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Inégalités de Sobolev optimales : approches dynamiques

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L'écriture des mathématiques pour la recherche est un exercice particulièrement codifié. Ces règles, intégrées au fil des années sans forcément s'en rendre compte, constituent la grammaire du vocabulaire mathématique. À l'exception de cette section, la totalité de ce document est écrit dans cette langue, étrangère pour beaucoup, et c'est pourquoi j'ai plaisir à écrire ces quelques paragraphes dans la langue qui m'est naturelle, avant que le « je » ne laisse ma place au « nous » impersonnel du mathématicien.

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Introduction en Français

Cette introduction en Français est délibérément courte, elle ne joue pas le même rôle que l'introduction en Anglais, plus détaillée. Je souhaite ici simplement contextualiser mon travail de recherche et quelques résultats. Pour une approche plus progressive, ne requiérant qu'un bagage mathématique modéré, le lecteur est invité à consulter les sections 1.3 à 1.6. Un index des notations (p. 95) est disponible à la fin du manuscrit. J'insiste sur le fait que les diverses notions (comme les espaces de Lebesgue ou la mesure de Hausdorff, par exemple) sont utilisées sans être nommées dans cette courte introduction.

Les inégalités de Sobolev portent le nom du mathématicien russe qui les a formalisées et prouvées en 1938, Sergueï Sobolev. Depuis cette date, elles ont été un sujet de recherche très actif, et ont permis de nombreuses applications dans des domaines variés de l'analyse et des probabilités. Elles sont notamment au cœur de l'étude des équations aux dérivées partielles, car elles traduisent des propriétés fondamentales des dérivées au sens faible : l'intégrabilité d'une dérivée au sens faible implique la régularité au sens fort. Les dérivées au sens faible, ou au sens des distributions, on le rappelle, constituent un cadre particulièrement efficace dans la résolution d'équations aux dérivées partielles, en permettant par exemple de donner du sens à la dérivée d'une onde de choc, phénomène se manifestant même dans des modèles physiques très simples.

De manière générale, pour une inégalité fonctionnelle, s'il existe des fonctions qui réalise l'égalité, on dit que l'inégalité est optimale (ces fonctions sont alors appelées optimales). Cependant, la plupart du temps, de telles fonctions n'existent pas : dans ce cas, on parle d'inégalité optimale lorsque l'égalité peut être *approchée* par une suite de fonctions.

Pour la majorité des applications, connaître la forme optimale d'une inégalité ne présente pas vraiment plus d'intérêt que d'avoir une estimation des constantes impliquées dans l'inégalité. Toutefois, la recherche d'inégalités optimales est une entreprise riche pour deux raisons. La première, c'est que les inégalités fonctionnelles sont naturellement liées aux équations aux dérivées partielles, et donc aux modèles physiques. Les fonctions optimales sont alors exactement les solutions d'énergie minimale du problème physique sous-jacent. Même s'il n'existe pas de fonction optimale, la connaissance des constantes optimales renseigne sur l'énergie du système physique. L'autre raison est le lien très fort entre les inégalités de Sobolev d'une part, et la géométrie de l'espace ambiant d'autre part. Par exemple, l'inégalité isopérimétrique dans l'espace euclidien est équivalente à un cas de l'inégalité de Sobolev classique (sous forme optimale). De manière plus générale, la théorie géométrique de Brunn-Minkowski offre des outils suprenamment efficaces dans l'étude des inégalités de Sobolev.

Le manuscrit de thèse est divisé en quatre chapitres, dont les trois derniers sont des reproductions, plus ou moins verbatim, d'articles écrits au cours de la thèse. Le premier chapitre, introductif, présente et définit tous les outils utilisés dans les chapitres suivants.

Pour un entier $d > 1$, il est bien connu des experts que l'inégalité isopérimétrique

$$\mathcal{H}^{d-1}(\partial A) \geq d\omega_d^{1/d}|A|^{(d-1)/d} \quad \text{pour tout compact } A \subset \mathbb{R}^d \text{ de classe } \mathcal{C}^1$$

est équivalente à l'inégalité de Sobolev pour $p = 1$,

$$\|\nabla u\|_1 \geq d\omega_d^{1/d} \|u\|_{1^*} \quad \text{pour toute fonction positive } u \subset \mathcal{C}_c^\infty(\mathbb{R}^d),$$

où $1^* = d/(d-1)$ et $\omega_d = |B(0,1)|$ est le volume de la boule unité de dimension d. Bien sûr, d'un point de vue logique, deux propositions vraies sont toujours équivalentes, mais il faut comprendre ici que ces deux inégalités sont équivalentes dans le sens où l'une se déduit de l'autre, et vice versa. Ce résultat est une manifestation du lien qui existe entre l'inégalité de Sobolev et la géométrie, comme mentionné plus haut. C'est ce lien qui motive l'approche du chapitre 2 : on entend extraire de nouvelles inégalités de nature analytique à partir d'inégalités géométriques. Plus riche que l'inégalité isopérimétrique puisque cette dernière en est une conséquence, l'inégalité de Brunn-Minkowski

$$|A + B|^{1/d} \geq |A|^{1/d} + |B|^{1/d} \quad \text{pour tout compacts } A, B \subset \mathbb{R}^d$$

et ses généralisations offrent un cadre efficace pour démontrer des inégalités de la famille de l'inégalité de Sobolev [BL08]. Plus exactement, Sergueï Bobkov et Michel Ledoux font appel à l'inégalité de Borell-Brascamp-Lieb, que l'on peut reformuler de la manière suivante : soit $t \in [0, 1]$. Si g, W, et H sont trois fonctions définies sur $\mathbb{R}^d$ et à valeurs dans $(0, +\infty]$, telles que $\int g^{-d} = \int W^{-d} = 1$ et

$$\forall x, y \in \mathbb{R}^d, \quad H((1-t)x + ty) \leq (1-t)g(x) + tW(y),$$

alors

$$\int H^{-d} \geq 1.$$

Grâce à la théorie du transport optimal, on peut prouver la version suivante de l'inégalité de Borell-Brascamp-Lieb [BCEF+17] : avec les mêmes hypothèses sur g, W et H,

$$\int H^{1-d} \geq (1-t) \int g^{1-d} + t \int W^{1-d}.$$

C'est cette inégalité qui est réellement le point de départ du chapitre 2. En utilisant à notre avantage les propriétés d'une l'inf-convolution, on peut non seulement retrouver l'inégalité de Sobolev sur $\mathbb{R}^d$ tout entier [BCEF+17], mais on peut aussi prouver une famille d'inégalités de Gagliardo-Nirenberg-Sobolev à trace sur les cônes convexes, notamment.

Les deux chapitres suivants, 3 et 4, reposent sur la méthode de Dominique Bakry et Michel Émery [BE85]. L'idée derrière cette méthode est séduisante de simplicité.

On dit qu'une variété riemannienne compacte M vérifie une inégalité de Sobolev logarithmique avec la constante $C > 0$ et pour la mesure riemannienne μ si pour toute fonction strictement positive $u \in \mathcal{C}^\infty(M)$,

$$\mathrm{Ent}_\mu(u) := \int_M u \log\left(\frac{u}{\|u\|_1}\right) d\mu \leq \frac{C}{2} \int_M \frac{\|\nabla u\|^2}{u} d\mu.$$

Dans ce cas, si $u_0 \in \mathcal{C}^\infty(M)$ et u désigne la solution de l'équation de la chaleur

$$\begin{cases} \partial_t u = \Delta u & \text{sur } \mathbb{R}_+^* \times M, \\ u(0,.) = u_0 & \text{sur } M, \end{cases}$$

où Δ désigne l'opérateur de Laplace-Beltrami sur M, alors l'inégalité de Sobolev logarithmique relie exactement l'entropie $\mathrm{Ent}_\mu(u) = \int_M u \log u d\mu$ et sa dérivée temporelle,

$$\frac{d}{dt}\,\mathrm{Ent}_\mu(u) = \int_M \Delta u(1+\log u)d\mu = -\int_M \frac{\|\nabla u\|^2}{u} d\mu,$$

ce qui implique une décroissance exponentielle de l'entropie vers 0, $\mathrm{Ent}_\mu(u) \leq e^{-Ct/2}\,\mathrm{Ent}_\mu(u_0)$. Remarquablement, dériver l'entropie une seconde fois permet de faire apparaître l'opérateur de carré du champ itéré, aussi plus sobrement appelé Γ_2 ([BGL14], voir aussi la section 1.6.2) :

$$\frac{d^2}{dt^2}\,\mathrm{Ent}_\mu(u) = 2\int_M u\Gamma_2(\log u)d\mu.$$

Dès lors, une inégalité du type $\Gamma_2(\log u) \geq \rho\|\nabla \log u\|^2$ pour un $\rho > 0$ conduit à

$$\frac{d^2}{dt^2}\,\mathrm{Ent}_\mu(u) \geq -2\rho\frac{d}{dt}\,\mathrm{Ent}_\mu(u),$$

ce qui, après intégration, et en utilisant le fait que l'entropie comme sa dérivée convergent vers 0 en $+\infty$, on retrouve $\mathrm{Ent}_\mu(u_0) \leq -\frac{1}{2\rho}\,\frac{d}{dt}\,\mathrm{Ent}_\mu(u)\big|_{t=0}$, c'est-à-dire l'inégalité de Sobolev logarithmique avec la constante $1/\rho$. La méthode de Bakry et Émery, c'est, en somme, « dériver l'entropie deux fois ».

L'inégalité utilisée pour comparer la dérivée seconde de l'entropie à la dérivée première est un cas particulier de *condition de courbure dimension*, notée $CD(\rho, n)$, qui relie Γ_2, Γ et l'opérateur différentiel L, dans qui dans notre exemple était Δ. On dit que L vérifie une condition $CD(\rho, n)$ si

$$\forall\,\phi \in \mathcal{C}^\infty(M),\ \Gamma_2(\phi) \geq \rho\Gamma(\phi) + \frac{1}{n}(L\phi)^2$$

Par exemple, l'opérateur de Laplace-Beltrami, défini sur une variété d-dimensionnelle (M, g) dont la courbure de Ricci est minorée par ρg, vérifie $CD(\rho, d)$. Lorsque ρ et n sont strictement positifs, la condition $CD(\rho, n)$ conduit facilement aux inégalités de Sobolev logarithmique, mais aussi de Poincaré, avec la constante $1/\rho$. Dans le chapitre 3, on démontre différentes familles d'inégalités de Beckner, qui interpolent entre l'inégalité de Poincaré et de Sobolev logarithmique, dans deux cas.

- Sous l'hypothèse $CD(0, n)$, avec $n > 0$, on démontre des inégalités de Poincaré à poids. L'idée est que le poids permet de compenser le manque de courbure de la variété.
- Sous l'hypothèse $CD(\rho, n)$, avec $\rho > 0$ et $n < -2$, on démontre des inégalités de Beckner, avec constante optimale $(n-1)/(\rho n)$.

Le dernier chapitre sort du cadre linéaire du chapitre 3, et ne permet donc pas le recours à la théorie des semigroupes de Markov. En voyant l'équation de diffusion rapide

$$\partial_t u = \Delta u^m \quad \text{sur } \mathbb{R}^d,$$

avec $m \in [1-1/d, 1)$, comme le flot-gradient d'une entropie généralisée dans l'espace de Wasserstein $\mathcal{W}_2(\mathbb{R}^d)$, la méthode de Bakry-Émery permet de retrouver non seulement l'inégalité de Sobolev classique, mais aussi de démontrer de nouvelles formes d'inégalités, comme une inégalité de Sobolev logarithmique à trace sur le demi-espace $\mathbb{R}^d_+$. Malgré la différence des techniques impliquées dans les chapitre 3 et 4, les expressions de la dérivée seconde de l'entropie s'avèrent très similaires.

La vie humaine ne dure qu'un instant. Passons-la donc à faire ce qui nous plaît. En ce monde fugace comme un songe, c'est folie que de vivre misérablement, adonné aux seules choses qui nous rebutent. [...] J'aime à dormir. Face à la situation actuelle du monde, je pense que je vais rester chez moi et dormir.

Jōchō Yamamoto, *Hagakure*

Contents

Chapter 1

Introduction

1.1 Preamble and outline

This thesis focuses on sharp Sobolev-type inequalities, both in the Euclidean space and in Riemannian manifolds. Ever since its initial proof by Sergei Sobolev in 1938 [Sob38], the eponymous inequality and its relatives have been at the center of very many applications, notably in the study of partial differential equations. Indeed, they allow to relate integrability and regularity of functions in a weak sense, which makes not only for a powerful framework, since weak derivatives behave much better than regular derivatives, but also perhaps for a more natural one, since many physical phenomena, like shock waves, are inherently discontinuous.

While it is true that for many, if not most applications, the inequality need not be sharp (that is, with the smallest constant possible), the study of their sharp versions is of particular interest for at least two reasons. A first one is that these inequalities, while pertaining to functions, are intricately geometrical in nature. For instance, the Euclidean isoperimetric inequality can be seen as a reformulation to the classical Sobolev inequality for functions with integrable gradient. This relationship between geometry and functional analysis will, hopefully, be made somewhat clearer throughout this manuscript, notably through the study of the implications of the so-called curvature-dimension conditions. Another reason to investigate sharp inequalities is that they may be used to compute the ground state energy in some physical models.

While the proof of the original Sobolev inequality goes back to 1938, its sharp version was only proved some forty years later by Thierry Aubin and Giorgio Talenti [Aub76, Tal76], independently. Since then, variants of the Sobolev inequality have been studied intensely, using tools borrowed from many different fields, including calculus of variations, operator theory, geometrical measure theory, probabilities, and, more recently, optimal transport.

The manuscript is divided in five chapters: the last three are reproductions of the articles written during the period of the PhD. Each of these may be read independently from the rest, they aim to be as self-contained as possible. As a result, a few definitions and basic results will be repeated multiple times; we believe that it provides further coherence to the text. The first two chapters make up the introduction, in French and then in English, in which we present many of the notions and the tools that will be used in the subsequent chapters. We chose to include some proofs in the introduction, some because we think they are beautiful, some others because they provide solid examples as well as motivation for the developments in the other chapters. Among those proofs, some are easy and of well-known facts, it is our hope that they will interest readers who are less familiar with the topics addressed in this work.

The introduction is organised as follows: in the next section, the main inequalities in the limelight of this manuscript will be defined, so that we can use names a bit more explicit than "Sobolev-type inequalities", and also because for each one of them, there exists a great number of variants bearing the same name. After this perhaps sligtly less interesting yet necessary part, we will succinctly present the Brunn-Minkowski theory and its relationship with Sobolev inequalities, working our way to the Borell-Brascamp-Lieb inequality, which is some kind of functional counterpart of the Brunn-Minkowski inequality. We will propose a proof of an improvement of that inequality, and to do so we will have to take a detour into the very relevant theory of optimal transport, which allow us to very briefly introduce the formalism named as Otto's calculus, a very useful tool for calculations when dealing with paths in the family of probabilities, or, equivalently, with nonnegative, unit-mass solutions to conservative partial differential equations.

In the remaining sections of the introduction, we will see that Sobolev inequalities may be used to study solutions to certain partial differential equations. Namely, the logarithmic Sobolev inequality may be used to deduce exponential decay of the entropy of the solution to the heat equation on compact manifolds. The method proposed by Bakry and Émery [BE85] consists of a spectacular reversal: studying the entropy allows to recover the logarithmic Sobolev inequality! In order to apply their method to various problems, we will have to use some classical results of the theory of parabolic partial differential equations. The linear case is rather well understood through the prism of operator theory, and we will, without proof this time, recall some important facts. In the nonlinear case however, a little bit more work is usually required.

Even though much of the mathematical vocabulary stems from the field of probability (Markov semigroups, carré du champ operator, diffusions, etc.), we will sadly never actually adopt a probabilistic viewpoint.

Among all the other chapters, chapter 2 stands out a little bit, as it is the only one which does not rely on the method proposed by Bakry and Émery. Instead, we work within the Brunn-Minkowski theory to prove some new sharp trace Sobolev inequalities in convex subsets of the Euclidean space. The keystone of that approach is a generalization of the Borell-Brascamp-Lieb inequality, a proof of which we reproduce in the introduction using optimal transport.

Chapter 3, which is a joint work with Ivan Gentil, deals with Riemannian manifolds equipped with differential operators satisfying so-called curvature-dimension conditions. Inequalities like the Poincaré inequality classically hold in positively curved spaces, in the sense of the Ricci curvature. The novelty of this chapter consists of proving Poincaré-like inequalities (or more specifically, Beckner inequalities) in the case where the curvature is positive and the effective dimension is negative, and also when the space is only assumed flat, and the effective dimension positive. All the results from the Markov semigroup theory needed in the chapter is recalled in the introduction.

Finally, we go back to the Euclidean space again in chapter 4, in which we study nonlinear diffusion equations. Since the Markov semigroup theory developped earlier is of no use in that case, we have to rely on standard partial differential equations techniques. In this chapter, we focus entirely on proving functional inequalities, and not on the exact resolution of the nonlinear parabolic equations considered, with the explicit goal of ultimately applying those same methods to manifolds.

All three chapters have this in common: to prove the proposed functional inequalities, the proofs all use a family of functions indexed on the real half-line. In chapter 2, the starting point is a family of infimal convolutions, whereas chapters 3 and 4 use solutions of parabolic equations, which we also call flows. In that sense, all of these approaches are dynamical, hence the title of this thesis.

1.2 The classical Sobolev inequality and its variants

1.2.1 Definitions of classical inequalities

It seems only right that this cornerstone of an inequality should be cited at the very beginning of the manuscript.

Theorem 1.1 (Sobolev inequality)**.** *Let $1 \leq p < d$. For any $u \in C_c^\infty(\mathbb{R}^d)$,*

$$\|u\|_{L^{p^*}} \leq S(d,p)\|\nabla u\|_p, \tag{1.1}$$

where $S(d,p)$ is an explicit constant, and

$$p^* = \frac{dp}{d-p}.$$

Inequality (1.1) extends naturally to the Sobolev space $W^{1,p}(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d), \nabla u \in L^p(\mathbb{R}^d)\}$ for $p > 1$, and to the functions with bounded variation, $BV(\mathbb{R}^d) = \{u \in L^1(\mathbb{R}^d), \|u\|_{BV} < +\infty\}$ for $p = 1$. Then, there is equality in (1.1) if, and only if,

$$p > 1 \text{ and } u(x) = \left(\frac{1}{a + b\|x - x_0\|^{p/(p-1)}}\right)^{(d-p)/p} \text{ for some } a,\, b > 0 \text{ and } x_0 \in \mathbb{R}^d$$

or

$$p = 1 \text{ and } u \text{ is the characteristic function of a ball.}$$

The original proof of this inequality for $p > 1$ is due to Sergei Sobolev in 1938 [Sob38], hence its name, but its sharp form and the equality case were only found some forty years later, in 1976, by Thierry Aubin [Aub76] and Giorgi Talenti [Tal76], independently. Even though one think of those results as purely analytical, the proofs by Sobolev, as well as those by Aubin and Talenti, all rely on a symmetrization technique, the so-called symmetric decreasing rearrangement (a simple yet thorough presentation of which can be foud in [LL01]). The fact that minimizers in the case

$p = 1$ are characteristic functions of balls further hints at a geometric nature of the Sobolev inequality. As it turns out, the Sobolev inequality for $p = 1$ is equivalent to the isoperimetric inequality theorem 1.14 below, which is a fact that we will prove in section 1.3.

So fix some ideas and to provide a bit of context to what we mean by Sobolev inequalities, we define here some classical inequalities in a very generic way. Let M be a smooth, connected Riemannian manifold.

Definition 1.2 (Log-Sobolev)**.** We say that the manifold M satisfies a *logarithmic Sobolev inequality* with constant $C > 0$ with respect to the measure μ if, for all nonnegative functions $u \in \mathcal{C}_c^\infty(M)$,

$$\int_M u^2 \log\left(\frac{u^2}{\|u\|_2^2}\right) d\mu \leq 2C \int_M \|\nabla u\|^2 d\mu. \tag{1.2}$$

Definition 1.3 (Poincaré)**.** We say that the manifold M satisfies a *Poincaré inequality* with constant $C > 0$ with respect to the measure μ if, for all nonnegative functions $u \in \mathcal{C}_c^\infty(M)$,

$$\int_M u^2 d\mu - \left(\int_M u d\mu\right)^2 \leq C \int_M \|\nabla u\|^2 d\mu \tag{1.3}$$

Definition 1.4 (Beckner)**.** We say that the manifold M satisfies a *Beckner inequality* with constant $C > 0$ and parameter $p \in (1, 2]$ with respect to the measure μ if, for all nonnegative functions $u \in \mathcal{C}_c^\infty(M)$,

$$\frac{p}{p-1}\left(\int_M u^2 d\mu - \left(\int_M u^{2/p} d\mu\right)^p\right) \leq 2C \int_M \|\nabla u\|^2 d\mu \tag{1.4}$$

Definition 1.5 (Gagliardo-Nirenberg-Sobolev)**.** We say that the manifold M satisfies a *Gagliardo-Nirenberg-Sobolev inequality* with constant $C > 0$ and parameters $p \in [1, d]$ and $q, r \in [1, +\infty]$ with respect to the measure μ if, for all nonnegative functions $u \in \mathcal{C}_c^\infty(M)$,

$$\left(\int_M u^r d\mu\right)^{1/r} \leq C\left(\int_M u^q d\mu\right)^{(1-\theta)/q} \left(\int_M \|\nabla u\|^p d\mu\right)^{\theta/p} \tag{1.5}$$

where θ is given by

$$\frac{1}{r} = \left(\frac{1}{p} - \frac{1}{d}\right)\theta + \frac{1-\theta}{q}. \tag{1.6}$$

For example, consider the Euclidean space $M = \mathbb{R}^d$. For the standard Gaussian measure μ defined by

$$d\mu(x) = \frac{e^{-\|x\|^2/2}}{(2\pi)^{d/2}} dx,$$

$\mathbb{R}^d$ satisfies a logarithmic Sobolev inequality as was proved by Leonard Gross [Gro75], a Poincaré inequality, and a Beckner inequality for all $p \in (1, 2)$ [Bec89], all of them with constant $C = 1$. More generally, if $V : \mathbb{R}^d \to \mathbb{R}$ is a twice differentiable function such that $\nabla^2 V \geq \rho I_d$ for some constant $\rho > 0$, then all three inequalities (Poincaré, logarithmic Sobolev, Beckner) are satisfied with constant $1/\rho$ for the measure $d\mu = e^{-V} dx$. This is a classical result for so-called log-concave measures, see for example [BGL14].

On the other hand, for the Lebesgue measure, the Gagliardo-Nirenberg-Sobolev inequality is satisfied for all $p \in [1, d)$ and $q, r \in [1, +\infty]$ with constant $S(d, p)^\theta$, where $S(d, p)$ is the constant from the Sobolev inequality (1.1), and θ is given by equation (1.6),

$$\frac{1}{r} = \frac{\theta}{p^*} + \frac{1-\theta}{q}. \tag{1.7}$$

Indeed, it is a simple consequence of the Sobolev inequality and interpolation: if θ is as in equation (1.7) and u is a smooth function with compact support, then

$$\|u\|_r \leq \|u\|_{p^*}^{\theta} \|u\|_q^{1-\theta} \leq S(d,p)^{\theta} \|\nabla u\|_p^{\theta} \|u\|_q^{1-\theta}.$$

This constant, however, is not sharp. A sharp one-parameter sub-family of Gagliardo-Nirenberg-Sobolev inequalities has been famously exhibited by Manuel del Pino and Jean Dolbeault [dPD02]. For $p = 2$ and $q = r/2 + 1$, not only is the sharp constant known, but the optimal functions are completely characterized, and are of the form

$$u(x) = \left(a + b\|x - x_0\|^2\right)^{\frac{2}{2-r}},$$

for some $a, b > 0$ and $x_0 \in \mathbb{R}^d$.

