relations, logical connectives (negation, conjunction, disjunction) and quantifiers (universal, existential),
we can build meaningful statements called *formulas*.
A formula without free variables is a *sentence*.

Examples

- $\forall x \forall y \forall z \ \ (x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- $\forall x \exists y \ \ x \cdot y = 1$
- $x \cdot y = y \cdot x$
- $\forall y \ \ x \cdot y = y \cdot x$.

These formulas are interpreted in $\mathfrak{M}$ in the natural way.
Note that we can only quantify over the elements of M.

Definable sets

A formula $\phi(\bar{x})$ with $\bar{x} = (x_1, \ldots, x_n)$ gives rise to a definable subset of M^n.

Examples

- $\forall y\ x \cdot y = y \cdot x$ defines the centre $Z(G)$ of a group G.
- $x \cdot g = g \cdot x$ defines the centralizer $C_G(g)$.
- $\exists y\ x = y + y$ defines the subgroup $2G$ in an abelian group G.

Note that in order to define $C_G(g)$, we have used a parameter g.
Some subsets are not definable, for instance the group generated by an element g of infinite order, or the subset of torsion elements (unless there is a bound on the exponent).

Type-definable sets

A subset of M^n is *type-definable* if it is the intersection of definable sets.

Examples

- The collection of divisible elements of G, given by $\exists y\ \underbrace{y+\cdots+y}_{n \text{ times}} = x$ for all $n > 0$.
- The *connected component* H^0 of a definable subgroup H of G, given by the intersection of all definable subgroups of finite index.

Note that type-definable sets may be accidentally too small. For instance, in $\mathbb{Z}$ the type-definable subgroup $\bigcap_{n>0} n\mathbb{Z}$ is trivial, even though it is the connected component of $\mathbb{Z}$ and should be of comparable size.

Compactness

We shall assume that our structure $\mathfrak{M}$ is *saturated*, i.e. compact in the following sense:

> *Any intersection of few definable sets is non-empty, provided its finite subintersections are.*

Few or *small* means of cardinality less than the cardinality of M. We restrict ourselves to small intersections, and to small parameter sets.

Theorem (Compactness Theorem)

Suppose $\Phi(\bar{x})$ and $\Psi(\bar{x})$ are small collections of formulas, and for all $\bar{m} \in M$, whenever $\phi(\bar{m})$ holds in $\mathfrak{M}$ for all $\phi \in \Phi$, then $\psi(\bar{m})$ holds in $\mathfrak{M}$ for some $\psi \in \Psi$.
Then there are finite $\Phi_0 \subseteq \Phi$ and $\Psi_0 \subseteq \Psi$ with the same property.

Hyperdefinable sets and types

A *hyperdefinable set* is a type-definable set, divided by a type-definable equivalence relation.

Example

Let $H = \bigcap_i H_i$ be a type-definable subgroup of G, and $N = \bigcap_j N_j$ a type-definable normal subgroup of H. Then H/N is hyperdefinable, given by the classes of elements in H modulo the type-definable equivalence relation $\bigwedge_j \; xy^{-1} \in N_j$.

The *type* $\mathrm{tp}(\bar{m}/A)$ of a tuple $\bar{m} \in M$ over a (small) subset $A \subset M$ is the set of all formulas with parameters in A which hold of $\bar{m}$.
By saturation, two tuples have the same type over A if and only if there is an automorphism of $\mathfrak{M}$ fixing A pointwise and moving one tuple to the other.

Orthogonality and internality

Let X and Y be hyperdefinable sets in $\mathfrak{M}$.

Definition

We say that X and Y are *orthogonal* if for any parameter set $A \subset X \cup Y$ and any $x \in X$ and $y \in Y$, $\mathrm{tp}(x/A) \cup \mathrm{tp}(y/A)$ determines $\mathrm{tp}(x, y/A)$.

Example: M and N are orthogonal in the product $\mathfrak{M} \times \mathfrak{N}$.

Definition

We say that Y is *(almost) X-internal* if there is a parameter set $A \subset X$ and an A-hyperdefinable injection (map with finite fibres) from Y to some set hyperdefinable in X.

Example: Any Cartesian power of X is X-internal.
It is easy to see that (almost) internality is transitive.

Properties of orthogonality

Theorem

Let X and Y be orthogonal hyperdefinable sets in $\mathfrak{M}$. Then any almost X-internal set is orthogonal to any almost Y-internal set.