Remark 1.6. The Beckner inequality with $p = 2$ is exactly the Poincaré inequality. Interestingly, if the Beckner inequality is valid for all $p \in (1, p_0)$ for some $p_0 \in (1, 2)$ with a constant C that is independent from p, then taking the limit $p \to 1$ in the Beckner inequality yields the logarithmic Sobolev inequality with constant C. In that sense, the Beckner inequality interpolates between the Poincaré inequality and the logarithmic Sobolev inequality.

Likewise, the Gagliardo-Nirenberg-Sobolev inequality with $r = p^*$ is exactly the Sobolev inequality. Furthermore, taking the limit $q, r \to 2$ in the del Pino-Dolbeault sub-family yields the Euclidean logarithmic Sobolev inequality [dPD03]: for all smooth function $u : \mathbb{R}^d \to \mathbb{R}_+$ such that $\|u\|_2 = 1$,

$$\int_{\mathbb{R}^d} u^2 \log(u^2) dx \leq \frac{d}{2} \log\left(\frac{2}{\pi d e} \|\nabla u\|_2^2\right).$$

This inequality is, in fact, equivalent to Gross' logarithmic inequality for the Gaussian measure. In that sense, again, the Gagliardo-Nirenberg-Sobolev inequalities can be seen as interpolating between the Sobolev inequality and the logarithmic Sobolev inequality.

1.2.2 Selected results from the manuscript

Let us gather, in this section, some new results proved in the subsequent chapters of this manuscript. Some of the inequalities at hand are called *trace* inequalities, because they involve the trace operator T, which maps functions u defined on a domain $\Omega \subset \mathbb{R}^d$ to their restriction to the boundary, $u|_{\partial\Omega}$. This restriction is obviously well-defined on the space of continuous functions $\mathcal{C}(\overline{\Omega})$, and admits extensions to certain Sobolev spaces, as the trace inequalities imply.

Theorem 1.7 (Sharp trace GNS inequality, theorem 2.2)**.** *Let $a \geq d > p > 1$, and $\Omega = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R},\ x_2 \geq \varphi(x_1)\}$ be a convex cone, that is when φ is convex and satisfies: for all $t > 0$ and $x_1 \in \mathbb{R}^{d-1}$, $\varphi(tx_1) = t\varphi(x_1)$. Then, there exists a positive constant $D_{d,p,a}(\Omega)$ such that for any non-negative function $f \in C_c^{\infty}(\Omega)$,*

$$\left(\int_{\mathbb{R}^{d-1}} f^q(x, \varphi(x)) dx\right)^{1/q} \leq D_{d,p,a}(\Omega) \|\nabla f\|_{L^p(\Omega)}^{\theta} \|f\|_{L^q(\Omega)}^{1-\theta}, \tag{1.8}$$

where

$$\theta = \frac{a-p}{p(a-d-1)+d}, \quad q = p\frac{a-1}{a-p}.$$

Furthermore, when $f(x) = \|(x_1, x_2 + 1)\|^{-\frac{a-p}{p-1}}$, then (1.8) is an equality.

The next two theorems involve $CD(\rho, n)$ conditions, which we introduce in section 1.6.2, along with Γ, the carré du champ operator.

Theorem 1.8 (Weighted Poincaré inequality, theorem 3.6)**.** *Let (M, g) be a smooth connected d-dimensional Riemannian manifold equipped with a diffusion operator L, and let φ be a positive function function such that $\nabla^2\varphi \geq cg$ for some constant $c > 0$. Assume that the operator L satisfies a $CD(0, n)$ condition, with $n \geq d$.*

Fix a real number $\beta \geq n + 1$ and assume that $\varphi^{-\beta} \in L^1(dx)$, where dx is the Riemannian measure, and then define $d\mu_{\varphi,\beta} = c_\beta \varphi^{-\beta} dx$, where c_β is a normalizing constant chosen so that $\mu_{\varphi,\beta}$ is a probability measure. Then for all smooth bounded functions u,

$$\mathrm{Var}_{\mu_{\varphi,\beta}}(f) = \int u^2 d\mu_{\varphi,\beta} - \left(\int u d\mu_{\varphi,\beta}\right)^2 \leq \frac{1}{c(\beta-1)} \int \Gamma(u)\varphi d\mu_{\varphi,\beta}. \tag{1.9}$$

Theorem 1.9 (Beckner inequalities, theorem 3.16)**.** *Let (M, g) be a smooth, connected, d-dimensional Riemannian manifold, equipped with a diffusion operator L that is symmetric with respect to a measure μ. Assume that the diffusion operator L satisfies a $CD(\rho, n)$ condition, with $\rho > 0$ and $n \in \mathbb{R}\backslash[-2, d)$. Then, M satisfies a Beckner inequality (1.4)*

$$\frac{p}{p-1}\left(\int u^2 d\mu - \left(\int u^{2/p} d\mu\right)^p\right) \leq 2\frac{n-1}{\rho n}\int \Gamma(u) d\mu.$$

in the following range of parameters:

- *for all $p \in (1, 2]$ if $n \geq d$,*
- *for all $p \in [p^*, 2]$ if $n < -2$, where $p^* = 1 + \dfrac{1-4n}{2n^2+1}$.*

Finally, we prove a trace version of the logarithmic Sobolev inequality, as well as a version of the (concave) Gagliardo-Nirenberg-Sobolev inequality on the half-space.

Theorem 1.10 (Trace logarithmic Sobolev inequality, corollary 4.7)**.** *For all $h \in \mathbb{R}$, and for all $u \in C^\infty(\mathbb{R}^d_+)$ such that $\int_{\mathbb{R}^d_+} u = 1$, the following inequality stands*

$$\int_{\mathbb{R}^d_+} u \log u \leq \frac{d}{2}\log\left(\frac{1}{2\pi d e}\int_{\mathbb{R}^d_+}\frac{\|\nabla u\|^2}{u}\right) - \log\gamma(\mathbb{R}^d_{+he}) - h\left(\int_{\partial\mathbb{R}^d_+} u\right)\left(\frac{1}{d}\int_{\mathbb{R}^d_+}\frac{\|\nabla u\|^2}{u}\right), \tag{1.10}$$

where γ stands for the standard Gaussian probability measure. Furthermore, there is equality when $u = C_h \exp\big(-\|x + he\|^2\big)$, where C_h is a normalizing constant.

Note that for $h = 0$, this is the standard optimal logarithmic Sobolev inequality on the half space. Interestingly, the parameter h can be chosen either positive or negative, allowing the trace term to be used as an upper or a lower bound.

Theorem 1.11 (Concave GNS inequality, corollary 4.8)**.** *Let $\alpha > 1$. For all functions $f \in C^\infty(\mathbb{R}^d_+)$, the following inequality stands*

$$\|f\|_{2\alpha/(2\alpha-1)} \leq C_\alpha \|\nabla f\|_2^\theta \|f\|_{2/(2\alpha-1)}^{1-\theta}, \tag{1.11}$$

where

$$\theta = \left(1 - \frac{1}{d(\alpha-1)+1}\right)\left(1 - \frac{1}{2\alpha}\right).$$

Furthermore, there is equality when $f = \big(1 - \|x\|^2\big)_+^{1/(\alpha-1)}$, up to multiplication by a constant, rescaling, and translation by a vector in $\mathbb{R}^{d-1} \times \{0\}$.

1.3 The Brunn-Minkowski theory

As alluded to in the previous sections, Sobolev inequalities are eminently geometric and measure theoretic. In this section, we want to put flesh on the bones of that allegation, as well as to introduce the Borell-Brascamp-Lieb inequality and an extended version of it, which will be used in chapter 2. Both of them are part of the so-called Brunn-Minkowski theory, which really is a wealth of inequalities, both functionnal and geometric, all related to the classical Brunn-Minkowski inequality, theorem 1.12.

1.3.1 The Brunn-Minkowski inequality and its relatives

First proved for convex sets by Hermann Brunn and Hermann Minkowski by the end of the 19th century, the Brunn-Minkowski inequality was extended to compact nonconvex sets by Lazar Lyusternik in 1935, and for that reason is also called the Brunn-Minkowski-Lyusternik inequality. It states that the d-dimensional volume, to the power $1/d$, is a concave function for the Minkowski sum of sets, defined in the following way: if A and B are subsets of $\mathbb{R}^d$,

$$A + B = \{x + y,\ (x, y) \in A \times B\}.$$

Theorem 1.12 (Brunn-Minkowski inequality). *Let A, $B \subset \mathbb{R}^d$ be nonempty compact sets. Then*

$$|A + B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}. \tag{1.12}$$

Remark 1.13. Inequality (1.12) is exactly a convacity inequality. Indeed, the Lebesgue measure is homogeneous of degree d: if $t \in [0, 1]$, $|tA| = t^d|A|$, so that using tA and $(1 - t)B$ instead of A and B in inequality (1.12) yields

$$|tA + (1 - t)B|^{1/d} \geq t|A|^{1/d} + (1 - t)|B|^{1/d}. \tag{1.13}$$

Proof. Assume that A and B are boxes in $\mathbb{R}^d$, that is $A = \prod_{i=1}^d [0, a_i]$ and $B = \prod_{i=1}^d [0, b_i]$. Note that the problem is invariant under translation of both sets, so we have assumed that A and B have a vertex at the origin. In that case, $A + B = \prod_{i=1}^d [0, a_i + b_i]$, so that

$$\begin{aligned}\frac{|A|^{1/d} + |B|^{1/d}}{|A + B|^{1/d}} &= \left(\prod_{i=1}^{d} \frac{a_i}{a_i + b_i}\right)^{1/d} + \left(\prod_{i=1}^{d} \frac{b_i}{a_i + b_i}\right)^{1/d} \\ &\leq \frac{1}{d}\sum_{i=1}^{d} \frac{a_i}{a_i + b_i} + \frac{1}{d}\sum_{i=1}^{d} \frac{b_i}{a_i + b_i} = 1,\end{aligned}$$

by the inequality of arithmetic and geometric means, and the theorem is proved for boxes.

Let us now extend this result to finite unions of disjoint boxes (they are allowed to overlap on their boundary, the measure of which is always zero). If A is such a set, let $\#A$ be the number of boxes constituting A. We proceed by induction: assume inequality (1.12) holds for unions of boxes of total cardinality $\leq N$, and consider A and B two unions of boxes such that $\#A + \#B = N + 1$. Without loss of generality, we may assume that the hyperplane $\{x_d = 0\}$ separates at least two boxes of A. We may then define $A_+ = A \cap \{x_d \geq 0\}$ and $A_- = A \cap \{x_d \leq 0\}$, both of which are unions of boxes such that $\max(\#A_-, \#A_+) < \#A$. Defining B_+ and B_- in the same manner as for A, we invoke once again the invariance by translation: shifting B if necessary, we may assume that

$$\frac{|A_+|}{|A|} = \frac{|B_+|}{|B|}.$$

Since $\#A_+ + \#B_+ < \#A + \#B = N+1$, the inductive hypothesis applies; the same goes for A_- and B_-. Notice now that $A_+ + B_+ \subset \{x_d \geq 0\}$ and $A_- + B_- \subset \{x_d \leq 0\}$, so that we finally get the following estimate

$$\begin{aligned} |A+B| \geq |A_+ + B_+| + |A_- + B_-| &\geq \left(|A_+|^{1/d} + |B_+|^{1/d}\right)^d + \left(|A_-|^{1/d} + |B_-|^{1/d}\right)^d \\ &= |A_+|\left(1 + \frac{|B|^{1/d}}{|A|^{1/d}}\right)^d + |A_-|\left(1 + \frac{|B|^{1/d}}{|A|^{1/d}}\right)^d \\ &= \left(|A|^{1/d} + |B|^{1/d}\right)^d, \end{aligned}$$

which concludes the induction. We may conclude in the general case by approximation. □

The isoperimetric inequality below is then an easy consequence of the Brunn-Minkowski inequality, theorem 1.12. The proof presented here is slightly informal, because we do not wish to define too many measure theoretic notions: instead of using the Hausdorff measure to quantify the surface area of the boundary, we shall use the outer-Minkowski content, which coincides with the Hausdorff measure for regular enough sets anyway (Lipshitz, for example).

Theorem 1.14 (Isoperimetric inequality)**.** *Let $A \subset \mathbb{R}^d$ be a set with Lipshitz boundary, and let B be the open ball centered in zero such that $|B| = |A|$. Then*

$$\mathcal{H}^{d-1}(\partial A) \geq \mathcal{H}^{d-1}(\partial B), \tag{1.14}$$

and if there is equality, then $A = B$ up to a translation and a set of zero measure.

Remark 1.15. The surface area of B may be explicitly calculated, so that inequality (1.14) becomes

$$\mathcal{H}^{d-1}(\partial A) \geq d\omega_d^{1/d}|A|^{(d-1)/d}, \tag{1.15}$$

where $\omega_d = |B(0,1)|$.

Proof. Let A be a bounded subset of $\mathbb{R}^d$. For $\varepsilon > 0$, define the dilated set $A_\varepsilon = A + B(0,\varepsilon)$; it is the set of all points at distance less than ε from A. Since A is assumed smooth enough, the surface area of the boundary may be calculated in the following manner

$$\mathcal{H}^{d-1}(\partial A) = \lim_{\varepsilon \to 0} \frac{|A_\varepsilon| - |A|}{\varepsilon}.$$

Applying the Brunn-Minkowski inequality (1.12) to A_ε, we get

$$\begin{aligned} |A_\varepsilon| &\geq \left(|A|^{1/d} + \varepsilon\omega_d^{1/d}\right)^d \\ &\geq |A| + \varepsilon d\omega_d^{1/d}|A|^{(d-1)/d}, \end{aligned}$$

where $\omega_d = |B(0,1)|$ is the volume of the d-dimensionnal unit ball. Taking the limit $\varepsilon \to 0$, we immediately conclude that

$$\mathcal{H}^{d-1}(\partial A) \geq d\omega_d^{1/d}|A|^{(d-1)/d},$$

where the right-hand side is nothing else than the surface area of a ball that has the same volume as A, and the proof is complete. □

Remark 1.16. It is interesting to note that the only inequality that we used in the proof of the isoperimetric inequality is the Brunn-Minkowski inequality, which, as we have seen, is really a consequence of the arithmetic-geometric inequality. Remarkably, this last inequality is also an isoperimetric property: indeed, it expresses the fact that amongst all boxes with a fixed volume, the one with smallest surface area is the cube. In fact, this is a rather interesting property of boxes: in dimension d, for any $k \in \{0, \dots, d\}$, the measure of the k-dimensional border of a box of unit volume is always greater then the measure of the k-dimensional border of the unit cube. For example, in dimension 3, a box of unit volume has both greater surface area and total edge length than the unit cube.

Now that these tools are defined, we turn to the proof of the following statement, where we claim equivalence between two inequalities. Obviously, from the point of view of logic, any pair of true statements are equivalent, what we mean here is that the one can be obtained from the other rather naturally, and vice versa.

Claim. In $\mathbb{R}^d$, with $d \geq 2$, the isoperimetric inequality theorem 1.14 and the Sobolev inequality for $p = 1$, theorem 1.1, are equivalent.

The proof does not go too much into measure-theoretic details, but hopefully gets the point across. We refer the interested reader to the reference book by Herbert Federer [Fed69], and also to a perhaps more easily digestible book by Evans and Garieppy [EG15]. Talenti's paper [Tal76] contains a nice short but slightly formal proof of a related fact, the Pólya-Szegő inequality.

Proof. Let us retrieve the Sobolev inequality for smooth functions from the isoperimetric inequality. We will use the coarea formula, a kind of generalized Fubini theorem, which relates the integral of the derivative of a function to the Hausdorff measure of its level sets. Before writing it down, let us first give some intuition for that formula: let $u \in C_c^\infty(\mathbb{R}^d)$ be nonnegative. If $t > 0$ is not a critical value of u, i.e. there is no $x \in \mathbb{R}^d$ such that $u(x) = t$ and $\nabla u(x) = 0$, then the level set $u^{-1}\{t\} = \{x \in \mathbb{R}^d,\, u(x) = t\}$ is a (finite) union of smooth hypersurfaces. By Sard's theorem, since u is smooth, the set of critical values is of zero measure, so that the following formula is true.

$$\int_{\mathbb{R}^d} \|\nabla u\| = \int_{\mathbb{R}_+} \mathcal{H}^{d-1}(u^{-1}\{t\})dt.$$

As it turns out, this formula, called the coarea formula, is valid in a more general case, when u is only assumed to be Lipshitz, see for instance [EG15].

For $t \geq 0$, we write $\{u > t\} = u^{-1}((t, +\infty])$ and $\{u = t\} = u^{-1}\{t\}$. The smoothness of u furthermore implies that $\{u = t\} = \partial\{u > t\}$ for almost every $t \geq 0$, so that, calling to the isoperimetric inequality (1.15), we find that

$$\int_{\mathbb{R}^d} \|\nabla u\| \geq d\omega_d^{1/d} \int_{\mathbb{R}_+} |\{u > t\}|^{(d-1)/d} dt.$$

The goal is now to relate this last integral to the L^{1^*} norm of u, where $1^* = \frac{d}{d-1}$. Note that, by Fubini's theorem, if χ_A denotes the characteristic function of a set A, then for all $x \in \mathbb{R}^d$,

$$u(x) = \int_0^{+\infty} \chi_{\{u>t\}}(x)dt.$$

Since, for a fixed $x \in \mathbb{R}^d$, the function $t \mapsto \chi_{\{u>t\}}(x)$ is piecewise constant, it is also Riemann-integrable. We may thus see u as the limit of Riemann sums, and the triangular inequality readily passes to the limit and implies that

$$\|u\|_{1^*} \leq \int_0^{+\infty} \left\|\chi_{\{u>t\}}\right\|_{1^*} dt = \int_{\mathbb{R}_+} |\{u > t\}|^{(d-1)/d} dt,$$

which immediately proves

$$\|\nabla u\|_1 \geq d\omega_d^{1/d} \|u\|_{1^*}. \tag{1.16}$$

It is not yet clear whether or not this inequality is sharp, information may have been lost in this little calculation. Nevertheless, we will see that it implies the isoperimetric inequality, which is sharp, and will also prove that the optimal constant in the Sobolev inequality is indeed $S(d,1) = \left(d\omega_d^{1/d}\right)^{-1}$. This implication is easier than the other one, as is usually the case when going between inequalities applying to sets and their functional counterparts. Consider a smooth bounded set $A \subset \mathbb{R}^d$. For $\varepsilon > 0$, let

$$u_\varepsilon : x \mapsto \begin{cases} 1 & \text{if } x \in A, \\ 1 - \frac{d(x,A)}{\varepsilon} & \text{if } x \in A_\varepsilon \backslash A, \\ 0 & \text{if } d(x,A) > \varepsilon, \end{cases}$$

where $A_\varepsilon = A + B(0, \varepsilon)$. Even though u_ε is not smooth, the Sobolev inequality still applies by approximation by smooth functions. Then, for ε small enough, we find that $\|\nabla u_\varepsilon\| = \frac{1}{\varepsilon}\chi_{A_\varepsilon \backslash A}$, so that inequality (1.16) reads

$$\|\nabla u_\varepsilon\|_1 = \frac{|A_\varepsilon| - |A|}{\varepsilon} \geq d\omega_d^{1/d} \|u_\varepsilon\|_{d/(d-1)}.$$

Since it is pretty clear that $\lim_{\varepsilon \to 0} u_\varepsilon = \chi_{\bar{A}}$, letting ε go to 0 ultimately yields exactly the isoperimetric inequality (1.15), which, being sharp, also implies that there can be no better constant in inequality (1.16). □

While we may derive Sobolev inequalities for $p \in (1, d)$ from the case $p = 1$ (1.16) using the substitution $u = f^{p(d-1)/(d-p)}$ and then applying Hölder's inequality to the integral where the gradient appears, the resulting inequality turns out not to be sharp. Nevertheless, it is still possible to recover the whole family of sharp Sobolev inequalities for $p \in [1, d)$ using only inequalities derived from the Brunn-Minkowski inequality. To that end, we will introduce the Borell-Brascamp-Lieb inequality, a variant of which was used by Sergei Bobkov and Michel Ledoux in [BL08] for a direct proof of Sobolev inequalities.

1.3.2 The Borell-Brascamp-Lieb inequality

Before tackling the inequalities in this subsection, let us briefly introduce the useful concept of p-means.

Definition 1.17 (p-means)**.** Let $p \in [-\infty, +\infty]$, and a, $b \geq 0$, $t \in [0, 1]$. If $ab > 0$, then

$$M_p(a, b\,; t) = \left((1-t)a^p + tb^p\right)^{1/p}, \tag{1.17}$$

and if $ab = 0$, then $M_p(a, b\,; t) = 0$. For $p = 0$, the value of $M_0(a, b\,; t)$ is to be understood as the limit of $M_p(a, b\,; t)$ when p goes to 0, and the same goes for $+\infty$ and $-\infty$. In particular, $M_0(a, b\,; t) = a^{1-t}b^t$ corresponds to the geometric mean, and $M_{+\infty}(a, b\,; t) = \max(a, b)$, and $M_{-\infty}(a, b\,; t) = \min(a, b)$.

Applying Jensen's inequality to p-means immediately yields the following important consequence :

$$\text{if } p \geq q, \text{ then } M_p(a, b\,; t) \geq M_q(a, b\,; t). \tag{1.18}$$

With this notation, Brunn-Minkowski's inequality (1.13) reads $|(1-t)A + tB| \geq M_{1/d}(|A|, |B|\,; t)$. The ordering of the p-means leads to a whole family of inequalities, for all $p \in [-\infty, 1/d]$, $|(1-t)A + tB| \geq M_p(|A|, |B|\,; t)$. However, as it turns out, all of them are equivalent: the case $p = -\infty$ can easily be used to recover the case $p = 1/d$, see for instance [Gar02, Section 4].

Perhaps the case $p = 0$ is of particular interest: for all nonempty compact subsets $A, B \subset \mathbb{R}^d$ and all $t \in [0, 1]$,

$$|(1-t)A + tB| \geq |A|^{1-t}|B|^t.$$

This multiplicative form of the Brunn-Minkowski inequality has a functional counterpart:

Theorem 1.18 (Prékopa-Leindler inequality)**.** *Let u, v, w be nonnegative integrable functions on $\mathbb{R}^d$. If, for all x, $y \in \mathbb{R}^d$,*

$$w\big((1-t)x + ty\big) \geq u(x)^{1-t}v(y)^t,$$

then

$$\|w\|_1 \geq \|u\|_1^{1-t}\|v\|_1^t.$$

If A, B are nonempty compact subsets of $\mathbb{R}^d$, applying theorem 1.18 to the functions $u = \chi_A$, $v = \chi_B$ readily yields the multiplicative version of the Brunn-Minkowski inequality, of which the Prékopa-Leindler is thus, in fact, a generalization. This inequality differs from the Brunn-Minkowski inequality in that it does not involve the dimension of the underlying space. Thus, it is well suited to study properties of the Gaussian measure, that is infinite dimensional in the sense that it can be seen as the limit of measures in spaces of increasing dimension. It was notably used by Bobkov and Ledoux to recover the sharp logarithmic Sobolev inequality for the Gaussian measure in [BL00].

The Prékopa-Leindler inequality has been extensively studied, notably by Herm Jan Brascamp and Elliott Lieb, and also Christer Borell independently, who further generalized the inequality [Bor75, BL76], as well as alleviated some integrability technicalities, linked to the fact that the Minkowski sum of two measurable sets might not be measurable. They proved the following theorem, which now bears their name. We refer to the survey [Gar02, Section 10] and the references therein for further historical details.

Theorem 1.19 (Borell-Brascamp-Lieb inequality)**.** *Let u, v, w be nonnegative integrable functions on $\mathbb{R}^d$, and let $p \in [-1/d, +\infty]$. If, for all x, $y \in \mathbb{R}^d$,*

$$w\big((1-t)x + ty\big) \geq M_p(u(x), v(y)\,; t)$$

then

$$\|w\|_1 \geq M_q(\|u\|_1, \|v\|_1\,; t),$$

where $q = p/(1+pd)$, with $q = -\infty$ when $p = -1/d$ and $q = 1/d$ when $p = +\infty$.

Note how the case $p = 0$ is exactly the Prékopa-Leindler inequality. Interestingly, the case $p = -1/d$ is the strongest in the sense that it can be used to recover all the other cases. Rewriting the theorem in that case after replacing w by w^{-d}, and doing the same with u and v, reads: if, for all $x, y \in \mathbb{R}^d$,

$$w\big((1-t)x + ty\big) \leq (1-t)u(x) + tv(y),$$

then

$$\int_{\mathbb{R}^d} w^{-d} \geq \min\left(\int_{\mathbb{R}^d} u^{-d}, \int_{\mathbb{R}^d} v^{-d}\right). \tag{1.19}$$

While this inequality is a definite improvement on the Prékopa-Leindler inequality, it is still not enough to reach the sharp Sobolev inequality. To that end, Bobkov and Ledoux used a strengthened version of the Borell-Brascamp-Lieb inequality for functions which projections onto a line have the same L^∞ norm [BL08]. They elegantly proved it using nothing else than an elementary lemma from the Brunn-Minkowski theory and smart tensorization.

In chapter 2, we will use a different strengthening of the Borell-Brascamp-Lieb inequality, that was proved with optimal transport in a recent paper by Bolley et al. [BCEF+17], and which we reproduce in the next section. The idea is to be able to use the parameter p in theorem 1.19 up to $-1/(d-1)$, or, equivalently, to be able to find a lower bound for $\int w^{1-d}$ in inequality (1.19). This is only possible at the cost of further assumptions on u and v, namely, that $\int u^{-d} = \int v^{-d} = 1$. If this assumption is fulfilled, and if

$$\forall\, x,\, y \in \mathbb{R}^d, \quad w((1-t)x + ty) \leq (1-t)u(x) + tv(y)$$

then

$$\int_{\mathbb{R}^d} w^{1-d} \geq (1-t)\int_{\mathbb{R}^d} u^{1-d} + t\int_{\mathbb{R}^d} v^{1-d}.$$

1.4 Optimal transport

1.4.1 Fundamentals

Somewhat recently, the theory of optimal transport has proved a very powerful tool in the study of sharp Sobolev inequalities, in that the proofs are usually quite short and simple. For a very nice introduction on this topic with a lot of historical context, we refer to Dario Cordero-Erausquin, Bruno Nazaret and Cédric Vilani's paper [CENV04]. While we do not use optimal transport in chapters 3 and 4, the inequality on which all of chapter 2 is based has only been proved, up to our knowledge, using optimal transport. Furthermore, optimal transport is very much relevant for all the other results as well, since it can usually provide an interesting viewpoint to interpret them. For that reason, we will briefly introduce the theory here, as well as provide a quick proof of the strengthened Borell-Brascamp-Lieb inequality, theorem 2.1 below, for the sake of completeness.

Geometry and analysis are at the very center of the theory of optimal transport. Seeing how both Sobolev inequalities are at the confluence of both of these fields, it makes sense that optimal transport is an appropriate tool to tackle problems related to such inequalities. Although the topic itself is very old and generally attributed to Gaspard Monge, it was not before 1987 that it started being immensely popular amongst many reasearchers. The work that sparked that revival is a note by Yann Brenier [Bre87], in which he proved that given two probability measures, requiring that the first one is absolutely continuous with respect to the Lebesgue measure is enough to ensure that there exists a mapping which pushes the first measure forward onto the second one, and that is the gradient of some convex function. It was later refined by various authors, including Robert McCann, who removed the assumption that the second order moments should be finite. We recall this theorem here without proof, and refer the interested reader to Cédric Villani's book [Vil03] for a presentation rich in context and insightful remarks.

Theorem 1.20 (Brenier's theorem)**.** *Let μ and ν be two probability measures on $\mathbb{R}^d$. If μ is absolutely continuous with respect to the Lebesgue measure, then there exists a (μ-almost everywhere unique) convex function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla\varphi_{\#}\mu = \nu$.*

Here, $T_\# \mu$ designates the measure obtained by pushing μ forward (or transporting) with the map T. In other words, for all Borel subset $B \subset \mathbb{R}^d$, $T_\#\mu(B) = \mu(T^{-1}(B))$. Now, let us assume that *both* measures μ and ν are absolutely continuous with respect to the Lebesgue measure, and abuse the notation slightly by denoting their densities by $\mu(x)$ and $\nu(x)$, respectively. The fact that $\nu = \nabla\varphi_\#\mu$ implies that for all nonnegative measurable functions $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\int_{\mathbb{R}^d} f(x)\nu(x)dx = \int_{\mathbb{R}^d} f(\nabla\varphi(x))\mu(x)dx. \tag{1.20}$$

Assuming that $\nabla\varphi$ is a $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}^d$ onto itself, the change of variables $x = \nabla\varphi(y)$ in equation (1.20) shows that φ is a solution of the so-called Monge-Ampère equation,

$$\mu(x) = \nu(\nabla\varphi(x)) \det(\nabla^2\varphi(x)). \tag{1.21}$$

To try to make sense of that equation without any regularity assumption, we rely on the fact that φ is convex. Hence, φ is locally Lipschitz, hence almost everywhere differentiable, according to Rademacher's theorem. But convexity is even stronger, and Aleksandrov's theorem asserts that the distributionnal Hessian of φ is actually almost everywhere absolutely continuous with respect to the Lebesgue measure, see for example [EG15, pp. 242–245] for a proof.

As Robert McCann proved in his paper [McC97, Remark 4.5], the absolute continuity of both measures μ and ν implies that the Monge-Ampère equation (1.21) is indeed satisfied μ-almost everywhere in the Aleksandrov sense, meaning that the second derivative $\nabla^2\varphi$ is to be understood as the absolutely continuous part (w.r.t the Lebesgue measure) of the distributional Hessian. Note that, under various conditions on μ and ν, Luis Cafarelli proved that equation (1.21) is actually satisfied in the classical sense, but we will not need such strong results in our applications.

Theorem 1.21 (Monge-Ampère)**.** *Let μ and ν be two absolutely continuous probability measures. If φ is a convex function such that $\nu = \nabla\varphi_\#\mu$, then, for μ-almost every $x \in \mathbb{R}^d$,*

$$\mu(x) = \nu(\nabla\varphi(x)) \det(\nabla^2\varphi(x)). \tag{1.22}$$

1.4.2 A proof of the improved Borell-Brascamp-Lieb inequality

We turn here to the proof of theorem 2.1 for the sake of completeness. Before tackling the proof itself, we will need two elementary lemmata.

The first lemma, which we will not prove, and refer to the last point in [Vil03, Theorem 2.12], is a classical result about Legendre transforms. It states that the inverse of the gradient of a convex function φ is the gradient of its Legendre transform φ^*, given by

$$\varphi^*(y) = \sup_{x\in\mathbb{R}^d} \{x \cdot y - \varphi(x)\}.$$

Lemma 1.22. *If μ and ν are both absolutely continuous probability measures, and $\nabla\varphi$ is the unique gradient of a convex function transporting μ onto ν, then*

$$\nabla\varphi^* \circ \nabla\varphi(x) = x, \qquad \nabla\varphi \circ \nabla\varphi^*(y) = y$$

for μ-almost every x and ν-almost every y.

Remark 1.23. Lemma 1.22 also implies that $\nabla\varphi^*$ is the unique gradient of a convex function transporting ν onto μ.

We will also use the following application of the arithmetic-geometric inequality, which is a very natural matrix version of the Brunn-Minkowski inequality, if one understands the determinant of a matrix as the (oriented) volume of the image of the unit cube. However, stating this is not enough to prove the lemma, since it is not entirely clear that the image of a (vector) sum of matrices is the same thing as the (Minkowski) sum of the images. The same result is used by McCann in his thesis to prove a similar inequality [McC94, Corollary D.4].

Lemma 1.24. *For any two symmetric positive matrices A, $B \in M_d(\mathbb{R})$,*

$$\det\big(A+B\big)^{1/d} \geq \det(A)^{1/d} + \det(B)^{1/d}.$$

Proof. Factorizing by A, it is sufficient to prove that $\det(I_d + C)^{1/d} \geq 1 + \det(C)^{1/d}$, where $C = A^{-1}B$. As a product of positive matrices, C is also positive, and it is similar to a symmetric matrix, $C = A^{-1/2}(A^{-1/2}BA^{-1/2})A^{1/2}$, and thus diagonalizable. If c_i, $i \in \{1, \dots, d\}$, are the eigenvalues of C, the lemma then boils down to proving that

$$\prod_{i=1}^{d}(1+c_i)^{1/d} \geq 1 + \prod_{i=1}^{d} c_i^{1/d},$$

which follows from the arithmetic-geometric inequality applied to

$$\left(\prod_{i=1}^{d}\frac{1}{1+c_i}\right)^{1/d} + \left(\prod_{i=1}^{d}\frac{c_i}{1+c_i}\right)^{1/d}.$$

□

Theorem 1.25 (Improved Borell-Brascamp-Lieb inequality)**.** *Let $d \geq 2$, and $t \in [0,1]$. Let u, v, and $w : \mathbb{R}^d \to (0, +\infty]$ be measurable functions such that $\int u^{-d} = \int v^{-d} = 1$ and*

$$\forall\, x,\, y \in \mathbb{R}^d, \quad w((1-t)x + ty) \leq (1-t)u(x) + tv(y) \tag{1.23}$$

then

$$\int_{\mathbb{R}^d} w^{1-d} \geq (1-t)\int_{\mathbb{R}^d} u^{1-d} + t\int_{\mathbb{R}^d} v^{1-d}.$$

Proof. u^{-d} and v^{-d} being probability measures, consider the convex function φ such that $\nabla\varphi$ transports u^{-d} onto v^{-d}. McCann's displacement interpolation is given by $\rho_t = \nabla\varphi_{t\#}u^{-d}$, where $\nabla\varphi_t = (1-t)x + t\nabla\varphi$. For all $t \in [0,1]$, almost everywhere, the Monge-Ampère equation holds:

$$u^{-d} = \rho_t(\nabla\varphi_t)\det(\nabla^2\varphi_t).$$

In particular, for $t = 1$, $u^{-d} = v^{-d}(\nabla\varphi)\det(\nabla^2\varphi)$ almost everywhere. Now, the concavity of the map $M \mapsto \det^{1/d}$ on positive symmetric matrices (lemma 1.24) implies that,

$$\begin{aligned}\det(\nabla^2\varphi_t)^{1/d} &= \det((1-t)I + t\nabla^2\varphi)^{1/d}\\ &\geq (1-t)\det(I)^{1/d} + t\det(\nabla^2\varphi)^{1/d},\end{aligned}$$

so that almost everywhere

$$\begin{aligned}\rho_t(\nabla\varphi_t) &\leq u^{-d}\Big((1-t) + t\det(\nabla^2\varphi)^{1/d}\Big)^{-d}\\ &= \Big((1-t)u + tu\det(\nabla^2\varphi)^{1/d}\Big)^{-d}\\ &= \big((1-t)u + tv(\nabla\varphi)\big)^{-d}.\end{aligned}$$

This estimate on ρ_t, together with hypothesis (1.23) on w, yields that almost everywhere,

$$((1-t)u + tv(\nabla\varphi))\rho_t(\nabla\varphi_t) \leq ((1-t)u + tv(\nabla\varphi))^{1-d} \leq w(\nabla\varphi_t)^{1-d}. \tag{1.24}$$

In particular, we may apply equation (1.24) to $\nabla\varphi_t^*(x)$ and use the fact that almost everywhere in $\mathbb{R}^d$, $\nabla\varphi_t \circ \nabla\varphi_t^* = I_d$ to find that, almost everywhere,

$$((1-t)u(\nabla\varphi_t^*) + tv(\nabla\varphi \circ \nabla\varphi_t^*))\rho_t \leq w^{1-d}. \tag{1.25}$$

Integrating inequality (1.25), we find, by definition of the transport,

$$\begin{aligned}
\int_{\mathbb{R}^d} w^{1-d} &\geq \int_{\mathbb{R}^d} ((1-t)u(\nabla\varphi_t^*) + tv(\nabla\varphi \circ \nabla\varphi_t^*))\rho_t \\
&= (1-t)\int_{\mathbb{R}^d} u(\nabla\varphi_t^* \circ \nabla\varphi_t)u^{-d} + t\int_{\mathbb{R}^d} v(\nabla\varphi \circ \nabla\varphi_t^* \circ \nabla\varphi_t)u^{-d} \\
&= (1-t)\int u^{1-d} + t\int_{\mathbb{R}^d} v(\nabla\varphi)u^{-d} \\
&= (1-t)\int u^{1-d} + t\int_{\mathbb{R}^d} v^{1-d},
\end{aligned}$$

which concludes the proof. □

Remark 1.26. This theorem can also be seen as the result of the convexity of the functional $\mathcal{F}(\rho) = \int_{\mathbb{R}^d} -\rho(x)^{1-1/d}dx$ along the McCann's displacement interpolation, a property that is called displacement convexity. The crucial step, which is already present in McCann's thesis [McC94], is the domination of the displacement interpolant $\rho_t \leq w^{-d}$.

1.4.3 Some words on Otto's calculus

To provide a bit of context to the equations considered in chapters 3 and 4, we introduce Otto's calculus. Its name, coined by Cédric Villani in [Vil09], refers to Felix Otto, who in his 2001 paper [Ott01] proposed an interesting way to see the space of probability measures, endowed with a distance derived from optimal transport, as a Riemannian manifold. While this point of view was, at the beginning, mostly formal, Ambrosio, Gigli and Savaré [AGS08] removed many technical problems since. Nevertheless, Otto's calculus remains an excellent heuristic tool to understand flows of probability measures.

Let us first define a few additional classical notions of optimal transport,

Definition 1.27. Let μ and ν be two probability measures on $\mathbb{R}^d$. We define $\Pi(\mu,\nu)$ as the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν, respectively. Equivalently, $\pi \in \Pi(\mu,\nu)$ if for all $A \subset \mathbb{R}^d$, $\pi(A \times \mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d \times A) = \nu(A)$. If μ and ν have finite second moment, that is $x \mapsto \|x\|^2 \in L^2(\mu) \cap L^2(\nu)$, then we define

$$W_2(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{\mathbb{R}^d} \|x-y\|^2 d\pi(x,y)\right)^{1/2}. \tag{1.26}$$

W_2 is a distance on the set of probability measure with finite second moment, $\mathcal{P}_2(\mathbb{R}^d)$, it is called the *Wasserstein distance*, and consequently, $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ is called the *Wasserstein space*.

Remark 1.28. Replacing 2 with some $p \geq 1$ in equation (1.26), one may define p-Wasserstein distances, and thus other Wasserstein spaces. However, since it is the only one we consider, the Wasserstein space will always designate $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ in this manuscript.

Remarkably, if μ is absolutely continuous with respect to the Lebesgue measure, which we write $\mu \in \mathcal{P}_{2,ac}(\mathbb{R}^d)$, then the infimum in equation (1.26) is reached for $\pi = (I_d \otimes \nabla\varphi)_{\#}\mu$, where $\nabla\varphi$ is Brenier's map from theorem 1.20. We recall this fact in the next theorem.

Theorem 1.29 (Brenier's theorem, continued)**.** *If $\mu, \nu \in \mathcal{P}_{2,ac}(\mathbb{R}^d)$, then there exists a μ-almost everywhere unique $\nabla\varphi$ with φ convex, such that $\nabla\varphi_{\#}\mu = \nu$, and furthermore,*

$$W_2^2(\mu, \nu) = \int_{\mathbb{R}^d} \|x - \nabla\varphi(x)\|^2 d\mu(x).$$

Note that, by the uniqueness of the map $\nabla\varphi$, given *any* convex function ψ such that $\nabla\psi_{\#}\mu = \nu$, then necessarily $W_2^2(\mu,\nu) = \int \|x - \nabla\psi(x)\|^2 d\mu(x)$. As a consequence, MacCann's displacement interpolation, $\mu_t = ((1-t)I_d + t\nabla\varphi)_{\#}\mu$ is a constant speed geodesic, since a straightforward computation shows that $W_2(\mu_t, \mu_s) = (t-s)W_2(\mu,\nu)$ for $0 \le s \le t \le 1$. It is thus pretty clear that the Wasserstein distance endows the probability space $\mathcal{P}_2(\mathbb{R}^d)$ with a very convenient structure.

Otto's idea was, at least heuristically, to see the space of probability measures with finite second moment as an infinite dimensional manifold, by defining a tangent space at every probability measure, as well as an inner product associated to that probability, so that the resulting Riemannian distance coincides with the Wasserstein distance. We would like to give the reader a sense of how this can work here, using Ivan Gentil's note [Gen19] (in French) as an inspiration, and also drawing parallels with fluid mechanics. In the rest of this subsection, we purposefully keep the presentation mostly formal, and we will also use the same notation for a measure and its density with respect to the Lebesgue measure. The first observation is the following: given a path $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}_2(\mathbb{R}^d)$, there exists a function ϕ_t such that

$$\partial_t \mu_t + \operatorname{div}(\mu_t \nabla\phi_t) = 0, \tag{1.27}$$

This equation, called the continuity equation, is a very natural consequence of *mass conservation*. In a fluid with density field ρ and velocity field v, mass conservation readily implies $\partial_t\rho + \operatorname{div}\rho v = 0$. Of course, some regularity hypothesis are needed on the path (μ_t) to have existence of the function ϕ_t, but we will not bother with technicalities now, and instead redirect the reader to [AGS08, Theorem 8.3.1].

Given a path (μ_t) satisfying the continuity equation, we may choose to define its tangent vectors to be $\dot{\mu_t} = \nabla\phi_t$. Thus, for a probability measure μ, we define the tangent space $T_\mu\mathcal{P}_2(\mathbb{R}^d) = \{\nabla\phi, \phi : \mathbb{R}^d \to \mathbb{R}\}$ as the space of velocities. A natural metric is then the quadratic form associated to the kinetic energy. For $\nabla\phi, \nabla\psi \in T_\mu\mathcal{P}(\mathbb{R}^d)$,

$$(\nabla\phi, \nabla\psi)_\mu = \int_{\mathbb{R}^d} \nabla\phi \cdot \nabla\psi d\mu.$$

The (formal) Riemannian distance associated to that metric is then

$$d(\mu,\nu) = \inf_{(\mu_t,\phi_t)\in C(\mu,\nu)} \left(\int_0^1 \left(\int_{\mathbb{R}^d} \|\nabla\phi_t\|^2 d\mu_t \right)^{1/2} dt \right)$$

where $C(\mu,\nu) = \{(\mu_t,\phi_t)_{t\in[0,1]} \mid \partial_t\mu_t = \operatorname{div}(\mu_t\nabla\phi_t),\ \mu_0 = \mu,\ \mu_1 = \nu\}$ designates the (continuous) paths between μ and ν. As announced, this coincides exactly with the Wasserstein distance, as Jean-David Benamou and Yann Brenier [BB00] have proved:

Proposition 1.30 (Benamou-Brenier formula)**.** *If $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, then*

$$W_2(\mu,\nu) = \inf_{(\mu_t,\phi_t)\in C(\mu,\nu)} \left(\int_0^1 \int_{\mathbb{R}^d} \|\nabla\phi_t\|^2 d\mu_t dt \right)^{1/2} = d(\mu,\nu).$$

Remark 1.31. It might not be completely obvious that finding minimizers of the energy functional

$$\int_0^1 \int_{\mathbb{R}^d} \|\nabla \phi_t\|^2 d\mu_t dt$$

yields paths with minimal length between μ and ν, but it is a classical fact of Riemannian geometry, still true in this context, as it is just a consequence of the Cauchy-Schwarz inequality in $L^2((0,1))$ equipped with the Lebesgue measure.

With that setting in place, we may identify the gradient of functionals defined on the Wasserstein space with the formula

$$(\operatorname{grad}_{\mu_t} \mathcal{F}, \dot{\mu_t})_{\mu_t} = \frac{d}{dt}\mathcal{F}(\mu_t).$$

For example, if $\mathcal{F}$ is Boltzmann's entropy defined by $\operatorname{Ent}(\mu) = \int_{\mathbb{R}^d} \mu \log(\mu)$, then

$$\begin{aligned}\frac{d}{dt}\operatorname{Ent}(\mu_t) = \frac{d}{dt}\int_{\mathbb{R}^d} \mu_t \log(\mu_t) &= -\int_{\mathbb{R}^d} \log(\mu_t)\operatorname{div}(\mu_t \nabla \phi_t) \\ &= \int_{\mathbb{R}^d} \nabla \log(\mu_t) \cdot \nabla \phi_t \mu_t = (\nabla \log \mu_t, \dot{\mu_t})_{\mu_t}\end{aligned}$$

so that $\operatorname{grad}_\mu \operatorname{Ent} = \nabla \log \mu$. This fact will prove very relevant, because then the heat equation $\partial_t \mu_t = \Delta \mu_t$ is exactly the gradient flow of the Boltzmann entropy: indeed,

$$\Delta \mu_t = \operatorname{div}(\mu_t \nabla(\log \mu_t)),$$

so that

$$\partial_t \mu_t = \Delta \mu_t \iff \dot{\mu_t} = -\operatorname{grad}_{\mu_t} \operatorname{Ent}(\mu_t).$$

Such an equation is an example of a *gradient flow*, which implies important consequences on its solutions. Notably, if we consider a functional $\mathcal{F}$, and a solution to $\dot{\mu_t} = -(\operatorname{grad}_{\mu_t} \mathcal{F}, \dot{u_t})_{\mu_t}$, then, by definition,

$$\frac{d}{dt}\mathcal{F}(\mu_t) = -\|\dot{\mu_t}\|_{\mu_t}^2 \leq 0.$$

We will come back to this with concrete examples in the next sections.

1.5 Long-term behavior of solutions to the heat equation

1.5.1 Convergence on compact manifolds

In this section, we stray apart from the more geometric considerations of the previous sections, and we expand on the analysis side of Sobolev inequalities. One of the popular uses of the Sobolev inequalities is to study long-term behavior of solutions of certain parabolic equations, with the use of particular entropies. To illustrate this phenomenon, let us look at the heat equation on a Riemannian manifold M, that we also assume compact and smooth. For some nonnegative initial condition u_0 such that $\|u_0\|_1 = 1$, let us study the heat equation

$$\begin{cases} \partial_t u = \Delta u, & t > 0, \\ u = u_0, & t = 0, \end{cases} \tag{1.28}$$

where Δ is the Laplace-Beltrami operator. Let us not bother with existence and regularity of solutions to this equation right now: assume there is a unique solution, and that it is smooth and positive. In what follows, we will take the liberty to consider this solution as a one-parameter family of functions $(u_t)_{t\in\mathbb{R}_+}$ defined by $u_t(x) = u(t,x)$ for all $x \in M$, which we call the flow. For positive functions u defined on M, define the entropy

$$\operatorname{Ent}_\mu(u) = \int_M u\log(u)d\mu,$$

where μ is the (only) probability measure proportionnal to the Riemannian volume form. Differentiating the entropy along the flow yields

$$\begin{aligned}\frac{d}{dt}\operatorname{Ent}_\mu(u_t) &= \int_M \Delta u_t(\log(u_t)+1)d\mu \\ &= -\int_M \frac{\|\nabla u_t\|^2}{u_t}d\mu < 0,\end{aligned}$$

the integration by parts being valid since μ is proportionnal to the Riemannian volume form. The fact that the entropy is always decreasing translates the fact that the physical phenomenon that the heat equation models is not reversible. In chapters 3 and 4, we will use functionnals which we will still call *entropy*, even though they will not necessarily be the physical (Boltzmann) entropy. That is because the entropy is to be thought of merely as a tool; a functionnal that is decreasing along a particular flow.