Theorem

Let X and Y be orthogonal type-definable sets in $\mathfrak{M}$. Then any relatively definable subset of $X^m \times Y^n$ is a finite union of rectangles, i.e. sets of the form $X' \times Y'$ with $X' \subseteq X^m$ and $Y' \subseteq Y^n$ relatively definable.

A subset of Z is *relatively definable* if it is the intersection of Z with a definable set.

Examples

Examples

- Let K be an algebraically closed Henselian valued field with residue field k and value group Γ. Then k and Γ are definable and orthogonal.
- Let K be a differentially closed field. Then any two minimal differentially algebraic sets are either almost bi-internal, or orthogonal.
- Let K be an existentially closed difference field. Then any two minimal transformally algebraic sets are either almost bi-internal, or orthogonal.

Products?

In view of the characterisation of subsets of $X^m \times Y^n$ for orthogonal X and Y as finite unions of rectangles, one may ask whether a similar property holds for groups:

Question 1. If X and Y are orthogonal hyperdefinable sets, is every hyperdefinable group in $X \times Y$ a product of an almost X-internal group and an almost Y-internal group?

The answer is no, as the following example shows:

Example

The universal cover $f : G \to H$ of a real Lie group H definable in an *o*-minimal expansion $\mathfrak{R}$ of the real field $\mathbb{R}$ is definable in $\langle \mathbb{Z}, +, -, 0 \rangle \times \mathfrak{R}$.

A totally ordered structure $\mathfrak{M}$ is *o-minimal* if every definable subset of M is a finite union of intervals.

Extensions?

In view of the above example, one can modify the question:

Question 2. If X and Y are orthogonal hyperdefinable sets, is every hyperdefinable group in $X \times Y$ an extension of an almost X-internal group by an almost Y-internal group?

Example

Let $X = Y = \langle \mathbb{R}, +, 0, < \rangle$. Then there is a hyperdefinable group in $X \times Y$ with no almost X- or Y-internal subgroup.

Proof.

Take $G = (X \times Y)/\Lambda$, where $\mathbb{Z}^2 \cong \Lambda < X \times Y$ is in sufficiently generic position. For a big enough rectangle $Z \subset X \times Y$ we have $Z + \Lambda = X \times Y$ and $Z \cap \Lambda$ is finite. We can then identify G with Z/Λ, which is definable in $X \times Y$. $\square$

Extensions!

Theorem (Berarducci, Mamino)

Let X and Y be orthogonal sets, where X is superstable of finite and definable Lascar rank. Then any group hyperdefinable in $X \cup Y$ is an extension of a Y-internal group by an X-internal group.

Theorem (W.)

Let X and Y be orthogonal hyperdefinable sets, where X is simple. Then any group hyperdefinable in $X \cup Y$ is an extension of a Y-internal group by an X-internal group.

Any structure of finite and definable Lascar rank is simple. A structure is *simple* if it has a well-behaved notion of independence (similar to, and generalizing, linear independence in vector spaces and algebraic independence in fields).

Products!

Corollary

Let X and Y be orthogonal hyperdefinable sets, where X is simple. If G is a group hyperdefinable in $X \cup Y$, then there is an X-internal hyperdefinable normal subgroup N such that G/N is Y-internal. Moreover:

- *N is unique up to commensurability.*
- *$G/C_G(N)$ is X-internal.*
- *$N \cdot C_G(N)$ has small index in G.*
- *$(N \cdot C_G(N))/Z(N) \cong N/Z(N) \times C_G(N)/Z(N)$, where $N/Z(N)$ is X-internal and $C_G(N)/Z(N)$ is Y-internal.*

Local groups

Definition

A *local group* is an ascending chain of sets $(X_n : n \in \mathbb{N})$, an element $1 \in X_0$, a unary function $i_n^{-1} : X_n \to X_n$ and a binary function $m_n : X_n \times X_n \to X_{n+1}$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} X_n$ is a group with inverse $\bigcup_{n\in\mathbb{N}} i_n^{-1}$ and multiplication $\bigcup_{n\in\mathbb{N}} m_n$.

We also say that X_0 is a local group if such a chain exists.

Question 3. Let X and Y be orthogonal hyperdefinable sets, and G a hyperdefinable group in $X \cup Y$. Is there an X-internal local group G_X, a Y-internal local group G_Y, and an equivalence relation E on $G_X \times G_Y$ with small classes, all hyperdefinable, such that G is isogenous, or even isomorphic, to $(G_X \times G_Y)/E$?