If the logarithmic Sobolev inequality (1.2) is valid on the manifold M for some constant $C > 0$, then note that it relates exactly the entropy and its derivative along the flow. Thus,

$$\operatorname{Ent}_\mu(u_t) \leq -C\frac{d}{dt}\operatorname{Ent}_\mu(u_t),$$

which, when integrated, leads to

$$\operatorname{Ent}_\mu(u_t) \leq e^{-Ct}\operatorname{Ent}_\mu(u_0).$$

Finally, we may use the classical Csiszár-Kullback-Pinsker inequality (see for example [Jü16, Theorem A.2] for a quick proof) to relate the entropy to the L^1 norm.

Lemma 1.32 (Csiszár-Kullback-Pinsker inequality)**.** *Let u, v be integrable functions, $u \geq 0$ and $v > 0$, such that $\int_M ud\mu = \int_M vd\mu = 1$. Then*

$$\|u - v\|_1^2 \leq 2\operatorname{Ent}_\mu(u \mid v) = \int_M u\log\Big(\frac{u}{v}\Big)d\mu.$$

Applying this lemma to $u = u_t$ and $v = 1$, we may conclude that u_t converges towards 1 in $L^1(\mu)$ with exponential speed.

Proposition 1.33. *If u_t is the solution to the heat equation (1.28) on a manifold M that satisfies a logarithmic Sobolev inequality (1.2) with constant C, then*

$$\|u_t - 1\|_{L^1(\mu)}^2 \leq 2\operatorname{Ent}_\mu(u_t) \leq 2e^{-Ct}\operatorname{Ent}_\mu(u_0).$$

Remark 1.34. Like we pointed out in the previous section, the heat flow is the gradient flow of the Boltzmann entropy, which makes this particular entropy a good candidate to study the equation. However, it is not the only possible choice: in subsection 1.6.3, we study the L^2 norm instead to prove the Poincaré inequality (1.3).

1.5.2 Relaxation to self-similarity

Even though we made use of the fact that the manifold M is compact in the previous paragraph, a similar behavior can be observed on $\mathbb{R}^d$. Again, we consider a solution u_t of the heat equation $\partial_t u_t = \Delta u_t$, with nonnegative, unit-mass initial condition u_0. In that particular case, the unique solution $u_t \in L^1(\mathbb{R}^d)$ is known, and explicitly given by

$$u_t(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\|x-y\|^2/(4t)} u_0(y) dx.$$

We observe that

$$\|u_t\|_\infty \leq \frac{1}{(4\pi t)^{d/2}},$$

so that, by interpolation, for all $p > 1$, the L^p norm of u_t goes to zero when time goes to infinity. Just like on a compact manifold, the solution converges towards a steady solution of the heat equation. However, in the case of of $\mathbb{R}^d$, the only *integrable* steady solution of the heat equation is the zero function. The fact that the heat equation is mass-preserving and that there is no steady solution of unit mass is already hinting to the fact that solutions of the heat equation need to be seen in a different context to study their long-term behavior. To circumvent this limitation, the trick is to not study u_t directly, but a time-dependent rescaling of it, namely

$$v(t,x) = e^{dt} u\left(\frac{e^{2t}-1}{2}, e^t x\right),$$

where we used $u(t,x) = u_t(x)$ for clarity. This rescaling is mass-preserving too, and v is then a solution of

$$\begin{cases} \partial_t v_t = \Delta v_t + \operatorname{div}(x v_t), \\ v_0 = u_0. \end{cases} \tag{1.29}$$

This equation, called the Fokker-Planck equation, is exactly the heat equation with an added confining potential, so that the mass of its solutions does not escape to infinity. To look for its steady states, and better understand the long-term behavior of its solutions, it is worth to rewrite it in the following manner: if v is a positive function,

$$\Delta v + \operatorname{div}(xv) = \operatorname{div}(\nabla v + xv) = \operatorname{div}(v \nabla(\log(v) - \log(v_\infty))), \tag{1.30}$$

where v_∞ is such that $\nabla \log(v_\infty) = -x$, so that we can choose v_∞ to be the standard Gaussian,

$$v_\infty(x) = \frac{1}{(2\pi)^{d/2}} e^{-\|x\|^2/2}.$$

Assuming v is a positive and integrable steady state of the Fokker-Planck equation, it is the rescaling of an integrable solution to the heat equation, and thus smooth and decreases sufficiently fast at infinity for the following integration by parts to be valid: multiplying equation (1.29) by $\log(v) - \log(v_\infty)$ and integrating yields

$$0 = \int_{\mathbb{R}^d} (\log(v) - \log(v_\infty)) \operatorname{div}(v \nabla(\log(v) - \log(v_\infty))) = -\int_{\mathbb{R}^d} v \|\nabla \log(v) - \nabla \log(v_\infty)\|^2,$$

so that $\log(v) = \log(v_\infty) + C$, where C is some real constant. Thus, we have determined all positive steady states to be multiples of v_∞, the standard Gaussian.

Let us now go back to our rescaled solution of the heat equation v_t. In the same manner as the compact manifold case, v_t has unit mass, and thus converges towards v_∞. Furthermore, the relative entropy decays exponentially:

$$\operatorname{Ent}_{dx}(v_t \mid v_\infty) = \int_{\mathbb{R}^d} v_t \log\left(\frac{v_t}{v_\infty}\right) \leq e^{-2t} \operatorname{Ent}_{dx}(v_0 \mid v_\infty).$$

This classical fact is a direct consequence of theorem 4.3 below, which generalizes Sobolev inequalities. It is otherwise formally proved in [Jü16, Theorem 2.1]. Finally, we may once again invoke the classical Csiszár-Kullback-Pinsker inequality to conclude exponential decay of $\|v_t - v_\infty\|_1^2$, whence, after changing variables, we find the following proposition.

Proposition 1.35. *If $u_t \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d))$ is the solution to the heat equation (1.28) on $\mathbb{R}^d$,*

$$\|u_t - U_t\|_1^2 \leq \frac{2}{2t+1} \operatorname{Ent}_{dx}(u_0 \mid v_\infty),$$

where

$$U_t(x) = \frac{1}{(2t+1)^{d/2}} v_\infty\left(\frac{x}{\sqrt{2t+1}}\right).$$

Remark 1.36. The function U_t is a particular solution of the heat equation called the *self-similar solution* of the heat equation. The solution u_t is thus said to relax to self-similarity.

The change of point of view that we operated here, going from the heat equation (1.28) to the Fokker-Planck (1.29), is particularly efficient, and will be the starting point of chapter 4. For example, instead of studying the fast diffusion equation $\partial_t u = \Delta u^m$, for $m \in [1/d, 1)$, we will instead study the following nonlinear Fokker-Planck equation

$$\partial_t v = \operatorname{div}(v(mv^{m-2}\nabla w + x)), \tag{1.31}$$

obtained through the change of variables

$$v(t,x) = e^{dt} u\left(\frac{e^{\lambda t} - 1}{\lambda}, e^t x\right), \quad \lambda = 2 - (1-m)d$$

Just like we did with equation (1.30), equation (1.31) can be rewritten in a perhaps more natural way using the function $\psi(x) = -\frac{m}{1-m} x^{m-1}$, as it allows for easy identification of steady states for example:

$$\partial_t v = \operatorname{div}\big(v\nabla(\psi(v) - \psi(v_\infty))\big), \quad v_\infty(x) = \left(1 + \frac{1-m}{m}\frac{\|x\|^2}{2}\right)^{-1/(1-m)}.$$

This point of view was adopted by various authors to study the long-term behavior of solutions of the fast diffusion equation, as well as its counterpart for exponents $m > 1$, the porous medium equation [dPD02, Ott01, CJM+01]. The other key of these approaches is the use of the Bakry-Émery method, which we introduce in the subsequent subsection.

1.6 The Bakry-Émery method

In their resonating paper of 1985, Dominique Bakry and Michel Émery [BE85] proposed a technique that is now commonly called after their names. To study hypercontractivity, a property of Markov semigroups intimately linked with logarithmic Sobolev inequalities, they computed not just the first derivative of the entropy along a flow, but the second. Under some convexity conditions on the entropy functional, and curvature conditions of the underlying space, they prove that the entropy is not only nonincreasing, but also convex. The convexity may, in turn, be used to prove Sobolev-type inequalities.

This is quite a shift of perspective: originally, the logarithmic Sobolev inequality was used to study the heat flow, and the Bakry-Émery method proposes to do the reverse. It is a striking property of the heat flow that the one relevant functional inequality to study its solutions would be a rather natural consequence of the convexity of the entropy along the flow. This approach is at the core of both chapters 3 and 4, where we use flows to prove Sobolev inequalities, and we skip the study of the long-term behavior of solutions altogether. The goal here really is the Sobolev inequality itself.

While Bakry and Émery's original article pertains to probabilities, we shall work in the field of partial differential equations instead. The two fields obviously share an intimate relationship, and some of the terminology and the concepts that we will use – such as the *carré du champ* operator, written Γ – are borrowed from the field of probability. In this section, we introduce the formalism known as Γ-calculus, the main and most thorough reference for which is probably the book written by Dominique Bakry, Ivan Gentil, and Michel Ledoux [BGL14]. This formalism will be used in the linear case, chapter 3, as well as the nonlinear one, chapter 4. It makes a lot of sense in the linear case, since that is the context for which it was developped. Indeed, Γ-calculus is used in the analysis of Markov diffusion (linear) operators, hence the name of the book [BGL14]. In a nonlinear setting, it might seem a bit out of place, however we believe it is an efficient formalism for multiple reasons: not only is it slightly more compact and easily adapts from the Euclidean space to Riemannian manifolds, it also emphasizes the fact that results in both cases are actually extremely similar, see for instance lemma 3.8 and proposition 4.14 below.

1.6.1 Markov semigroups: a crashcourse

This subsection is an introduction to the linear case, where we define the carré du champ and related operators, and we recall some classical results about linear parabolic partial differential equations.

Let (M, g) be a smooth, connected, d-dimensionnal Riemannian manifold, equipped with the differential operator

$$L = \Delta - \nabla W \cdot \nabla,$$

where Δ is the Laplace-Beltrami operator, and $W : M \to \mathbb{R}$ is a smooth function. Equivalently, in a local chart, L can be written

$$L = \sum_{ij} g^{ij} \partial^2_{ij} + \sum_i b^i \partial_i,$$

where g^{ij} are the components of the inverse metric tensor, and the b^i are smooth, real functions. Then, L is symmetric with respect to the measure $d\mu = e^{-W} dx$, where dx is the Riemannian measure on (M, g). In other words, for all functions $u, v \in \mathcal{C}^\infty_c(M)$,

$$\int_M uLvd\mu = \int_M vLud\mu.$$

Although unbounded measures like the Lebesgue measure on $\mathbb{R}^d$ are of particular importance, let us restrict ourselves to probability measures, and assume from now on that μ is of mass 1. Some subtle considerations are required for the unbounded case, and we do not want to dive into too much detail in this somewhat long section. For a fixed initial condition u_0 to be specified, consider the system

$$\begin{cases} \partial_t u = Lu, & t > 0, \\ u = u_0, & t = 0. \end{cases} \tag{1.32}$$

The question of existence and uniqueness of solutions to problem (1.32) is classically solved with tools from the theory of operators. It is a very interesting topic, albeit way out of the scope of this manuscript, so we only try to give a taste of some classical results here, even more so since the operators we will work with are rather tame. We refer to the book [BGL14] and references therein for further details.

Assume just in this paragraph that L is a symmetric matrix of size n, and that we are looking for solutions to equation (1.32) with initial condition $u_0 \in \mathbb{R}^n$. Because it is self-adjoint, L is diagonalizable with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. In a basis made out of eigenvectors of L, the solution $u = (u_1, \ldots, u_n)$ simply writes

$$u_i(t, .) = e^{\lambda_i t} u_i^0,$$

or, in matrix notation and independently of the basis, $u(t, .) = e^{tL} u^0$. This example might seem extremely basic, but it is essentially what we try to do when L is not a matrix, but an operator acting on a infinite dimensional Hilbert space. Of course, the tools involved are more technical, and one has to be more careful with the setting, but that is the gist of it.

Let us now return to the differential operator $L = \Delta - \nabla W \cdot \nabla$, seen as an operator acting on the real Hilbert space $L^2(\mu)$, and defined on the dense subset of smooth bounded functions $\mathcal{C}_c^\infty(M) \subset L^2(\mu)$. For $t \geq 0$, we would like to define the operator e^{tL}, to construct the function $t \mapsto e^{tL} u_0$, which is a good candidate to solve equation (1.32). This is exactly what the spectral theorem allows to do for self-adjoint nonnegative operators, but while L is nonpositive (so that we could potentially apply the spectral theorem to $-L$), a discussion is to be had regarding self-adjointness. First, we prove that L is indeed nonpositive. To that end, we introduce the carré du champ operator Γ, which will also be the focus of the next subsection.

Definition 1.37 (Carré du champ)**.** For $u, v \in C_c^\infty(M)$, define the nonnegative symmetric bilinear map Γ by

$$\Gamma(u, v) = \frac{1}{2}(L(uv) - uLv - vLu).$$

For compactness of the notations, we will also write $\Gamma(u, u) = \Gamma(u)$

It is readily seen that, in a local chart,

$$\Gamma(u, v) = \sum_{ij} g^{ij} \partial_i u \partial_j v, \tag{1.33}$$

so that the positivity of the metric implies the nonnegativity of Γ. But then, since $L1 = 0$ and L is symmetric with respect to μ, we have the integration by parts formula

$$\int_M uLv d\mu = \frac{1}{2} \int_M uLv d\mu + \frac{1}{2} \int_M vLu d\mu - \int_M L(uv) d\mu = - \int_M \Gamma(u, v) d\mu. \tag{1.34}$$

In particular, $(u, Lu)_{L^2(\mu)} = -\int_M \Gamma(u) d\mu \leq 0$, where $(., .)_{L^2(\mu)}$ is the scalar product in the Hilbert space $L^2(\mu)$. This is, however, where a specificity of the infinite dimensional setting kicks in: in general, symmetric operators are not self-adjoint. These two notions are, in fact, quite different for unbounded operators, which differential operators usually are. Let us recall their definition here, and refer to the book by Michael Reed and Barry Simon [RS80] for a more thorough introduction to unbounded operators and their properties.

Definition 1.38. Let A be an operator densely defined on a Hilbert space $\mathcal{H}$, and let $\mathcal{D}(A)$ be its domain. Let $y \in \mathcal{H}$ for which there exists $z \in \mathcal{H}$ such that

$$\forall x \in \mathcal{D}(A),\ (y, Ax)_{\mathcal{H}} = (z, x)_{\mathcal{H}}.$$

We note $z = A^*y$, where A^* is called the *adjoint* of A, and the set of all such y is noted $\mathcal{D}(A^*)$. By Riesz's lemma, $y \in \mathcal{D}(A^*)$ if, and only if, there exists $C > 0$ such that $|(y, Ax)_{\mathcal{H}}| \leq C\|x\|_{\mathcal{H}}$ for all $x \in \mathcal{D}(A)$.

The operator A is said *symmetric* if, for all $x, y \in \mathcal{D}(A)$, $(y, Ax)_{\mathcal{H}} = (Ay, x)_{\mathcal{H}}$. Equivalently, A is symmetric if, and only if, A^* is an *extension* of A, that is $\mathcal{D}(A) \subset \mathcal{D}(A^*)$, and A and A^* coincide on $\mathcal{D}(A)$.

A symmetric operator is said *self-adjoint* if $A = A^*$, or equivalently, if $\mathcal{D}(A^*) = \mathcal{D}(A)$.

While the operator $-L = -\Delta + \nabla W \cdot \nabla$ is indeed positive and symmetric on $\mathcal{C}_c^\infty(M)$, it is never self-adjoint. Indeed, it is pretty clear that L^* is well-defined on the set of bounded functions of class C^2, so that there is no way that $\mathcal{C}_c^\infty(M) = \mathcal{D}(L^*)$. In practice, this is however not a problem: instead of working with L, we may work with an extension of L. If that extension is self-adjoint, then the spectral theorem still allows the construction of e^L, and we thus recover a solution to the system (1.32). Furthermore, Kurt Friedrichs proposed a canonical construction of a self-adjoint extension to any nonnegative symmetric densely defined operator. The main problem is that, in general, there is not uniciqueness of the self-adjoint extension. In other words, as far as the partial differential equation (1.32) is concerned, existence of a solution is generally not a problem, but uniciqueness is. The following notion is thus of prime importance:

Definition 1.39. A symmetric operator A is said *essentially self-adjoint* if it has a unique self-adjoint extension.

Note that self-adjoint operators are also essentially self-adjoint, their only self-adjoint extension being themselves. To better put this notion in perspective in the context of differential operators, let us consider the following example: let $\mathcal{H} = L^2([0,1])$ with the standard Lebesgue measure, and $L = -d^2/dx^2$, with domain

$$\mathcal{D} = \{u \in \mathcal{C}^\infty([0,1]),\ u(0) = u'(0) = u(1) = u'(1) = 0\}.$$

A straightforward integration by parts shows that L is symmetric and nonnegative. We may define the extensions $L_d = -d^2/dx^2$ and $L_n = -d^2/dx^2$ on their respective domain

$$\mathcal{D}_d = \{u \in \mathcal{C}^\infty([0,1]),\ u(1) = u(0) = 0\}, \quad \mathcal{D}_n = \{u \in \mathcal{C}^\infty([0,1]),\ u'(1) = u'(0) = 0\}.$$

The operators L_d and L_n are still symmetric and nonnegative, but remarkably, do not admit a common symmetric extension. Indeed, such an extension would be symmetric on $\mathcal{D}_d \cup \mathcal{D}_n$, and consequently on $\mathcal{C}^\infty([0,1])$, which is impossible. As a consequence, L must admit more than one self-adjoint extension, which translates to non-uniqueness of the solutions to the equation $\partial_t u = Lu$. Specifically, it appears that the boundary conditions play a crucial role in the unicity of solutions to parabolic differential equations. Chapter 3 focuses on manifolds that do not have boundaries, so that the theory may apply. In chapter 4 however, we study equations defined on bounded subsets of $\mathbb{R}^d$, and none of this is applicable, but we will come back to this in the next subsection. Classical criteria for essential self-adjointness lead to the following result:

Theorem 1.40. *Let M be a smooth, connected Riemannian manifold, and $L = \Delta - \nabla W \cdot \nabla$, where Δ is the Laplace-Beltrami operator and $W : M \to \mathbb{R}$ is a smooth function. Assume that the measure $e^{-W}dx$, with respect to which L is symmetric, is finite, and let define μ to be the* *probability measure proportionnal to* $e^{-W}dx$. *Then* L *is essentially self-adjoint on* $\mathcal{C}_c^\infty(M)$, *and there exists a unique family of bounded operators* $(P_t)_{t\in\mathbb{R}_+}$ *defined on* $L^2(\mu)$ *such that* $P_t u_0$ *is the unique solution to the problem* (1.32) *in* $L^2(\mu)$.

Remark 1.41. This result may be extended to unbounded measures, but additional hypotheses on the space are needed to get the full conclusions of proposition 1.42 below.

In the following proposition, we list a number of essential properties of the family of operators $(P_t)_{t\in\mathbb{R}_+}$. We will not prove any of them, but refer again to classical material on the subject, such as the book [BGL14] and the cited work therein for proofs. Most of it simply follows from construction, but some other properties still require more work.

Proposition 1.42. *Let* $(P_t)_{t\in\mathbb{R}_+}$ *be the family of operators given by theorem 1.40. It satisfies the following properties:*

(i) $P_0 = id$.

(ii) For all $t, s \in \mathbb{R}_+$, $P_{t+s} = P_t P_s = P_s P_t$. *(semigroup property)*

(iii) For $u \in L^2(\mu)$, $t \mapsto P_t u$ *is a continuous map from* $\mathbb{R}_+$ *to* $L^2(\mu)$, *and* $\|P_t u\|_2 \leq \|u\|_2$. *(contraction property)*

(iv) $P_t 1 = 1$. *(mass conservation)*

(v) If $u \geq 0$, *then* $P_t u \geq 0$. *(positivity preservation)*

(vi) For all $u, v \in L^2(\mu)$, $\int_M u P_t v d\mu = \int_M v P_t u d\mu$. *(reversibility)*

(vii) For all $u \in L^2(\mu)$, $\lim_{t\to+\infty} P_t u = \int_M u d\mu$ *in* $L^2(\mu)$. *(ergodicity)*

Generally, a family of operators $(P_t)_{t\in\mathbb{R}_+}$ satisfying conditions (i) to (v) is called a *Markov semigroup of operators*. Note that properties (i) to (iv), as well as (vi), are direct consequences of the construction of P_t. Property (v) is linked to the fact that L is a diffusion operator, and property (vii) follows from property (iv) and the fact that μ is a probability measure.

1.6.2 Introduction to Γ-calculus

Before showcasing, in an example, one use of the notions developped in the previous subsection, we define and motivate some notions introduced by Bakry and Émery, and developped in the book [BGL14], to which we refer for many more details and discussions.

The Γ-calculus formalism is called that because of its eponymous operator introduced in definition 1.37. We still assume that (M, g) is a smooth, connected Riemannian manifold, and the operator $L = \Delta - \nabla W \cdot \nabla$, with smooth W. According to equation (1.33), $\Gamma(u, v) = g(\nabla u, \nabla v)$, where g designates the metric tensor of M. The following iterated version of Γ is then a natural operator to consider in order to meaningfully study the geometry of M.

Definition 1.43 (Iterated carré du champ operator)**.** For $u, v \in C_c^\infty(M)$, define the symmetric bilinear map Γ_2 by

$$\Gamma_2(u, v) = \frac{1}{2}(L\Gamma(u, v) - \Gamma(u, Lv) - \Gamma(v, Lu)).$$

We will also write $\Gamma_2(u, u) = \Gamma_2(u)$ for brevity.

Remark 1.44. $L\Gamma(u, v)$ is well-defined because $C_c^\infty(M)$ is a stable subset of the operator L.

The operator Γ_2 is, in general, not necessarily nonnegative. However, lower bounds are also not an unreasonable expectation. On the Euclidean space equipped with the standard Laplacian, for example, given two smooth functions u and v, $\Gamma(u,v) = \nabla u \cdot \nabla v$, and $\Gamma_2(u,v) = \mathrm{tr}((\nabla^2 u)^t \nabla^2 v)$, which *is* nonnegative. Actually, on the Euclidean space, the Γ_2 operator is slightly more than just nonnegative: indeed, by the Cauchy-Schwarz inequality,

$$\Gamma_2(u) = \sum_{ij} (\nabla^2 u)_{ij}^2 \geq \sum_i (\nabla^2 u)_{ii}^2 \geq \frac{1}{d}\left(\sum_i (\nabla^2 u)_{ii}\right)^2 = \frac{1}{d}(\Delta u)^2.$$

Moreover, the fact that the Γ_2 operator writes so nicely is not a coincidence. Its definition, as matter of fact, was motivated by the following identity:

Proposition 1.45 (Bochner-Lichnerowicz formula)**.** *If u is a smooth function u defined on M,*

$$\Gamma_2(u) = \left\|\nabla^2 u\right\|_{H.S.}^2 + (\mathrm{Ric}_g + \nabla^2 W)(\nabla u, \nabla u), \tag{1.35}$$

where $\|.\|_{H.S.}$ designates the Hilbert-Schmidt norm, and Ric_g is the Ricci tensor of M equipped with the metric tensor g.

The metric tensor of a manifold encapsulates its curvature, so one may be tempted to compare it, in local coordinates, to the identity matrix I_d. However, such a comparison is not intrinsic: it is dependent on the choice of local coordinates. For that reason, it is preferable to work with intrinsic quantities, such as the scalar curvature, or the Ricci curvature appearing in equation (1.35).

Assume just in this paragraph that $W = 0$, so that $L = \Delta$ is the Laplace-Beltrami operator. When the Ricci curvature is bounded below by some constant $\rho \in \mathbb{R}$ times the metric g, as is the case for instance on compact manifolds, then $\Gamma_2(u) \geq \rho g(\nabla u, \nabla u) = \rho\Gamma(u)$. This inequality is, in fact, equivalent to the lower bound of the Ricci tensor, since the first-order and second-order terms in the Bochner-Lichnerowicz formula (1.35) may be considered independent from one another. The Cauchy-Schwarz inequality, just like in the Euclidean case, further shows that $\left\|\nabla^2 u\right\|_{H.S.}^2 \geq \frac{1}{d}(\Delta u)^2$, since $\Delta u = \mathrm{tr}(\nabla^2 u)$. Putting both of those inequalities together, we find that

$$\Gamma_2(u) \geq \rho\Gamma(u) + \frac{1}{d}(\Delta u)^2.$$

Just as announced, while Γ_2 is not always nonegative (in the case where $\rho < 0$), it does have a lower bound. As we shall see in the next subsection, this inequality is crucial in proving the convexity of various functionnals defined on M. This motivates the following definition, which we will extensively use in chapter 3.

Definition 1.46 (Curvature-dimension condition)**.** The operator L is said to satisfy a *curvature-dimension condition* with constants $\rho \in \mathbb{R}$ and $n \in \mathbb{R}^* \cup \{\infty\}$, written $CD(\rho, n)$, if, for all smooth functions $u : M \to \mathbb{R}$,

$$\Gamma_2(u) \geq \rho\Gamma(u) + \frac{1}{n}(Lu)^2. \tag{1.36}$$

We have already seen that the Laplace-Beltrami operator, on a d-dimensional manifold with a Ricci tensor bounded below by ρg, satisfies a $CD(\rho, d)$ condition. Of particular interest are the manifolds with *constant* Ricci curvature, also called Einstein manifolds. For instance, the Euclidean space has a constant zero Ricci tensore, but more generally, any manifold with constant sectional curvature is an Einstein manifold. An example of particular interest is the d-dimensional sphere $\mathbb{S}^d$, that satisfies $\mathrm{Ric}_g = (d-1)g$ for the round metric g. Thus, Laplace-Beltrami operator on the d-sphere with the round metric satisfies a $CD(d-1,d)$ condition.

Note also that there is a hierarchy in the curvature-dimension conditions, in that the $CD(\rho,n)$ condition implies the $CD(\rho',n')$ condition for all $\rho \in (-\infty,\rho]$ and all $n' \in \mathbb{R}^* \cup \{\infty\}$ such that $1/n' \leq 1/n$.

1.6.3 An introductory example

In this subsection, we showcase the Bakry-Émery method together with the Γ-calculus formalism in an elementary yet already rich example. On a smooth, connected manifold M, assuming that the differential operator $L = \Delta - \nabla W \cdot \nabla$ satisfies a $CD(\rho,\infty)$ condition for some $\rho > 0$, we prove the Poincaré inequality for the measure $d\mu = Ze^{-W}dx$, under the assumption that it is indeed a probability.

Proposition 1.47. *Let $L = \Delta - \nabla W \cdot \nabla$ be a differential operator defined on M, symmetric with respect to the measure μ. If μ is a probability measure, and if L satisfies a $CD(\rho,\infty)$ condition for some $\rho > 0$, then, for all nonnegative $u \in \mathcal{C}_c^\infty(M)$,*

$$\int_M u^2 d\mu - \left(\int_M u d\mu\right)^2 \leq \frac{1}{\rho}\int_M \Gamma(u) d\mu. \tag{1.37}$$

Proof. Consider the evolution equation

$$\begin{cases} \partial_t u_t = L u_t & \text{on } M \times (0,+\infty) \\ u_0 = u & \text{on } M. \end{cases}$$

Since L is assumed nice, there exists a solution to this problem, and it is given by a Markov semigroup, $u_t = P_t u$. Now, consider the variance of u_t along the flow:

$$\mathrm{Var}_\mu(P_t u) = \int_M (P_t u)^2 d\mu - \left(\int_M P_t u d\mu\right)^2,$$

The reversivility, together with mass conservation from proposition 1.42, imply that $\int_M P_t u d\mu = \int_M u d\mu$. Furthermore, ergodicity (from that same theorem) implies that $P_t u$ converges in $L^2(\mu)$ towards its mean, $\int_M u d\mu$. As a result,

$$\lim_{t\to+\infty} \mathrm{Var}_\mu(P_t u) = \lim_{t\to+\infty}\left(\int_M (P_t u)^2 d\mu - \left(\int_M u d\mu\right)^2\right) = 0$$

In this example, the variance plays the role of the entropy in section 1.5, and we shall even be so bold as to actually call it *entropy* in the upcoming chapters. Differentiating the variance with respect to time, we find

$$\begin{aligned} \frac{d}{dt}\mathrm{Var}_\mu(P_t u) &= 2\int_M P_t u L(P_t u) d\mu \\ &= -2\int_M \Gamma(P_t u) d\mu \leq 0. \end{aligned} \tag{1.38}$$

At this stage, take a moment to notice that comparing the variance to its derivative along the flow would exactly amount to prove the Poincaré inequality 1.37. We proceed with the Bakry-Émery method, differentiating again:

$$\begin{aligned}\frac{d^2}{dt^2}\operatorname{Var}_\mu(P_tu) = -4\int_M \Gamma(P_tu, L(P_tu)) &= 4\int_M \left[\frac{1}{2}L(\Gamma(P_tu)) - \Gamma(P_tu, L(P_tu))\right]d\mu \\ &= 4\int_M \Gamma_2(P_tu)d\mu,\end{aligned} \tag{1.39}$$

where we used the fact that $\int_M L(\Gamma(P_tu))d\mu = 0$, which is a direct consequence of the integration by parts formula, equation (1.34). Carefully note that the integration by parts in both equations (1.38) and (1.39) are, a priori, *not* justified, since it is not true in general that $P_tu \in \mathcal{C}_c^\infty$. For example, in the case of the Euclidean heat semigroup, P_tu_0 is strictly positive everywhere on $\mathbb{R}^d$ for all $t > 0$. However, we assumed that L satisfies a $CD(\rho, n)$ condition, which is sufficient to extend the integration by parts formula to functions that are not in $\mathcal{C}_c^\infty$. As is by now usual, we refer to the book [BGL14, Section 3.2.3] for details.

Finally, we may invoke the $CD(\rho, \infty)$ condition to prove that

$$\frac{d^2}{dt^2}\operatorname{Var}_\mu(P_tu) \geq 4\rho\int_M \Gamma(P_tu)d\mu = -2\rho\frac{d}{dt}\operatorname{Var}_\mu(P_tu),$$

which proves that the variance is convex along the flow, since its first derivative is nonpositive. However, the inequality we have is stronger than just a convexity inequality, since we can integrate it between 0 and $+\infty$ to find

$$\operatorname{Var}_\mu(u) - \lim_{t\to+\infty}\operatorname{Var}_\mu(P_tu) \leq \frac{1}{2\rho}\left(-\frac{d}{dt}\operatorname{Var}_\mu(P_tu)\Big|_{t=0} + \lim_{t\to+\infty}\frac{d}{dt}\operatorname{Var}_\mu(P_tu)\right).$$

Now, we already established that $\lim_{t\to+\infty}\operatorname{Var}_\mu(P_tu) = 0$, the other limit being 0 too is a consequence of the variance being convex, decreasing and having a lower bound. We are thus left with

$$\operatorname{Var}_\mu(u) \leq \frac{1}{\rho}\int \Gamma(u)d\mu, \tag{1.40}$$

which is exactly the Poincaré inequality and concludes the proof. □

Equation (1.40) is a type of inequality also called *entropy - entropy production* inequality. Indeed, the variance playing the role of the entropy here, it is an inequality between the entropy and its derivative along the flow, the entropy production.

1.6.4 Nonlinear parabolic equations

In chapter 4, we want to apply the Bakry-Émery method to flows that are nonlinear, but also have boundary conditions. For both of these reasons, the Markov semigroup tools developped for the linear case completely fall apart. We have to rely on tools purely designed for the study of parabolic partial differential equations. While the tools used to study both of those are different, the results are extremely similar. Essentially, after differentiating the entropy twice, we find something related to what we found in the introductory example, equation (1.39), only with two more terms involving $\Gamma(\phi)$ and $(L\phi)^2$ for some function ϕ depending on the flow u. The goal will thus be to recover the properties of a standard Markov semigroup for the final calculations to work. Specifically, we will need mass conservation and ergodicity if the Bakry-Émery method is ever going to work.

In their 2002 article [dPD02], Manuel del Pino and Jean Dolbeault used the Gagliardo-Nirenberg-Sobolev inequality to study the asymptotic behavior of solutions to the fast-diffusion equation on $\mathbb{R}^d$,

$$\begin{cases} \partial_t u_t = \Delta u_t^m, & t > 0, \\ u_0 = u, & t = 0, \end{cases} \tag{1.41}$$

with $m \in (1 - 1/d, 1)$. In particular, they showed that the Gagliardo-Nirenberg-Sobolev inequality could be rewritten as an entropy - entropy production inequality: for all smooth, nonnegative and integrable u of unit mass,

$$\mathcal{F}(u \mid u_\infty) \leq -C \, \frac{d}{dt}\mathcal{F}(u_t \mid u_\infty)\Big|_{t=0},$$

for some sharp constant $C > 0$, for a particular entropy functional $\mathcal{F}$, and along a particular flow, all of which we will come back to in due time. Just like in the heat flow – logarithmic inequality case of section 1.5.2, the interesting flow to consider is not the fast diffusion one, but a time-dependent rescaling of it. In particular, if we define the function

$$\psi(x) = -\frac{m}{1-m} x^{m-1}$$

on $\mathbb{R}_+^*$, then the function $v_t = v(t, .)$, defined for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$ by

$$v(t,x) = e^{dt} u\left(\frac{e^{\lambda t} - 1}{\lambda}, e^t x\right), \quad \lambda = 2 + (m-1)d,$$

solves the following generalized Fokker-Planck equation

$$\begin{cases} \partial_t v_t = -\operatorname{div}\big(v_t \nabla(\psi(v_\infty) - \psi(v_t))\big), & t > 0, \\ v_0 = u, & t = 0. \end{cases} \tag{1.42}$$

Again, instead of studying long-term behavior of solutions to (1.42) using Sobolev inequalities, we may do the opposite with the Bakry-Émery method. Equation (1.42) will be the starting point of chapter 4.

Chapter 2

Sharp trace Gagliardo-Nirenberg-Sobolev inequalities for convex cones, and convex domains

Abstract

We find a new sharp trace Gagliardo-Nirenberg-Sobolev inequality on convex cones, as well as a sharp weighted trace Sobolev inequality on epigraphs of convex functions. This is done by using a generalized Borell-Brascamp-Lieb inequality, coming from the Brunn-Minkowski theory.

Contents

2.1 Introduction and main results

The classical Sobolev inequality states that, for any function f sufficiently smooth and decaying fast enough at infinity, defined on the Euclidean space $\mathbb{R}^d$ with $d \geq 2$ (for instance, $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$), and for any $p \in [1, d)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^d)}, \quad p^* = \frac{pn}{n-p}, \tag{2.1}$$

Furthermore, equality is reached in inequality (2.1) if f can be written

$$f(x) = \left(1 + \|x\|^{p/(p-1)}\right)^{\frac{p-d}{p}},$$

up to a translation, a rescaling, and multiplication by a constant, where $\|.\|$ is the Euclidean norm. This was proved by Talenti [Tal76] and Aubin [Aub76] independently for $p = 2$. The Sobolev inequality can be seen as a corollary of a more general inequality, the Gagliardo-Nirenberg inequality, which states that

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C\|\nabla f\|^{\theta}_{L^p(\mathbb{R}^d)}\|f\|^{1-\theta}_{L^r(\mathbb{R}^d)}, \tag{2.2}$$

for any $p \in [1, d)$, $q, r \in [1, +\infty]$, $\theta \in [0, 1]$ such that

$$\frac{1}{q} = \left(\frac{1}{p} - \frac{1}{d}\right)\theta + \frac{1-\theta}{r};$$

whence the case $\theta = 1$ is exactly the Sobolev inequality. This family of inequalities has been notably investigated by del Pino and Dolbeault [dPD02], who, studying the 1-parameter subfamily given by $p = 2$ and $r = q/2 + 1$, have not only found an explicit sharp constant, but also proved that there is equality if, and only if, f has the form

$$f(x) = \left(1 + \|x\|^2\right)^{\frac{2}{2-q}},$$

up to, once again, a translation, a rescaling, and multiplication by a constant.

As Bobkov and Ledoux [BL08] showed, these sharp inequalities can be reached within the framework of the Brunn-Minkovski theory [Sch14]. With this approach, the sharp inequality follows in the more general case where the Euclidean norm is replaced by a generic norm on $\mathbb{R}^d$, which is a result already proved by Cordero-Erausquin, Nazaret, and Villani using optimal transport [CENV04]. This makes sense, since the Brunn-Minkovski inequality directly implies the isoperimetric inequality, which is famously equivalent to the sharp Sobolev inequality with $p = 1$ (for a nice overview on this subject, see Osserman's article on the isoperimetric inequality [Oss78]).

The key tool Bobkov and Ledoux use is an extended Borell-Brascamp-Lieb inequality, a quick proof of which using optimal transport is given by Bolley, Cordero-Erausquin, Fujita, Gentil and Guillin [BCEF+17]. For a bit of context, let us state the Brunn-Minkoski inequality: for any compact nonempty subsets A and B in $\mathbb{R}^d$, and any $t \in [0, 1]$

$$|tA + (1-t)B|^{1/d} \geq t|A|^{1/d} + (1-t)|B|^{1/d},$$

where $|.|$ denotes the Lebesgue measure on $\mathbb{R}^d$. This is to say that the volume, to the power $1/d$, is concave with respect to the Minkowski sum, defined by $A + B = \{a + b, (a, b) \in A \times B\}$. The classical Borell-Brascamp-Lieb inequality [Bor75][BL76], just like the isoperimetric inequality, follows from the Brunn-Minkowski inequality. It is, in some sense, its functional counterpart: let $t \in [0, 1]$ and $u, v, w : \mathbb{R}^d \to (0\,; +\infty]$ such that for all x, $y \in \mathbb{R}^d$,

$$w((1-t)x + ty) \geq \left((1-t)(u(x))^{-1/d} + t(v(y))^{-1/d}\right)^{-d},$$

then

$$\int w \geq \min\left(\int u, \int v\right).$$

Playing with the exponents and normalizing this inequality gives the following reformulation of the Borell-Brascamp-Lieb inequality: let g, W, and $H : \mathbb{R}^d \to (0, +\infty]$, and $t \in [0, 1]$, such that $\int g^{-d} = \int W^{-d} = 1$ and

$$\forall x, y \in \mathbb{R}^d, \quad H((1-t)x + ty) \leq (1-t)g(x) + tW(y),$$

then

$$\int H^{-d} \geq 1. \tag{2.3}$$

Note that the choice to use g, W and H is not random, we want to stress the fact that unlike in the classical Borell-Brascamp-Lieb inequality, the functions will not have symmetrical roles. Furthermore, W will generally be assumed convex.

Applying inequality (2.3) to the greatest function H meeting these criteria allows us to prove that

$$\int W^*(\nabla g) g^{-d-1} \geq 0, \tag{2.4}$$

where W^* is the Legendre transform of W. This inequality, as we will see in the next section, turns out to be equivalent to the Borell-Brascamp-Lieb inequality we use here. This might look like it is to be expected, because of the semigroup structure that underlies the theorem, but is actually a little bit surprising, because said semigroup is not quite linear. The equivalence between the more general theorems with which we work here remains an open question.

Inequality (2.4) can, in turn, be used to prove sharp Sobolev-type inequalities, but in the end proves to be limited as it does not allow to reach the full range of Gagliardo-Nirenberg inequalities showcased by del Pino and Dolbeault [dPD02]. Thus, a better inequality to work with is the following extension of the Borell-Brascamp-Lieb inequality, which was proved by Bolley et al. [BCEF+17].

Theorem 2.1. *Let $d \geq 2$, and $t \in [0,1]$. Let g, W, and $H : \mathbb{R}^d \to (0,+\infty]$ be measurable functions such that $\int g^{-d} = \int W^{-d} = 1$ and*

$$\forall\, x,\, y \in \mathbb{R}^d, \quad H((1-t)x + ty) \leq (1-t)g(x) + tW(y) \tag{2.5}$$

then

$$\int H^{1-d} \geq (1-t)\int g^{1-d} + t\int W^{1-d}.$$

With this theorem, which is proved in the introductory subsection 1.4.2, we are able to prove sharp trace-Sobolev inequalities on convex domains. More specifically, we prove sharp trace Sobolev in some convex domains, and sharp trace Gagliardo-Nirenberg inequalities in convex cones. In what follows, $\|.\|$ is a norm on $\mathbb{R}^d$, and $\|.\|_*$ is the dual norm, defined by $\|x\|_* = \sup_{\|y\|=1} x\cdot y$. In L^q norms of vector functions, the dual norm $\|.\|_*$ will be used. Let $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a convex function such that $\varphi(0) = 0$. We consider functions defined on φ's epigraph, that is $\Omega = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R},\ x_2 \geq \varphi(x_1)\}$. We say that Ω is a convex cone whenever φ is positive homogeneous of degree 1: for all $t > 0$ and $x_1 \in \mathbb{R}^{d-1}$, $\varphi(tx_1) = t\varphi(x_1)$.

Theorem 2.2 (Sharp trace Gagliardo-Nirenberg inequality)**.** *Let $a \geq d > p > 1$, and $\Omega = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R},\ x_2 \geq \varphi(x_1)\}$ be a convex cone. There exists a positive constant $D_{d,p,a}(\Omega)$ such that for any non-negative function $f \in C_c^\infty(\Omega)$,*

$$\left(\int_{\mathbb{R}^{d-1}} f^q(x, \varphi(x))dx\right)^{1/q} \leq D_{d,p,a}(\Omega)\|\nabla f\|^{\theta}_{L^p(\Omega)}\|f\|^{1-\theta}_{L^q(\Omega)}, \tag{2.6}$$

where

$$\theta = \frac{a-p}{p(a-d-1)+d}, \quad q = p\frac{a-1}{a-p}.$$

Furthermore, when $f(x) = \|(x_1, x_2+1)\|^{-\frac{a-p}{p-1}}$, then (2.6) is an equality.

The fact that there exists a function for which the equality is reached means that the constant $D_{d,p,a}(\Omega)$ may be computed explicitly. Choosing $a = d$, Theorem 2.2 immediately yields the sharp trace Sobolev inequality as a corollary:

Corollary 2.3 (Sharp trace Sobolev inequality)**.** *Let $d > p > 1$, and $\Omega = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_2 \geq \varphi(x_1)\}$ be a convex cone. There exists a positive constant $D_{d,p}(\Omega) = D_{d,p,d}(\Omega)$ such that for any non-negative function $f \in C_c^\infty(\Omega)$,*

$$\left(\int_{\mathbb{R}^{d-1}} f^{p\frac{d-1}{d-p}}(x, \varphi(x))dx\right)^{\frac{d-p}{p(d-1)}} \leq D_{d,p}(\Omega)\|\nabla f\|_{L^p(\Omega)}, \tag{2.7}$$

Furthermore, when $f(x) = \|(x_1, x_2 + 1)\|^{-\frac{d-p}{p-1}}$, then (2.7) is an equality.

The case $\Omega = \mathbb{R}^d_+$ has already been studied by Nazaret [Naz06].

If we only assume Ω to be convex, we prove, under some growth criteria on Ω, the following sharp weighted trace Sobolev inequality:

Theorem 2.4 (Sharp trace Sobolev inequality)**.** *Let $d > p > 1$, and $\Omega = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_2 \geq \varphi(x_1)\}$ be a convex set. There exists a positive constant $D'_{d,p}(\Omega)$ such that for any nonnegative function $f \in C_c^\infty(\Omega)$,*

$$\int_{\mathbb{R}^{d-1}} f^{p\frac{d-1}{d-p}}(x_1, \varphi(x_1))P(x_1)dx_1 \leq D'_{d,p}(\Omega)\left(\int_\Omega \|\nabla f\|_*^p\right)^{\frac{d-1}{d-p}} \tag{2.8}$$

where $P(x_1) = 1 + \varphi(x_1) - x_1 \cdot \nabla\varphi(x_1)$. Furthermore, when $f(x) = \|(x_1, x_2 + 1)\|^{-\frac{d-p}{p-1}}$, then (2.8) is an equality.

Once again, $D'_{d,p}(\Omega)$ can be computed explicitly. This inequality may be surprising, since the weight P can (and usually is, whenever Ω is not a cone) negative outside a compact neighbourhood of 0, but it is still sharp. For instance, with the set defined by $\varphi(x) = \|x\|^2$, the weight becomes $P(x) = 1 - \|x\|^2$, which happens to be negative outside the unit ball. One may define $\partial\Omega_+ \subset \partial\Omega$ such that $\partial\Omega_+ = \{(x_1, \varphi(x_1)), P(x_1) > 0\}$. In that case, inequality (2.8) restricted to functions $f \in C_c^\infty(\mathring{\Omega} \cup \partial\Omega_+)$ becomes a regular weighted inequality, with a positive weight.

In the next section, we first study the infimal convolution, which is the key tool in the proof of Theorems 2.2 and 2.4. Once some crucial properties are established, we prove the claimed equivalence between the classical Borell-Brascamp-Lieb inequality (2.3) and its differentiated formulation (2.4), within some limitations. Next, in section 2.3, we move on to prove the main Theorems 2.2 and 2.4, starting from an improved version of the Borell-Brascamp-Lieb inequality. The technical details, which will be glided over in these sections, can be found in the comprehensive appendix 2.4, at the end of the chapter.

2.2 Generalities

Let $t \in [0, 1)$. To use Theorem 2.1, instead of considering any H such that

$$\forall\, x,\, y \in \mathbb{R}^d, \quad H((1-t)x + ty) \leq (1-t)g(x) + tW(y),$$

we may well choose the greatest such function. That is,

$$H(z) = \inf_{\substack{x,y\in\mathbb{R}^d \\ (1-t)x+ty=z}} \{(1-t)g(x) + tW(y)\},$$

or, writing $h = t/(1-t)$,

$$\frac{H(z)}{1-t} = \inf_{y\in\mathbb{R}^d} \left\{ g\left(\frac{z}{1-t} - hy\right) + hW(y) \right\}.$$

This formula, being explicit, allows for some properties to be brought to light. It motivates the definition, and the study, of the so-called infimal convolution:

Definition 2.5. Let $f, g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. Their infimal convolute $f \,\square\, g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$(f \,\square\, g)(x) = \inf_{y,z\in\mathbb{R}^d} \{f(y) + g(z),\, y + z = x\} = \inf_{y\in\mathbb{R}^d} \{f(y) + g(x-y)\}.$$

The infimal convolution of f with g is said to be *exact at x* if the infimum is achieved, and *exact* if it is exact everywhere.

With this definition, and whenever $h = t/(1-t) > 0$, the greatest function H in Theorem 2.1 is given by

$$H(z) = (1-t) \inf_{y\in\mathbb{R}^d} \left\{ g\left(\frac{z}{1-t} - y\right) + hW(y/h) \right\} = (1-t)\left(g \,\square\, hW(./h)\right)(z/(1-t)),$$

we thus define

$$Q_h^W(g) = g \,\square\, hW(./h) = x \mapsto \inf_{y\in\mathbb{R}^d} \{g(x-y) + hW(y/h)\}.$$

Using Q_h^W in Theorem 2.1, inequality (2.5) becomes

$$\int Q_h^W(g)^{1-d} \geq \int g^{1-d} + h \int W^{1-d} \tag{2.9}$$

but there exists a slightly more general version of this inequality, namely Theorem 2.16, which we will use in section 2.3.

To begin with, let us first showcase some properties of the infimal convolution.

2.2.1 The general infimal convolution

This subsection is here to build some intuition about infimal convolution, before proving specific results useful for the study of Q_h^W.