Approximate groups

Definition

Let $K \geq 1$. A K-approximate subgroup of a local group G is a subset A such that $A \cdot A \subseteq X \cdot A$ for some $X \subseteq G$ with $|X| \leq K$.

Finite approximate groups have recently been classified by Breuillard, Green and Tao.

Question 4. Let X and Y be orthogonal hyperdefinable sets, and G a hyperdefinable group in $X \cup Y$.
Is there an X-internal approximate subgroup A_X and a Y-internal approximate subgroup A_Y, both hyperdefinable, with $G = A_X \cdot A_Y$?

Note. If there are X-internal A_X and Y-internal A_Y with $G = A_X \cdot A_Y$, then A_X and A_Y are automatically approximate subgroups of G.

Weak elimination

The following results show that we only need consider equivalence relations with small classes.

Theorem (Berarducci, Mamino)

Let X and Y be orthogonal hyperdefinable sets, and Z hyperdefinable in $X \cup Y$. Then there is X-internal X' and Y-internal Y', and a map $f : Z \to X' \times Y'$ with small fibres, all hyperdefinable.

Theorem (W.)

Let X and Y be orthogonal hyperdefinable sets, and Z hyperdefinable in $X \cup Y$. Then there is X-internal X' and Y-internal Y', a subset $Z' \subseteq X' \times Y'$ and a surjection $g : Z' \to Z$ with small fibres, all hyperdefinable.

This provides us with quasi-coordinates for elements of Z.

A criterion for (almost) internality

The quasi-coordinates from the previous slide can be used to characterise (almost) internality.

Lemma

Let X and Y be orthogonal hyperdefinable sets, and Z hyperdefinable in $X \cup Y$.
Let X', Y' and $f : Z \to X' \times Y'$ be as in the first elimination theorem. The following are equivalent:

- *Z is almost X-internal.*
- *Z is X-internal.*
- *The projection of $f(Z)$ to Y' has small image.*

One could also use the function g from the second elimination theorem, and consider the projection of $g^{-1}(Z)$ to Y'.

Uniqueness

Recall the Main Theorem:

Theorem

Let X and Y be orthogonal hyperdefinable sets, where X is simple. If G is a group hyperdefinable in $X \cup Y$, then there is an X-internal hyperdefinable normal subgroup N such that G/N is Y-internal.

Proof: Uniqueness.

Let N and N' be two such groups. Then $N/(N \cap N')$ and $N'/(N \cap N')$ are both X-internal and Y-internal, and must be small, as the projection to both quasi-coordinates is small. Thus N and N' are commensurable. □

Existence

For every $g \in G$ let (g_X, g_Y) be its quasi-coordinates. So there is a type-definable equivalence relation E with small classes such that $g = (g_X, g_Y)_E$. For fixed tp(g) and tp(h),

- $(gh)_X$ only depends on g_X and h_X,
- $(gh)_Y$ only depends on g_Y and h_Y.

Now $\text{tp}(g/g_Y)$ is X-internal, and therefore simple, for any $g \in G$. We may thus use our notion of independence. Choosing g and h of maximal dimension and independent over g_Y, h_Y, we see that g, h and gh are pairwise independent over $g_Y, h_Y, (gh)_Y$.
Then g is generic in the coset $S \cdot g$ over g_Y, where

$$S = \{s \in G : \text{tp}(sg/g_Y) = \text{tp}(g/g_Y)\}$$

is the stabilizer of $\text{tp}(g/g_Y)$, a hyperdefinable subgroup.

Existence

Since $\text{tp}(g/g_Y)$ is X-internal, so is S.

Moreover, the dimension of S equals the dimension of $\text{tp}(g/g_Y)$.

By maximality of dimension, any X-internal hyperdefinable subset $Z \subseteq G$ is covered by few cosets of S, as otherwise SZS would still be X-internal, but have larger dimension than S, and hence larger dimension than $\text{tp}(g/g_Y)$.

In particular, S is commensurable with any G-conjugate, and its connected component $S^0 =: N$ is normal in G.

Finally, for any $g' \in G$, the type $\text{tp}(g'/g'_Y)$ is covered by few cosets of N, so the orbit of $g'N$ over g'_Y is small.

Thus $\text{tp}(g'N)$ is almost $\text{tp}(g'_Y)$-internal, and hence Y-internal, as is G/N.

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