Definition 2.6. With any function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, we associate its

- essential domain (usually shortened to domain), $\operatorname{dom} f = \{x \in \mathbb{R}^d,\, f(x) < +\infty\}$;
- epigraph, $\operatorname{epi} f = \{(x, \alpha) \in \mathbb{R}^d \times \mathbb{R},\, f(x) \leq \alpha\}$;
- strict epigraph, $\operatorname{epi}_s f = \{(x, \alpha) \in \mathbb{R}^d \times \mathbb{R},\, f(x) < \alpha\}$.

Furthermore, the function f is said to be proper if it is not equal to the constant $+\infty$.

With these definitions, we highlight in the next proposition the link between infimal convolution of functions and Minkowski sum of sets, classically defined for two sets A, B by $A + B = \{a + b,\, (a, b) \in A \times B\}$.

Proposition 2.7. *Let $f, g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. Then*

- $\operatorname{dom} f \,\square\, g = \operatorname{dom} f + \operatorname{dom} g$;
- $\operatorname{epi}_s f \,\square\, g = \operatorname{epi}_s f + \operatorname{epi}_s g$;
- $\operatorname{epi} f \,\square\, g \supset \operatorname{epi} f + \operatorname{epi} g$, *and equality holds if, and only if, the infimal convolution is exact at each* $x \in \operatorname{dom} f \,\square\, g$.

Proof of this proposition and more in-depth details on infimal convolutions can be found in Thomas Strömberg's thesis [Str96]. The more delicate question of regularity of the infimal convolution is only addressed in subsection 2.2.2 in the particular study of $Q_h^W(g)$. That is because there is not *one* natural set of assumptions ensuring regularity, so it really depends on the goal, which, here, is that $Q_h^W(g)$ should be smooth enough to prove Sobolev inequalities. We only prove the following lemma in the most general case, since it is very useful.

Lemma 2.8. *Let* $f, g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *be lower semicontinuous functions. If* f *is nonnegative and* g *is coercive, that is,*

$$\lim_{\|x\| \to +\infty} g(x) = +\infty,$$

then $f \,\square\, g$ *is exact.*

Proof. Fix $x \in \mathbb{R}^d$. Consider $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, $y \mapsto f(x-y) + g(y)$ and assume that there exists y_0 such that $\psi(y_0) < +\infty$: ψ is lower semicontinuous, and greater than g, thus tends to $+\infty$ as $\|y\|$ goes to $+\infty$. As such, $\{y \in \mathbb{R}^d, \psi(y) \leq \psi(y_0)\}$ is closed and bounded, thus compact. Now, let $(y_n) \subset \{\psi \leq \psi(y_0)\}$ be a minimizing sequence, $\lim_{n\to+\infty} \psi(y_n) = \inf_{y \in \mathbb{R}^d}\{\psi(y)\}$. By compactness, we can assume that the sequence (y_n) converges towards $z \in \mathbb{R}^d$, and by lower semicontinuity, $-\infty < \psi(z) \leq \lim_{n\to+\infty} \psi(y_n) = \inf_{y\in\mathbb{R}^d}\{\psi(y)\}$, thus the infimum is finite and is actually a minimum. If such a y_0 does not exist, then $f \,\square\, g(x) = +\infty$, and the infimum is also reached. $\square$

2.2.2 Regularity of the inf-convolution $Q_h^W(g)$

We begin here the specific study of $Q_h^W(g) = g \,\square\, hW(./h)$. The study of the regularity of $Q_h^W(g)$ with respect to $h > 0$ is crucial, because we would like to differentiate inequality (2.9) with respect to h. Let us first state some classical results about the Legendre transform. The proofs can be found in Evans' book, [Eva98, p.120], and Brézis' book, [Bre99, p.10].

Definition 2.9. The Legendre transform of W is defined by

$$W^*(y) = \sup_{x \in \mathbb{R}^d} \{x \cdot y - W(x)\} \in \overline{\mathbb{R}}.$$

By definition, W^* is a lower semicontinuous convex function, but it is not always proper. For W^* to be well behaved, we have to assume a little bit more about W. In fact, it is enough to assume W to be lower semicontinuous: indeed, if $W : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous proper convex function, then W^* is also a lower semicontinuous proper convex function, and $(W^*)^* = W$. The infimal convolution is not only closely related to Minkovski sums, but also to Legendre transforms, as the next lemma shows.

Lemma 2.10. *Let* $g, W : \mathbb{R}^d \to (-\infty, +\infty]$ *be two measurable functions. If* g *is nonnegative and almost everywhere differentiable on its domain* $\operatorname{dom} g = \Omega_0$ *(with nonempty interior), and* W *grows superlinearly,*

$$\lim_{\|x\| \to +\infty} \frac{W(x)}{\|x\|} = +\infty,$$

then for almost every $x \in \mathring{\Omega}_0$, $h \mapsto Q_h^W(g)(x)$ is differentiable at $h = 0$, and

$$\frac{\partial}{\partial h}\bigg|_{h=0} Q_h^W(g)(x) = -W^*(\nabla g(x)),$$

where W^* is the Legendre transform of W.

Proof. Let $\Omega_1 = \operatorname{dom} W$, and fix $x \in \mathring{\Omega}_0$ such that the differential of g at x exists. Let $y \in \Omega_1$. For $h > 0$ sufficiently small, $x - hy \in \Omega_0$, and we get, by definition of $Q_h^W(g)$,

$$\frac{Q_h^W(g)(x) - g(x)}{h} \leq \frac{g(x - hy) - g(x)}{h} + W(y).$$

Taking the superior limit when $h \to 0$ yields

$$\limsup_{h \to 0} \frac{Q_h^W(g)(x) - g(x)}{h} \leq -\nabla g(x) \cdot y + W(y).$$

This being true for any $y \in \Omega_1$, we may take the infimum to find that

$$\limsup_{h \to 0} \frac{Q_h^W(g)(x) - g(x)}{h} \leq -W^*(\nabla g(x)).$$

Conversely, fix $e \in \Omega_1$, and $h_0 > 0$ such that $\overline{B(x, h_0\|e\|)} \in \mathring{\Omega}_0$. For $h \in (0, h_0)$, define

$$\Omega_{x,h} = \{y \in \Omega_1,\, hW(y) \leq g(x - he) + hW(e)\};$$

note that $e \in \Omega_{x,h}$. We claim that $\limsup_{h \to 0}\{h\|y\|,\, y \in \Omega_{x,h}\} = 0$. Indeed, if $y \in \Omega_{x,h}$, then

$$h\|y\|\frac{W(y)}{\|y\|} \leq g(x - he) + hW(e) \leq \sup_{z \in \overline{B(x, h_0\|e\|)}} g(z) + h_0 W(e).$$

Now, when h goes to 0, either $\limsup\|y\| < +\infty$, or $\limsup\|y\| = +\infty$; in both cases, since $\lim_{\|y\| \to +\infty} \frac{W(y)}{\|y\|} = +\infty$, the claim is proved. Notice now that for all $h \in (0, h_0)$, $Q_h^W(g)(x) \leq g(x - he) + hW(e)$, hence $Q_h^W(g)(x) = \inf_{y \in \Omega_{x,h}}\{\ldots\}$. Thus,

$$\begin{aligned}
\frac{Q_h^W(g)(x) - g(x)}{h} &= \inf_{y \in \Omega_{x,h}} \left\{\frac{g(x - hy) - g(x)}{h} + W(y)\right\} \\
&= \inf_{y \in \Omega_{x,h}} \{-\nabla g(x) \cdot y + y \cdot \varepsilon_x(hy) + W(y)\}
\end{aligned}$$

where $\varepsilon_x(z) \to 0$ when $\|z\| \to 0$. Let $1 \geq \eta > 0$; the claim proves that there exists $h_\eta \in (0, h_0)$ such that for all $0 < h < h_\eta$, $\forall y \in \Omega_{x,h}$, $\|\varepsilon_x(hy)\| \leq \eta$. Thus,

$$\begin{aligned}
\frac{Q_h^W(g)(x) - g(x)}{h} &\geq \inf_{y \in \Omega_{x,h}} \{-\nabla g(x) \cdot y - \eta\|y\| + W(y)\} \\
&= \inf_{\substack{y \in \Omega_{x,h} \\ y \in B(0,R)}} \{\ldots\} \\
&\geq \inf_{y \in \Omega_{x,h}} \{-\nabla g(x) \cdot y + W(y)\} - R\eta \\
&\geq -W^*(\nabla g(x)) - R\eta,
\end{aligned}$$

where R was chosen such that $\|y\| \geq R \implies W(y) \geq (\|\nabla g(x)\| + 1)\|y\| + W(e) - \nabla g(x) \cdot e$. Finally, taking the inferior limit of this inequality, and noticing that the result stays true for any $0 < \eta \leq 1$, we may conclude (since R is independent from η) that

$$\lim_{h \to 0} \frac{Q_h^W(g)(x) - g(x)}{h} = -W^*(\nabla g(x)).$$

□

This differentiation result is enough to prove the main theorems contained in section 2.3, but we can go a little bit further with more assumptions on g and W. Assuming W to be convex bestows upon Q_h^W a semigroup structure:

Lemma 2.11. *Assume that $g : \mathbb{R}^d \to [0, +\infty]$ is lower semicontinuous, and that W is a lower semicontinuous proper convex function such that $\lim_{\|x\| \to +\infty} W(x) = +\infty$. Then, for all $x \in \mathbb{R}^d$ and $0 < s < h$,*

$$\begin{aligned} Q_h^W(g)(x) &= \min_{y \in \mathbb{R}^d} \{g(x - hy) + hW(y)\} \\ &= Q_{h-s}^W(Q_s^W(g))(x). \end{aligned}$$

Proof. Exactness was already proved in Lemma 2.8. Notice that

$$\begin{aligned} Q_{h-s}^W(Q_s^W(g))(x) &= \inf_{y \in \mathbb{R}^d} \inf_{z \in \mathbb{R}^d} \{g(x - (h-s)y - sz) + (h-s)W(y) + sW(z)\} \\ &\leq \inf_{y \in \mathbb{R}^d} \{g(x - hy) + hW(y)\} = Q_h^W(g)(x). \end{aligned}$$

Conversely, let $y \in \mathbb{R}^d$, and choose $z \in \mathbb{R}^d$ such that

$$Q_s^W(g)(x - (t-s)y) = g(x - sz) + sW(z).$$

Then, by convexity,

$$\begin{aligned} Q_t^W(g)(x) &\leq g(x - (t-s)y - sz) + tW\left(\frac{t-s}{t}y + \frac{s}{t}z\right) \\ &\leq g(x - (t-s)y - sz) + (t-s)W(y) + sW(z) \\ &= (t-s)W(y) + Q_s^W(g)(x - (t-s)y). \end{aligned}$$

Taking the infimum over $y \in \mathbb{R}^n$ proves that $Q_t^W(g)(x) \leq Q_{h-s}^W(Q_s^W(g))(x)$, and thus there is equality. □

We want to investigate if some kind of regularity is preserved under the operation of infimal convolution. The answer is yes, under certain specific conditions. We will also provide an example showcasing regularity loss, emphasizing the delicate nature of this question. Work on this subject already exists, notably in Evans' book [Eva98, p. 128], where there is a global Lipschitz assumption, or in Villani's book [Vil09, Theorem 30.30], where functions are bounded. However, such assumptions are at odds with the goals we aim for here, as ultimately, we want $g^{-\alpha}$ to be integrable for some exponant $\alpha > 0$.

Let us study the case where g and W are finite *everywhere*.

Lemma 2.12. *Let $g, W : \mathbb{R}^d \to \mathbb{R}$. If g is nonnegative, locally Lipschitz continuous, and W is convex and coercive, then $(h, x) \mapsto Q_h^W(g)$ is locally Lipschitz continuous.*

Proof. In order to prove the full local Lipschitz continuity, we must first localize the arginf of the infimal convolution. Fix $\rho > 0$, $\eta > 0$, and let $x, x' \in B(0,\rho)$ and $0 < h < \eta$. Consider the set

$$\Omega_{x,h} := \{y \in \mathbb{R}^d, g(x-y) + hW(y/h) \leq g(x) + hW(0)\}.$$

We claim that, by positivity of g, and convexity of W, the set is bounded. Indeed, since W is convex and coercive, there exists $R > 0$ and $m > 0$ such that

$$\|y\| > R \implies W(y) \geq m\|y\|.$$

If $y \in \Omega_{x,h}$, then either $\|y\| \leq hR \leq \eta R$, or $\|y\| > hR$ and then $g(x) + hW(0) \geq hW(y/h) \geq m\|y\|$. Invoking continuity of g, we may prove the claim, and conclude that there exists $R_{\rho,\eta}$, independent from x and h, such that $\Omega_{x,h} \subset B(0, R_{\rho,\eta})$.

Let us now prove the local Lipschitz continuity with respect to x. The functions g and W are assumed continuous, and so the infimal convolution is exact, and there exists $y \in \mathbb{R}^n$ such that $Q_h^W(g)(x) = g(x-y) + hW(y/h)$. Necessarily, $\|y\| \leq R_{\rho,\eta}$, so

$$\begin{aligned}
Q_h^W(g)(x') - Q_h^W(g)(x) &= \inf_{y' \in \mathbb{R}^d} \{g(x'-y') + hW(y'/h)\} - g(x-y) - hW(y/h) \\
&\leq g(x'-y) - g(x-y) \\
&\leq \left(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})}\, g\right) \|x - x'\|,
\end{aligned}$$

where $\mathrm{Lip}_A f := \sup_{x \neq x' \in A}\{|f(x) - f(x')|/\|x - x'\|\}$. By symmetry, we conclude that

$$\left|Q_h^W(g)(x') - Q_h^W(g)(x)\right| \leq \left(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})}\, g\right) \|x - x'\|,$$

hence the local Lipschitz continuity with respect to x.

Now,

$$\begin{aligned}
Q_h^W(g)(x) - g(x) &= \inf_{y \in B(0,R_{\rho,n})} \{g(x-y) - g(x) + hW(y/h)\} \\
&\geq \inf_{y \in B(0,R_{\rho,\eta})} \left\{-(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})}\, g)\|y\| + hW(y/h)\right\} \\
&= h \inf_{z \in B(0,R_{\rho,\eta}/h)} \{-\lambda\|z\| + W(z)\} \\
&\geq -h \sup_{z \in \mathbb{R}^d} \{\lambda\|z\| - W(z)\} \\
&\geq -h \sup_{t \in B(0,\lambda)} W^*(t),
\end{aligned}$$

where $\lambda = \mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})}\, g$. Conversely, by definition,

$$Q_h^W(g)(x) - g(x) \leq hW(0),$$

and thus $\left|Q_h^W(g)(x) - g(x)\right| \leq Ch$, where $C = \max\{W(0), \sup_{t \in B(0,\lambda)} W^*(t)\}$. Note that C is finite because W^* is, by definition, convex and finite on $\mathbb{R}^d$, thus continuous. Finally, using the semigroup property $Q_{h+s}^W(g) = Q_h^W(Q_s^W(g))$ and the fact that the Lipschitz constant with respect to x is uniformly bounded by $\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})}$ for $0 < h < \eta$, we may conclude for the full local Lipschitz continuity. □

The above lemma is a slight generalization of the following proposition:

Proposition 2.13. *Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be lower semicontinuous functions. If f is nonnegative, locally Lipschitz continuous, and g is coercive, then $f \,\square\, g$ is locally Lipschitz continuous.*

Here, we do not need any convexity assumption, which was only used to prove Lipschitz continuity with respect to the $(n+1)$th variable, h. Also, note here that it is important for f and g to be finite *everywhere*, which will not be the case in sections 2.3 and appendix 2.4. In order for $f \,\square\, g$ to be locally Lipschitz continuous, further assumptions are needed on f and g, in particular on their domain. For example, if $\operatorname{dom} f = \{x_0\}$, then $f \,\square\, g = f(x_0) + g(\,.\, - x_0)$, so it already seems necessary that both f and g be at least locally Lipschitz continuous. However, this is not sufficient. Consider for example the following functions f and g, defined on $\mathbb{R}^2$ by

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in [0,1],\ x_2 = 0, \\ 1 - x_2 & \text{if } x_1 = 0,\ x_2 \in [0,1], \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 \in [0,1],\ x_2 = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

then

$$(f \,\square\, g)(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in (0,1],\ x_2 \in [0,1], \\ 1 - x_2 & \text{if } x_1 = 0,\ x_2 \in [0,1], \\ 0 & \text{if } x_1 = 0,\ x_2 \in [1,2], \\ +\infty & \text{otherwise} \end{cases}$$

is not a continuous function. This example can easily be adapted to obtain a discontinuous infimal convolution for smooth functions f and g. We conjecture that if the domain is assumed convex, and if both functions are Lipschitz continuous, and their domain is of non-empty interior, then their infimal convolution is Lipschitz continuous.

Lemma 2.12, together with Lemma 2.10 and Rademacher's theorem, prove the following proposition:

Proposition 2.14 (Hamilton-Jacobi)**.** *Let $g, W : \mathbb{R}^d \to \mathbb{R}$. If g is nonnegative, locally Lipschitz continuous, and W is convex and grows superlinearly,*

$$\lim_{\|x\| \to +\infty} \frac{W(x)}{\|x\|} = +\infty,$$

then, for almost every $h \geq 0$ and $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial h} Q_h^W(g)(x) = -W^*(\nabla Q_h^W g(x)).$$

2.2.3 An equivalent formulation of the classical Borell-Brascamp-Lieb inequality

In this subsection, we prove an interesting equivalence between the classical Borell-Brascamp-Lieb inequality and its differentiated expression, as announced in the introduction. It is also a good presentation of what is to come in the following sections.

Proposition 2.15. *Let $g, W : \mathbb{R}^d \to \mathbb{R}$. If g is nonnegative, locally Lipschitz continuous, and W is convex and grows superlinearly,*

$$\lim_{\|x\| \to +\infty} \frac{W(x)}{\|x\|} = +\infty,$$

and are such that $\int g^{-d} = \int W^{-d} = 1$, and if (g, W) is admissible in the sense of Definition 2.20, then the following statements are equivalent:

1. *The Borell-Brascamp-Lieb inequality holds: for every $t \in [0,1]$ and $H : \mathbb{R}^d \to \mathbb{R}$ such that*
$$\forall x, y \in \mathbb{R}^d, \quad H((1-t)x + ty) \leq (1-t)g(x) + tW(y),$$
there holds
$$\int H^{-d} \geq 1.$$

2. *The following inequality stands:*
$$\int \frac{W^*(\nabla g)}{g^{n+1}} \geq 0.$$

Proof. By definition of the infimal convolution $Q_h^W(g)$, it is actually sufficient to only consider the function $H = (1-t)Q_h^W(g)(\,.\,/(1-t))$, where $h = t/(1-t)$, in statement a. In fact, this leads to the statement a'.:
$$\int Q_h^W(g)^{-d} \geq 1,$$
which we prove is equivalent to b.

Let us consider the function $\phi : h \mapsto \int Q_h^W(g)^{-d}$, which is continuous and almost everywhere differentiable in light of Lemma 2.12 and Theorem 2.21 in the Appendix. Its derivative is given by
$$\phi'(h) = n \int \frac{W^*(\nabla g)}{g^{n+1}}.$$
The implication $a'. \implies b.$ follows from the fact that $\phi(0) = 1$, and $\phi(h) \geq 1$ for $h \geq 0$. Then, necessarily, $\phi'(0) \geq 0$.

Conversely, assume that b. holds. Then, whenever $h > 0$ is such that $\phi(h) = \int Q_h^W(g)^{-d} = 1$, statement b. applied to the function $\tilde{g} = Q_h^W(g)$ and the corresponding function $\tilde{\phi}$ implies that $\tilde{\phi}'(0) = \phi'(h) \geq 0$ thanks to the semigroup property proved in Lemma 2.11. This, together with the fact that $\phi(0) = 1$, proves that ϕ stays above 1, which is exactly statement a. □

Once again, we insist on the fact that the semigroup Q_h^W is not linear, and not Markov, which means, in particular, that there is no mass conservation. As such, this result stands as a bit unusual among similar results.

2.3 Sharp Gagliardo-Nirenberg-Sobolev inequalities

2.3.1 Borell-Brascamp-Lieb

Let us start from Theorem 8 in the recent paper by Bolley et al. [BCEF+17], the dynamical formulation of Borell-Brascamp-Lieb inequality. The proof from the introduction 1.4.2 may be easily adapted, swapping the exponent d for the parameter $a \geq d$, then giving it the same treatment than the one required to reach inequality (2.9).

Theorem 2.16 ([BCEF+17])**.** *Let $a > 1$ and $d \in \mathbb{N}^*$ such that $a \geq d$, and $g, W : \mathbb{R}^d \to (0, +\infty]$ be measurable functions such that $\int g^{-a} = \int W^{-a} = 1$. Then, for any $h \geq 0$,*
$$(1+h)^{a-d} \int_{\mathbb{R}^d} Q_h^W(g)^{1-a} \geq \int_{\mathbb{R}^d} g^{1-a} + h \int_{\mathbb{R}^d} W^{1-a}, \tag{2.10}$$
where
$$Q_h^W(g)(x) = \inf_{y \in \mathbb{R}^d} \{g(x - hy) + hW(y)\} \in (0, +\infty].$$
Furthermore, when g is equal to W and is convex, there is equality.

To see that there is equality whenever $g = W$ is convex, fix $x \in \mathbb{R}^d$. For any $y \in \mathbb{R}^d$, since $\frac{x}{1+h} = \frac{1}{1+h}(x - hy) + \frac{h}{1+h}y$,

$$(1+h)\left(\frac{W(x-hy)}{1+h} + \frac{h}{1+h}W(y)\right) \geq (1+h)W\left(\frac{x}{1+h}\right).$$

Conversely, $Q_h^W(g)(x)$ is achieved at $y = x/(1+h)$. In particular, for all $x \in \mathbb{R}^d$, $h \geq 0$,

$$Q_h^W(W)(x) = (1+h)W\left(\frac{x}{1+h}\right),$$

and equality in (2.10) is a straightforward computation.

In [BCEF+17], subsection 3.2, Bolley, Cordero-Erausquin, Fujita, Gentil, and Guillin use Theorem 2.16 to prove optimal Sobolev and Gagliardo-Nirenberg-Sobolev type inequalities in the half-space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$. We want to extend these results to more general domains Ω in $\mathbb{R}^d$, where $d \geq 2$. Let us assume that Ω is the epigraph of a continuous function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\varphi(0) = 0$. In other words,

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_2 \geq \varphi(x_1)\}.$$

Let $e = (0, 1) \in \mathbb{R}^{d-1} \times \mathbb{R}$, and for $h \geq 0$, define

$$\Omega_h = \Omega + \{he\} = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_2 \geq \varphi(x_1) + h\}.$$

Let $a \geq d$, and consider $g : \Omega \to (0, +\infty)$ and $W : \Omega_1 \to (0, +\infty)$, two measurable functions such that $\int_\Omega g^{-a} = \int_{\Omega_1} W^{-a} = 1$. After extending these functions by $+\infty$ outside of their respective domain, inequality (2.10) yields

$$(1+h)^{a-n} \int_{B_h} Q_h^W(g)^{1-a} \geq \int_\Omega g^{1-a} + h \int_{\Omega_1} W^{1-a} \tag{2.11}$$

where

$$B_h = \mathrm{dom}(Q_h^W(g)).$$

When $g(x) = W(x+e)$ and W is convex, then

$$Q_h^W(g)(x) = (1+h)W\left(\frac{x+e}{1+h}\right)$$

and equality is reached in the inequality above.

To get a sense of what is to follow, notice that there is equality in inequality (2.11) when $h = 0$. Now, when $\Omega = \mathbb{R}^d_+$, the interesting fact that $\Omega_h = B_h$ allows us, under certain admissibility criteria for W and g, to compute the derivative of inequality (2.11) with respect to h, at $h = 0$. By doing so, the term $\int_{\partial\mathbb{R}^d_+} Q_0^W(g)^{1-a} = \int_{\partial\mathbb{R}^d_+} g^{1-a}$ appears in the left hand side, thus leading to trace inequalities.

Before going any further, let us investigate under which condition the two sets Ω_h and B_h coincide. We have the following lemma:

Lemma 2.17. *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$, $B_h = \Omega_h$ if, and only if, Ω is a convex cone. In that case, B_h and Ω_h coincide for all $h \geq 0$.*

Proof. First, note that $Q_h^W(g)(x) < +\infty$ if, and only if, there exists $y \in \Omega_1$ such that $x - hy \in \Omega$. By definition of Ω, this is equivalent to

$$\begin{aligned} &\exists\,(y_1, y_2) \in \mathbb{R}^{d-1} \times \mathbb{R} \text{ s.t. } \begin{cases} y_2 \geq \varphi(y_1) + 1 \\ x_2 - hy_2 \geq \varphi(x_1 - hy_1) \end{cases} \\ \iff &\Big(\exists\, y_1 \in \mathbb{R}^{d-1} \text{ s.t. } x_2 \geq \varphi(x_1 - hy_1) + h\varphi(y_1) + h\Big). \end{aligned}$$

If $x \in \Omega_h$, then choosing $y_1 = 0$ proves that $x \in B_h$, so $\Omega_h \subset B_h$. If $h > 0$, $\Omega_h = B_h$ if, and only if, for all $x_1, y_1 \in \mathbb{R}^{d-1}$,

$$\varphi\left(\frac{x_1 - y_1}{h}\right) \geq \frac{\varphi(x_1) - \varphi(y_1)}{h}. \tag{2.12}$$

Indeed, if $\Omega_h \supset B_h$, then, for any $x_1, y_1 \in \mathbb{R}^{d-1}$,

$$x_2 := \varphi(x_1 - hy_1) + h\varphi(y_1) + h \geq \varphi(x_1) + h$$

and thus, replacing y_1 by $(x_1 - y_1)/h$, we get the stated inequality. The reciprocal is immediate.

Now, let $z \in \mathbb{R}^{d-1}$, $\|z\| = 1$. Inequality (2.12), for $y_1 = 0$, becomes

$$\varphi(z) \geq \frac{1}{h}\varphi(hz)$$

for any h smaller than h_0. Let $\alpha = \limsup_{h\to 0} \varphi(hz)/h$. Using inequality (2.12) once again, we get, for any $s \geq 0$,

$$\varphi(sz) \geq \frac{s}{sh}\varphi(shz),$$

for any sufficiently small $h > 0$. Taking the inferior limit when $h \to 0$ proves that for any $s \geq 0$

$$\varphi(sz) \geq s\alpha. \tag{2.13}$$

The set $\{s \geq 0, \varphi(sz) = s\alpha\}$ is non-empty because it contains 0, and it is closed by continuity. Let $s \geq 0$ be such that $\varphi(sz) = s\alpha$. Then, invoking inequality (2.12), and then inequality (2.13), we get

$$\begin{aligned} \varphi\left(\frac{(1+h)sz - sz}{h}\right) = \varphi(sz) = s\alpha &\geq \frac{\varphi((1+h)sz) - \varphi(sz)}{h} \\ &= \frac{\varphi((1+h)sz) - s\alpha}{h} \\ &\geq \frac{(1+h)s\alpha - s\alpha}{h} = s\alpha \end{aligned}$$

so there is actually equality, and $\varphi((1+h)sz) = (1+h)s\alpha$ for any sufficiently small $h > 0$. This shows that the connected component of $\{s \geq 0, \varphi(sz) = s\alpha\}$ containing 0 is open in $\mathbb{R}_+$. Since it is also closed, it is the half real line $\mathbb{R}_+$. Thus, φ is linear over half-lines with initial point 0. Inequality (2.12) then becomes

$$\varphi(x_1 - y_1) \geq \varphi(x_1) - \varphi(y_1)$$

for any $x_1, y_1 \in \mathbb{R}^{d-1}$. Let $t \in [0, 1]$; replacing x_1 by $(1-t)x_1 + ty_1$ and y_1 by ty_1, and using linearity, the inequality becomes exactly the convexity inequality, that is

$$\varphi((1-t)x_1 + ty_1) \leq (1-t)\varphi(x_1) + t\varphi(y_1).$$

The reciprocal is trivial. It is also clear that in this case, $B_h = \Omega_h$ for *any* $h \geq 0$. $\square$

This lemma will be used in section 2.3.2 to prove the trace Sobolev and the trace Gagliardo-Nirenberg-Sobolev inequalities in convex cones. We can go a bit further, and impose only φ to be convex.

Lemma 2.18. *If φ is convex, then*

$$B_h = \left\{(x_1, x_2) \in \mathbb{R}^d, x_2 \geq h + (1+h)\varphi\left(\frac{x_1}{1+h}\right)\right\}.$$

Proof. One may notice that setting $\omega(x) = 0$ if $x \in \Omega$ and $+\infty$ if $x \in \Omega^c$, and $W(x) = \omega(x-e)$, then ω is convex, thus

$$B_h = \mathrm{dom}(Q_h^W(\omega)) = \mathrm{dom}\left(x \mapsto (1+h)\, W\left(\frac{x+e}{1+h}\right)\right),$$

and

$$\begin{aligned} W\left(\frac{x+e}{1+h}\right) < +\infty \iff & \frac{x+e}{1+h} - e \in \Omega \\ \iff & x_2 \geq h + (1+h)\varphi\left(\frac{x_1}{1+h}\right). \end{aligned}$$

□

2.3.2 Convex cones

In this subsection, we assume that Ω is a convex cone. In that case, invoking Lemma 2.17, inequality (2.10) becomes

$$(1+h)^{a-d} \int_{\Omega_h} Q_h^W(g)^{1-a} \geq \int_\Omega g^{1-a} + h \int_{\Omega_1} W^{1-a}, \tag{2.14}$$

for any $h > 0$, and there is equality when $h = 0$. Taking the derivative of this inequality with respect to h, under the admissibility conditions for g and W exposed in full details in Appendix 2.4, and evaluating at $h = 0$, we prove that

$$(a-d) \int_\Omega g^{1-a} + (a-1) \int_\Omega \frac{W^*(\nabla g)}{g^a} - \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1, \varphi(x_1))dx_1 \geq \int_{\Omega_1} W^{1-a}. \tag{2.15}$$

There, we used Lemma 2.10, and the fact that

$$\begin{aligned} \frac{1}{h}\left(\int_{\Omega_h} Q_h^W(g)^{1-a} - \int_\Omega g^{1-a}\right) &= \int_{\Omega_h} \frac{Q_h^W(g)^{1-a} - g^{1-a}}{h} + \frac{1}{h}\left(\int_{\Omega_h} g^{1-a} - \int_\Omega g^{1-a}\right) \\ &= \int_{\Omega_h} \frac{Q_h^W(g)^{1-a} - g^{1-a}}{h} - \frac{1}{h}\left(\int_{\mathbb{R}^{d-1}} \int_{\varphi(x_1)}^{h+\varphi(x_1)} g^{1-a}(x_1, x_2)dx_2dx_1\right) \\ &\xrightarrow[h\to 0]{} (1-a) \int_\Omega \frac{W^*(\nabla g)}{g^a} - \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1, \varphi(x_1))dx_1, \end{aligned}$$

see Theorem 2.21.

Let $p \in (1, d)$, and q its conjugate exponent, $1/p + 1/q = 1$. Applying inequality (2.15) to the function W defined by $W(x) = C\|x\|^q/q$, where $C > 0$ is such that $\int W^{-a} = 1$, which happens to be admissible for this choice of q, in the sense of Definition 2.20 in the Appendix. We find

$$(a-d) \int_\Omega g^{1-a} + C^{1-p} \frac{a-1}{p} \int_\Omega \frac{\|\nabla g\|_*^p}{g^a} - \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1, \varphi(x_1))dx_1 \geq \int_{\Omega_1} W^{1-a}$$

for any admissible g, where

$$\|x\|_* = \sup_{\|y\|=1} x \cdot y \tag{2.16}$$

is the dual norm of x. Next, we extend the above inequality to all functions g such that $f = g^{(p-a)/p} \in \mathcal{C}_c^\infty(\Omega)$. This can be done by approximation by admissible functions, we refer to the Appendix 2.4. Rewriting the quantities in terms of $f = g^{-(a-p)/p}$ yields

$$\int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x,\varphi(x))dx \le C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_\Omega \|\nabla f\|_*^p - \int_{\Omega_1} W^{1-a} + (a-d)\int_\Omega f^{p\frac{a-1}{a-p}}.$$

We may then remove the normalization to find that inequality (2.15) becomes

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x,\varphi(x))dx \le C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \left(\int_\Omega \|\nabla f\|_*^p\right)\beta^{p\frac{p-1}{a-p}} - \left(\int_{\Omega_1} W^{1-a}\right)\beta^{p\frac{a-1}{a-p}}\\ + (a-d)\int_\Omega f^{p\frac{a-1}{a-p}}\end{aligned} \tag{2.17}$$

where

$$\beta = \left(\int_\Omega f^{\frac{pa}{a-p}}\right)^{\frac{a-p}{ap}}.$$

Now, define $u = \frac{a-1}{a-p}$ and $v = u' = \frac{a-1}{p-1}$, so that $u, v > 1$ and $1/u + 1/v = 1$. By Young's inequality, we find

$$\begin{aligned}A\int_\Omega \|\nabla f\|_*^p \beta^{p\frac{p-1}{a-p}} - \left(\int_{\Omega_1} W^{1-a}\right)\beta^{p\frac{a-1}{a-p}} &= Bv\left(\frac{A}{Bv}\int_\Omega \|\nabla f\|_*^p \beta^{p\frac{p-1}{a-p}} - \frac{1}{v}\beta^{p\frac{a-1}{a-p}}\right)\\ &\le D\left(\int_\Omega \|\nabla f\|_*^p\right)^u,\end{aligned} \tag{2.18}$$

where

$$A = C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p, \quad B = \int_{\Omega_1} W^{1-a} \quad \text{and} \quad D = \frac{A^u}{(Bv)^{u-1}}\frac{1}{u}.$$

In order to find a more compact inequality, we consider, for $\lambda > 0$, $f_\lambda : x \mapsto f(\lambda x)$. By linearity of φ, applying (2.18) to f_λ leads to

$$\int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x,\varphi(x))dx \le \lambda^{(a-d)\frac{p-1}{a-p}}\frac{A^u}{(Bv)^{u-1}}\frac{1}{u}\left(\int_\Omega \|\nabla f\|_*^p\right)^u + \frac{a-d}{\lambda}\int_\Omega f^{p\frac{a-1}{a-p}}.$$

Optimizing this inequality with respect to $\lambda > 0$ finally yields inequality (2.6) of Theorem 2.2

It remains to show that inequality (2.6) is optimal. The function for which equality is reached does not have compact support, but this technicality does not bear much relevance. To prove optimality, note that there is equality in (2.15) when $g(x) = W(x+e)$, which implies equality in (2.17) when $f(x) = \|x+e\|^{-\frac{a-p}{p-1}}$. If Young's inequality (2.18) is an equality, then the optimization with respect to parameter λ necessarily preserves the equality. Thus, it is enough to show that for $f(x) = \|x+e\|^{-\frac{a-p}{p-1}}$, there is equality in (2.18). This is the case if, and only if,

$$\frac{A}{Bv}\int_\Omega \|\nabla f\|_*^p = \left(\beta^{p\frac{p-1}{a-p}}\right)^{v-1}.$$

Let us now write, for $\alpha > 0$

$$I_\alpha := \int_\Omega \|x+e\|^{-\alpha}.$$

Then,

$$C = q \left(\int_\Omega \|x+e\|^{-qa} \right)^{\frac{1}{a}} = \frac{p}{p-1} I_{ap/(p-1)}^{1/a}$$

hence

$$A = \frac{(a-1)(p-1)^{p-1}}{(a-p)^p} I_{ap/(p-1)}^{(1-p)/a}, \quad B = I_{ap/(p-1)}^{(1-a)/a} I_{p(a-1)/(p-1)}, \quad \text{and} \quad \left(\beta^{p\frac{p-1}{a-p}}\right)^{v-1} = I_{ap/(p-1)}^{(a-p)/a}.$$

Claim. For $\gamma \in \mathbb{R}$, let $\delta : \mathbb{R}^d \backslash \{0\} \to]0, +\infty[\, , x \mapsto \|x\|^\gamma$. Then, almost everywhere, δ is differentiable, and $\|\nabla \delta(x)\|_*^p = |\gamma| \|x\|^{\gamma - 1}$.

Using this, we conclude that there is indeed equality in (2.18), since then

$$\int_\Omega \|\nabla f\|_*^p = \left(\frac{a-p}{p-1} \right)^p I_{p(a-1)/(p-1)}.$$

Proof of the claim. Consider $\phi : x \mapsto \|x\|$ and $\psi : \rho \mapsto \rho^\gamma$. ϕ is convex, hence almost everywhere differentiable by Rademacher's theorem, and ψ smooth on $]0, +\infty[$, hence the claimed regularity of $\delta = \psi \circ \phi$. For almost every x, $\nabla \delta(x) = \gamma \nabla \phi(x) \|x\|^{\gamma-1}$, so

$$\|\nabla \delta(x)\|_* = |\gamma| \|x\|^{\gamma-1} \|\nabla \phi(x)\|_*$$

If $x \neq 0$ is a point of differentiability of ϕ, and $t > 0$, then

$$1 = \frac{\|x + tx/\|x\|\| - \|x\|}{t} \xrightarrow[t \to 0]{} \nabla \phi(x) \cdot \frac{x}{\|x\|},$$

so $\|\nabla \phi(x)\|_* \geq 1$. Conversely, if $\|v\| = 1$, then

$$\nabla \phi(x) \cdot v = \lim_{t \to 0^+} \frac{\|x + tv\| - \|x\|}{t} \leq \lim_{t \to 0^+} \|v\| = 1,$$

so $\|\nabla \phi(x)\|_* = 1$ and the claim is proved. $\square$

2.3.3 Convex sets

Let us now assume that Ω is the epigraph of a convex function φ, with $\varphi(0) = 0$. Then, according to Lemma 2.18, for $h \geq 0$,

$$B_h = \mathrm{dom}(Q_h^W(g)) = \left\{ (x_1, x_2) \in \mathbb{R}^d, x_2 \geq h + (1+h)\varphi\left(\frac{x_1}{1+h} \right) \right\}.$$

Inequality (2.10) becomes

$$(1+h)^{a-d} \int_{B_h} Q_h^W(g)^{1-a} \geq \int_\Omega g^{1-a} + h \int_{\Omega_1} W^{1-a}, \tag{2.19}$$

and there still is equality for all $h > 0$ whenever $g(x) = W(x+e)$ and is convex. However, it is slightly trickier to compute the derivative at $h = 0$, since $B_h \neq \Omega_h$, and their symmetric difference depends heavily on φ. Effectively, a third term appears when trying to differentiate $\int_{B_h} Q_W^h(g)^{1-a}$:

$$\frac{1}{h}\left(\int_{B_h} Q_h^W(g)^{1-a} - \int_\Omega g^{1-a} \right) = \int_{\Omega_h} \frac{Q_h^W(g)^{1-a} - g^{1-a}}{h} - \frac{1}{h} \int_{\Omega \backslash \Omega_h} g^{1-a} + \frac{1}{h} \int_{B_h \backslash \Omega_h} Q_h^W(g)^{1-a}.$$

Taking the derivative at $h = 0$, when possible, yields

$$(a-d)\int_\Omega g^{1-a} + (a-1)\int_\Omega \frac{W^*(\nabla g)}{g^a} - \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1, \varphi(x_1))P(x_1)dx_1 \geq \int_{\Omega_1} W^{1-a}, \quad (2.20)$$

where

$$P(x_1) = 1 + \varphi(x_1) - x_1 \cdot \nabla\varphi(x_1).$$

Using inequality (2.20) with $W = C\|.\|^q/q$, and extending it for all $f = g^{-(a-p)/p} \in \mathcal{C}_c^\infty(\Omega)$ just like we did for convex cones, and finally invoking Young's inequality, we get the theorem

Theorem 2.19. *Let $a \geq n > p > 1$, and $\Omega = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_2 \geq \varphi(x_1)\}$ be a convex set. There exists a positive constant $D'_{n,p,a}(\Omega)$ such that for any positive function $f \in C_c^\infty(\Omega)$,*

$$\int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1, \varphi(x_1))P(x_1)dx_1 \leq D'_{n,p,a}(\Omega)\left(\int_\Omega \|\nabla f\|_*^p\right)^{\frac{a-1}{a-p}} + (a-d)\int_\Omega f^{p\frac{a-1}{a-p}}, \quad (2.21)$$

where $P(x_1) = 1 + \varphi(x_1) - x_1 \cdot \nabla\varphi(x_1)$. Furthermore, when $f(x) = \|x + e\|^{-\frac{a-p}{p-1}}$, then (2.21) is an equality.

Applying this theorem for $a = d$, we find a new version of the trace Sobolev inequality, Theorem 2.4, with $D'_{d,p}(\Omega) = D'_{d,p,d}(\Omega)$. It is important to note that in Theorem 2.4, as well as in Theorem 2.19, the left-hand side can be negative. The weight P itself generally *is* negative outside of a compact neighbourhood of the origin, but the inequality is still optimal.

2.4 Admissibility

In this section, we prove that the results are true for a class of admissible functions, and we extend these results to the appropriate, more general setting, by approximation by admissible functions. The difficulty here lies in that g must not be bounded or even Lipschitz, since g^{-a} has to be integrable. The case of the half-plane has already been investigated (in [BCEF+17]), and easily extends to convex cones. Here, we will only tackle convex sets, which, although more technical, follows the same general idea.

Throughout this section, $\varphi : \mathbb{R}^{d-1} \to [0, +\infty)$ is a convex function such that $\varphi(0) = 0$, $g : \Omega \to (0, +\infty)$ is assumed to be locally Lipschitz continuous, and $W : \Omega_1 \to (0, +\infty)$ is convex.

2.4.1 Differentiating the Borell-Brascamp-Lieb inequality

Inequality (2.19),

$$(1+h)^{a-d}\int_{B_h} Q_h^W(g)^{1-a} \geq \int_\Omega g^{1-a} + h\int_{\Omega_1} W^{1-a},$$

is trivially an equality for $h = 0$, we thus ask compute its derivative. Let us first give a non-rigorous proof for clarity. The most difficult part is computing the derivative of $\int_{B_h} Q_h^W(g)^{1-a}$, so let us start with that. Notice that $\Omega_h \subset B_h \cap \Omega$, thus

$$\frac{1}{h}\left(\int_{B_h} Q_h^W(g)^{1-a} - \int_\Omega g^{1-a}\right) = \underbrace{\int_{\Omega_h} \frac{Q_h^W(g)^{1-a} - g^{1-a}}{h}}_{(i)} - \underbrace{\frac{1}{h}\int_{\Omega\setminus\Omega_h} g^{1-a}}_{(ii)} + \underbrace{\frac{1}{h}\int_{B_h\setminus\Omega_h} Q_h^W(g)^{1-a}}_{(iii)}.$$

Recalling Lemma 2.10, almost everyhere,

$$\lim_{h\to 0} \frac{Q_h^W(g)(x) - g(x)}{h} = -W^*(\nabla g(x)),$$

thus (i) should converge towards

$$(a-1)\int_\Omega \frac{W^*(\nabla g)}{g^a}.$$

Next, (ii) can be rewritten in a way such that the convergence is quite clear:

$$(ii) = \int_{\mathbb{R}^{d-1}} \left(\frac{1}{h}\int_{\varphi(x_1)}^{\varphi(x_1)+h} g^{1-a}(x_1,x_2)dx_2\right)dx_1 \xrightarrow[h\to 0]{} \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1,\varphi(x_1))dx_1$$

as $h \to 0$. Finally, giving (iii) the same treatment,

$$\begin{aligned}(iii) &= \int_{\mathbb{R}^{d-1}} \left(\frac{1}{h}\int_{h+(1+h)\varphi(x_1/(1+h))}^{h+\varphi(x_1)} Q_h^W(g)^{1-a}(x_1,x_2)dx_2\right)dx_1 \\ &\xrightarrow[h\to 0]{} \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1,\varphi(x_1))(x_1\cdot\nabla\varphi(x_1) - \varphi(x_1))dx_1,\end{aligned}$$

since $Q_0^W(g) = g$ and

$$\lim_{h\to 0}\frac{1}{h}\left(\varphi(x_1) - (1+h)\varphi\left(\frac{x_1}{1+h}\right)\right) = x_1\cdot\nabla\varphi(x_1) - \varphi(x_1).$$

Summing these results up, we find the claimed derivative at $h = 0$. Whenever Ω is a convex cone, $B_h\backslash\Omega_h = \emptyset$, and thus $(iii) = 0$. In that case, the argument is much more succinct, but since it is also a corollary of the more general case, we will not address it. The conditions for the convergence to play out nicely are summed up in the following definition. They are mostly growth conditions on g and W, and will come into play later on.

Definition 2.20. The couple of functions (g, W) is said to be admissible if the following conditions are satisfied for some constant γ:

(C0) $\gamma > \max\left(\frac{a}{d-1}, 1\right)$;

(C1) there exists $A_1 > 0$ such that $W(x) \geq A_1\|x\|^\gamma$ for all $x \in \Omega_1$;

(C2) there exists $A_2 > 0$ such that $W(x) \leq A_2(1 + \|x\|^\gamma)$ for all $x \in \Omega_1$;

(C3) there exists $A_3 > 0$ such that $g(x) \geq A_3(1 + \|x\|^\gamma)$ for all $x \in \Omega$;

(C4) there exists $A_4 > 0$ such that $\|\nabla g(x)\| \leq A_4(1 + \|x\|^{\gamma-1})$ for all $x \in \Omega$.

The challenge is to prove that under these conditions, $Q_h^W(g)$ converges towards g in a controlled manner as $h \to 0$. The main result of this section is the following:

Theorem 2.21. *Assume that the couple (g, W) is admissible, and that there exist some constants $C > 0$ and $R > 0$ such that*

$$\forall\,\|x_1\| > R, \quad |x_1\cdot\nabla\varphi(x_1)| \leq C\|(x_1,\varphi(x_1))\|. \tag{2.22}$$

Then

$$\lim_{h\to 0}\frac{1}{h}\left(\int_{B_h} Q_h^W(g)^{1-a}(g) - \int_\Omega g^{1-a}\right) = (a-1)\int_\Omega \frac{W^*(\nabla g)}{g^a} - \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1,\varphi(x_1))P(x_1)dx_1, \tag{2.23}$$

where $P(x_1) = 1 + \varphi(x_1) - x_1\cdot\nabla\varphi(x_1)$.

In what follows, we will use a good number of different positive constants, which will all be written C for convenience. They will not depend on $x \in \mathbb{R}^d$, or $h > 0$, but might depend on A_i, $i \in \{1, 2, 3, 4\}$, γ.

Convergence of (*i*) and (*ii*)

Lemma 2.22. *If (g, W) is admissible, there exist constants $C > 0$ and $h_0 > 0$, such that for all $0 < h < h_0$, and $x \in \Omega_h$,*

$$\left|Q_h^W(g)(x) - g(x)\right| \leq Ch(1 + \|x\|^\gamma).$$

Proof. First, let $x', x \in \Omega$. Then, we may estimate $|g(x') - g(x)|$ using hypothesis (C4):

$$\begin{aligned}
|g(x') - g(x)| &\leq \int_0^1 \left\| \frac{\partial}{\partial \theta} g(x + \theta(x' - x)) \right\| d\theta \\
&\leq \|x' - x\| \int_0^1 A_4(1 + \|x + \theta(x' - x)\|^{\gamma-1}) d\theta \\
&\leq C\|x' - x\| \left(1 + \|x\|^{\gamma-1} + \|x' - x\|^{\gamma-1}\right). \qquad (2.24)
\end{aligned}$$

Now, let $0 < h \leq 1$ and $x \in \Omega_h$. Then, $x - he \in \Omega$, so

$$\begin{aligned}
Q_h^W(g)(x) - g(x) &\leq g(x - he) + hW(e) - g(x) \\
&\leq Ch(1 + \|x\|^{\gamma-1} + h^{\gamma-1}) + hW(e) \\
&\leq Ch(1 + \|x\|^{\gamma-1}).
\end{aligned}$$

For the converse inequality, we will of course use hypotheses (C1) and (C3), but we first have to localize the point where the infimum $Q_h^W(g)(x)$ is reached. Let $y \in \Omega_1$ be such that $Q_h^W(g)(x) = g(x - hy) + hW(y)$. Then, invoking hypothesis (C1) and inequality (2.24),

$$\begin{aligned}
hA_1\|y\|^\gamma \leq hW(y) &= Q_h^W(g)(x) - g(x - hy) \\
&= Q_h^W(g)(x) - g(x) + g(x) - g(x - hy) \\
&\leq Ch(1 + \|x\|^{\gamma-1}) + Ch\|y\|(1 + \|x\|^{\gamma-1} + (h\|y\|)^{\gamma-1}).
\end{aligned}$$

We thus choose $h_0 \in (0, 1)$ such that for any $h \in (0, h_0)$, $A_1 - Ch^{\gamma-1} > h^{\gamma-1}$. Then, for any $h \in (0, h_0)$,

$$\begin{aligned}
h^{\gamma-1}\|y\|^\gamma &< (A_1 - Ch^{\gamma-1})\|y\|^\gamma \\
&\leq C(1 + \|y\|)(1 + \|x\|^{\gamma-1}),
\end{aligned}$$

which implies that

$$h^{\gamma-1}\|y\|^{\gamma-1} \leq C(1 + \|x\|^{\gamma-1}),$$

since $\|y\|^{\gamma-1} \leq \max\left(1, 2\frac{\|y\|^\gamma}{1+\|y\|}\right)$. Now, using inequality (2.24) once again,

$$\begin{aligned}
|g(x - hy) - g(x)| &\leq Ch\|y\|(1 + \|x\|^{\gamma-1} + h^{\gamma-1}\|y\|^{\gamma-1}) \\
&\leq Ch\|y\|(1 + \|x\|^{\gamma-1}).
\end{aligned}$$

Plugging this in the definition of $Q_h^W(g)(x)$, we find

$$\begin{aligned} Q_h^W(g)(x) - g(x) &\geq \inf_{y\in\Omega_1} \left\{ -Ch\|y\|(1+\|x\|^{\gamma-1}) + hA_1\|y\|^\gamma \right\} \\ &\geq \inf_{y\in\mathbb{R}^d} \{\dots\} = -Ch(1+\|x\|^\gamma). \end{aligned}$$

To conclude, it is enough to notice that $1+\|x\|^{\gamma-1} \leq 2+\|x\|^\gamma$ since $\gamma > 1$. $\square$

Now that we have this estimation, we may estimate the speed of convergence of $Q_h^W(g)^{1-a}$ towards g^{1-a}.

Proposition 2.23. *If (g, W) is admissible, there exist constants $C > 0$ and $h_0 > 0$, such that for all $0 < h < h_0$, and $x \in \Omega_h$,*

$$\frac{\left|Q_h^W(g)^{1-a}(x) - g^{1-a}(x)\right|}{h} \leq \frac{C}{1+\|x\|^{\gamma(a-1)}}.$$

Proof. First, let $\alpha, \beta > 0$. Then,

$$\left|\int_\alpha^\beta t^{-a} dt\right| = \left|\frac{1}{1-a}(\beta^{1-a} - \alpha^{1-a})\right| \leq \max(\alpha^{-a}, \beta^{-a})|\alpha - \beta|,$$

implying that

$$\left|\alpha^{1-a} - \beta^{1-a}\right| \leq (a-1)|\alpha-\beta|(\alpha^{-a} + \beta^{-a}). \tag{2.25}$$

Then, according to Lemma 2.22, there exists $h_0 > 0$ such that for any $h \in (0, h_0)$, and any $x \in \Omega_h$,

$$\begin{aligned} \frac{\left|Q_h^W(g)^{1-a}(x) - g^{1-a}(x)\right|}{h} &\leq C \frac{\left|Q_h^W(g)(x) - g(x)\right|}{h} \left(Q_h^W(g)^{-a}(x) + g^{-a}(x)\right) \\ &\leq C(1+\|x\|^\gamma)\left(Q_h^W(g)^{-a}(x) + g^{-a}(x)\right). \end{aligned} \tag{2.26}$$

Now, hypotheses (C1) and (C3) and a straightforward computation yield

$$\begin{aligned} Q_h^W(g)(x) &\geq \inf_{y\in\Omega_1} \{A_3(1+\|x-hy\|^\gamma) + hA_1\|y\|^\gamma\} \\ &\geq \inf_{y\in\mathbb{R}^d} \{A_3(1+|\|x\| - h\|y\||^\gamma) + hA_1\|y\|^\gamma\} \\ &\geq C(1+\|x\|^\gamma). \end{aligned}$$

Using (C3) once again, we know that

$$g^{-a}(x) \leq (A_3(1+\|x\|^\gamma))^{-a};$$

putting these two inequalities together with inequality (2.26), we finally obtain

$$\begin{aligned} \frac{\left|Q_h^W(g)^{1-a}(x) - g^{1-a}(x)\right|}{h} &\leq C\frac{1+\|x\|^\gamma}{(1+\|x\|^\gamma)^a} \\ &\leq \frac{C}{1+\|x\|^{\gamma(a-1)}}. \end{aligned}$$

$\square$

Proposition 2.23, together with Lemma 2.10, proves the dominated convergence, and

$$\lim_{h\to 0}(i) = (a-1)\int_\Omega \frac{W^*(\nabla g)}{g^a},$$

as claimed. The convergence of (ii) is straightforward, as it is a direct implication of the local Lipschitz continuity of g and hypothesis (C3).

Convergence of (iii)

This term is a bit trickier, because comparing $Q_h^W(g)$ to g is not possible on the entirety of B_h, g being defined only on Ω. For many functions φ, $B_h \not\subset \Omega$ as is showcased on figure 2.1 below. Thus, we prove the following result:

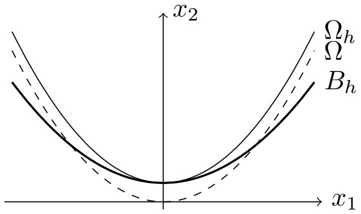

Figure 2.1: Graph of Ω, Ω_h, and B_h for $\varphi(x_1) = \|x_1\|^2$ and $h = 0.5$

Lemma 2.24. *If (g, W) is admissible, there exist constants $C > 0$ and $h_1 > 0$, such that for all $0 < h < h_1$, and $(x_1, x_2) \in B_h \backslash \Omega_h$,*

$$\left|Q_h^W(g)(x_1,x_2) - g(x_1,\varphi(x_1))\right| \le hC(1 + \|(x_1,\varphi(x_1))\|^\gamma + |x_1\cdot\nabla\varphi(x_1)|^\gamma). \tag{2.27}$$

The proof follows the same logic as the proof of Lemma 2.22.

Proof. Recall that, according to Lemma 2.18

$$\Omega_h = \left\{(x_1,x_2)\in\mathbb{R}^d,\, x_2 \ge h + \varphi(x_1)\right\}, \quad B_h = \left\{(x_1,x_2)\in\mathbb{R}^d,\, x_2 \ge h + (1+h)\varphi\left(\frac{x_1}{1+h}\right)\right\},$$

and that

$$|g(x') - g(x)| \le C\|x'-x\|\left(1 + \|x\|^{\gamma-1} + \|x'-x\|^{\gamma-1}\right) \qquad \text{(2.24 revisited)}$$

for any $x', x \in \Omega$.

1. Fix $h \in (0,1)$, $x = (x_1,x_2) \in B_h\backslash\Omega_h$, and define $p(x_1,x_2) = (x_1,\varphi(x_1))$, its projection onto $\partial\Omega$. Letting $y = \left(\frac{x_1}{1+h}, 1 + \varphi\left(\frac{x_1}{1+h}\right)\right)$, we find that $y \in \Omega_1$, and also that $x - hy \in \Omega$, thus, with hypothesis (C2) and inequality (2.24),

$$\begin{aligned} Q_h^W(x) - g(p(x)) &\le g(x-hy) - g(p(x)) + hW(y) \\ &\le C\|x-hy-p(x)\|\left(1 + \|p(x)\|^{\gamma-1} + \|x-hy-p(x)\|^{\gamma-1}\right) + hA_2(1+\|y\|^\gamma). \end{aligned}$$

For brevity, let us write $u = \|p(x)\| = \|(x_1,\varphi(x_1)\|$, and $v = x_1\cdot\varphi(x_1) = |x_1\cdot\varphi(x_1)|$. Now, notice that

$$\|x-hy-p(x)\| = h\left\|\left(\frac{x_1}{1+h}, \frac{x_2-h-\varphi(x_1)}{h} - \varphi\left(\frac{x_1}{1+h}\right)\right)\right\|.$$

From the definition of Ω_h and B_h, we find out that

$$\begin{aligned}0 \leq \frac{h+\varphi(x_1)-x_2}{h} &\leq \frac{\varphi(x_1)-(1+h)\varphi(x_1/(1+h))}{h}\\ &\leq x_1\cdot\nabla\varphi(x_1)=v,\end{aligned}$$

since φ is convex and nonnegative. Thus,

$$\begin{aligned}\|x-hy-p(x)\| &\leq h\left\|\left(\frac{x_1}{1+h},\varphi\left(\frac{x_1}{1+h}\right)\right)\right\|+h|x_1\cdot\nabla\varphi(x_1)|\\ &\leq h(\|p(x)\|+|x_1\cdot\nabla\varphi(x_1)|)=h(u+v),\end{aligned}$$

so, since $h<1$,

$$\begin{aligned}1+\|p(x)\|^{\gamma-1}+\|x-hy-p(x)\|^{\gamma-1} &\leq 1+u^{\gamma-1}+(h(u+v))^{\gamma-1}\\ &\leq C(1+u^{\gamma-1}+v^{\gamma-1}).\end{aligned}$$

Finally,

$$A_2(1+\|y\|^\gamma)=A_2\left(1+\left\|\frac{x_1}{1+h},1+\varphi\left(\frac{x_1}{1+h}\right)\right\|^\gamma\right)\leq C(1+u^\gamma).$$

Putting all these inequalities together, we find

$$\begin{aligned}Q_h^W(x)-g(p(x)) &\leq hC(u+v)(1+u^{\gamma-1}+v^{\gamma-1})+hC(1+u^\gamma)\\ &\leq hC(1+u^\gamma+v^\gamma).\end{aligned}\tag{2.28}$$

2. Conversely, let $y\in\Omega_1$ be such that $Q_h^W(g)(x)=g(x-hy)+hW(y)$. As before, we localize y. Using hypothesis A_1 and inequalities (2.24) and (2.28),

$$\begin{aligned}hA_1\|y\|^\gamma &\leq hW(y)=Q_h^W(g)(x)-g(p(x))+g(p(x))-g(x-hy)\\ &\leq hC(1+u^\gamma+v^\gamma)+C\|x-hy-p(x)\|\left(1+u^{\gamma-1}+\|x-hy-p(x)\|^{\gamma-1}\right)\\ &\leq hC(1+u^\gamma+v^\gamma)+hC(\|y\|+v)\left(1+u^{\gamma-1}+h^{\gamma-1}(\|y\|+v)^{\gamma-1}\right)\\ &\leq hC(1+u^\gamma+v^\gamma)+hC(\|y\|+v)\left(1+u^{\gamma-1}+h^{\gamma-1}\|y\|^{\gamma-1}+v^{\gamma-1}\right).\end{aligned}$$

Rearranging the terms and dividing by h yields

$$\begin{aligned}A_1\|y\|^\gamma-Ch^{\gamma-1}\|y\|^{\gamma-1}(\|y\|+v) &\leq C(1+u^\gamma+v^\gamma)+C(\|y\|+v)\left(1+u^{\gamma-1}+v^{\gamma-1}\right)\\ &\leq C(1+u+v+\|y\|)\left(1+u^{\gamma-1}+v^{\gamma-1}\right).\end{aligned}$$

We must now split the reasoning in two cases: either $\|y\|\leq v$, in which case the conclusion follows, or $\|y\|\geq v$, and then $A_1\|y\|^\gamma-Ch^{\gamma-1}\|y\|^{\gamma-1}(\|y\|+v)\geq A_1\|y\|^\gamma-2Ch^{\gamma-1}\|y\|^\gamma$. We thus choose $0<h_1<1$ such that for all $h\in(0,h_1)$, $A_2-2Ch^{\gamma-1}\geq h^{\gamma-1}$. Then, we have, for any $h\in(0,h_1)$,

$$\frac{h^{\gamma-1}\|y\|^\gamma}{1+u+v+\|y\|}\leq C\left(1+u^{\gamma-1}+v^{\gamma-1}\right).$$

Once again, either $\|y\|\leq 1+u+v$, or

$$h^{\gamma-1}\|y\|^{\gamma-1}\leq\frac{2h^{\gamma-1}\|y\|^\gamma}{1+u+v+\|y\|}.$$

Taking the greatest of the constants in those two cases, we may conclude that

$$h^{\gamma-1}\|y\|^{\gamma-1} \leq C(1+u^{\gamma-1}+v^{\gamma-1}). \tag{2.29}$$

3. We may now proceed with the converse inequality. Invoking once again inequality (2.24), and then inequality (2.29),

$$\begin{aligned}|g(x-hy)-g(p(x))| &\leq hC(\|y\|+v)(1+u^{\gamma-1}+h^{\gamma-1}\|y\|^{\gamma-1}+v^{\gamma-1})\\ &\leq hC(\|y\|+v)(1+u^{\gamma-1}+v^{\gamma-1}).\end{aligned}$$

Finally,

$$\begin{aligned}Q_h^W(g)(x)-g(p(x)) &= g(x-hy)-g(p(x))+hW(y)\\ &\geq -hC(\|y\|+v)(1+u^{\gamma-1}+v^{\gamma-1})+hA_2\|y\|^{\gamma}\\ &\geq h\inf_{y\in\mathbb{R}^d}\{-C(\|y\|+v)(1+u^{\gamma-1}+v^{\gamma-1})+A_2\|y\|^{\gamma}\}\\ &\geq -hC\left(1+u^{\gamma-1}+v^{\gamma-1}\right)^{\gamma/(\gamma-1)},\end{aligned}$$

and we may conclude. □

We may now prove Theorem 2.21: using the same notations as in the proof above, that is $u=\|p(x)\|=\|(x_1,\varphi(x_1)\|$, and $v=x_1\cdot\varphi(x_1)=|x_1\cdot\varphi(x_1)|$, hypothesis (C2) immediately yields, for all $h>0$ and all $x\in B_h\backslash\Omega_h$,

$$g^{-a}(p(x))\leq\frac{C}{(1+u^{\gamma})^a}.$$

Furthermore, inequality (2.27) and hypothesis (C2) yield

$$Q_h^W(g)(x)\geq -hC(1+u^{\gamma}+v^{\gamma})+C(1+u^{\gamma})$$

for all $x\in B_h\backslash\Omega_h$ and $0<h<h_1$. Now, assumption (2.22) reads: for all $x_1\in\mathbb{R}^{d-1}$ such that $\|x_1\|>R$,

$$v\leq Cu.$$

Since both u and v are bounded functions of x on the set $\{(x_1,x_2)\in B_h\backslash\Omega_h,\ \|x_1\|\leq R\}$, there exists $h_2>0$ such that, for all $0<h<h_2$,

$$Q_h^W(g)(x)\geq\begin{cases}C>0 & \text{whenever } \|x_1\|\leq R\\ C(1+u^{\gamma}) & \text{whenever } \|x_1\|>R\end{cases}$$

Thus, for all $0<h<h_2$ and all $x\in B_h\backslash\Omega_h$,

$$Q_h^W(g)^{-a}(x)\leq\frac{C}{(1+u^{\gamma})^a}.$$

Finally, invoking inequality (2.25) together with assumption (2.22) yields, for any $0<h<h_2$ and $x=(x_1,x_2)\in B_h\backslash\Omega_h$,

$$\begin{aligned}\frac{\left|Q_h^W(g)^{1-a}(x)-g^{1-a}(p(x))\right|}{h} &\leq C\frac{\left|Q_h^W(g)(x)-g(p(x))\right|}{h}\left(Q_h^W(g)^{-a}(x)+g^{-a}(p(x))\right)\\ &\leq C(1+u^{\gamma}+v^{\gamma})\frac{1}{(1+u^{\gamma})^a}\\ &\leq C\frac{1}{1+u^{(a-1)\gamma}}.\end{aligned}$$

Note that $u \geq \|x_1\|$, and we chose a such that $(a-1)\gamma > n$, hence $q(a-1)\gamma - q > q(n-1)$, thus the dominated convergence theorem applies, and we may conclude that

$$\lim_{h\to 0}\int_{B_h\setminus\Omega_h}\frac{1}{h}Q_h^W(g)^{1-a} = \lim_{h\to 0}\int_{\mathbb{R}^{d-1}}\left(\frac{1}{h}\int_{h+(1+h)\varphi(x_1/(1+h))}^{h+\varphi(x_1)} g^{1-a}(x_1,\varphi(x_1))dx_2\right)dx_1$$
$$= \int_{\mathbb{R}^{d-1}}(x_1\cdot\nabla\varphi(x_1)-\varphi(x_1))g^{1-a}(x_1,\varphi(x_1))dx_1,$$

this last equality also being a dominated convergence result, using the hypotheses on g.

2.4.2 Extending the differentiated inequality

We just proved that whenever (g, W) is admissible, with $\int_\Omega g^{-a} = \int_{\Omega_1} W^{-a} = 1$, and φ satisfies the asymptotic growth condition (2.22), then

$$(a-d)\int_\Omega g^{1-a} + (a-1)\int_\Omega \frac{W^*(\nabla g)}{g^a} - \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1,\varphi(x_1))P(x_1)dx_1 \geq \int_{\Omega_1} W^{1-a}. \quad (2.30)$$

Let $q > 1$. We want to use this inequality with $W(x) = C\|x\|^q/q$, where $C > 0$ is such that $\int_{\Omega_1} W^{-a} = 1$. The goal being to prove Sobolev-type inequalities, we may consider only the real q such that their conjugate exponent $p = q/(q-1)$, which will appear in W^*, is strictly less than n. Thus, we assume that $q > n/(n-1)$, and conditions (C0), (C1) and (C2) are automatically satisfied with $\gamma = q$.

We now compute W^*:

$$\begin{aligned} W^*(y) = \sup_{x\in\Omega_1}\{x\cdot y - C\|x\|^q/q\} &\leq \sup_{x\in\mathbb{R}^d}\{x\cdot y - C\|x\|^q/q\} \\ &= \sup_{R\geq 0}\sup_{\|x\|=R}\{x\cdot y - C\|x\|^q/q\} \\ &= \sup_{R\geq 0}\{R\|y\|_* - CR^q/q\} \\ &= C^{1-p}\|y\|_*^p/p. \end{aligned} \quad (2.31)$$

It is important to note that (2.31) becomes an equality for $y = \nabla g(z)$ whenever $g(\,.\,) = W(\,.+e)$, since in that case,

$$W^*(\nabla g(z)) = \sup_{x\in\Omega_1}\{x\cdot\nabla g(z) - W(x)\} = \sup_{x\in\Omega}\{(x+e)\cdot\nabla g(z) - g(x)\} = e\cdot\nabla g(z) + g^*(\nabla g(z))$$

and the supremum is indeed reached inside the right set. Optimality is not lost, and inequality (2.30) then becomes

$$(a-d)\int_\Omega g^{1-a} + C^{1-p}\left(\frac{a-1}{p}\right)\int_\Omega\frac{\|\nabla g\|^p}{g^a} - \int_{\mathbb{R}^{d-1}} g^{1-a}(x_1,\varphi(x_1))P(x_1)dx_1 \geq \int_{\Omega_1} W^{1-a}. \quad (2.32)$$

The next step is to lift the restrictions on the function g, extending the results to more general functions. Our tool here will be approximation by admissible functions.

$f = g^{(p-a)/p}$ is a smooth function with compact support

Let $f \in C_c^\infty(\Omega)$ be a nonnegative function such that $\int_\Omega f^{ap/(a-p)} = 1$. Let us fix some $\gamma > \max\{1, a/(n-1)\}$ and consider, for $\varepsilon > 0$,

$$f_\varepsilon(x) = \left(\varepsilon\|x+e\|^{-\gamma(a-p)/p} + C_\varepsilon f\right),$$

where C_ε is such that $\int_\Omega f_\varepsilon^{ap/(a-p)} = 1$, whenever ε is small enough for C_ε to exist. It is not difficult to see that the corresponding functions $g_\varepsilon = f_\varepsilon^{p/(p-a)}$ satisfy conditions (C3) and (C4), and that $\int_\Omega g_\varepsilon^{-a} = 1$. Furthermore, C_ε increases strictly as ε decreases towards 0, and an argument of continuity shows that $\lim_{\varepsilon\to 0} C_\varepsilon = 1$, meaning that, pointwise, $\lim_{\varepsilon\to 0} g_\varepsilon = f^{(p-a)/p} =: g$. Finally, the dominated convergence theorem, applied to $g_\varepsilon^{1-a} = f_\varepsilon^{(a-1)p/(a-p)}$, proves that inequality (2.32) is indeed valid for g. Rewriting it with f yields

$$(a-d)\int_\Omega f^{p\frac{a-1}{a-p}} + C^{1-p}\left(\frac{a-1}{p}\right)\left(\frac{p}{a-p}\right)^p\int_\Omega \|\nabla f\|^p - \int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi(x_1))P(x_1)dx_1 \geq \int_{\Omega_1} W^{1-a}. \tag{2.33}$$

Ω is the epigraph of a general convex function

Finally, we may lift the growth condition on φ (2.22) and prove theorem 2.4 in its full generality.

Let φ be a convex function with $\varphi(0) = 0$, and let Ω be its epigraph. The subdifferential of φ at point $x \in \mathbb{R}^{d-1}$ is the convex set

$$\partial\varphi(x) = \{v \in \mathbb{R}^{d-1} \mid \forall x' \in \mathbb{R}^{d-1},\ \varphi(x) - \varphi(x') \geq v\cdot(x-x')\}.$$

Whenever φ is differentiable, the subdifferential coincides with the gradient. Next, given $x \in \mathbb{R}^{d-1}$ and $v \in \partial\varphi(x)$, we consider the tangent half-space

$$H_{x,v} = \{(y_1,y_2) \in \mathbb{R}^{d-1}\times\mathbb{R},\ y_2 - \varphi(x) \geq v\cdot(y_1 - x)\}.$$

For $R > 0$, define

$$\Omega_R = \bigcap_{\substack{x\in B(0,R)\\ v\in\partial\varphi(x)}} H_{x,v}.$$

Ω_R is the epigraph of a convex function φ_R that coincides with the function φ on the ball $B(0,R) \in \mathbb{R}^{d-1}$, and its gradient is uniformly bounded by $\sup_{x\in B(0,R)}|\nabla\varphi(x)| < +\infty$, so that it verifies the condition (2.22). Now, fix a function $f \in \mathcal{C}_c^\infty(\Omega)$. The support of f is inside a ball of radius R_0, so for any $R \geq R_0$, we may apply inequality (2.33) to the function f:

$$\begin{aligned}&(a-d)\int_{\Omega_R} f^{p\frac{a-1}{a-p}} + A_R\int_{\Omega_R}\|\nabla f\|^p - \int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi_R(x_1))P_R(x_1)dx_1\\ =&(a-d)\int_\Omega f^{p\frac{a-1}{a-p}} + A_R\int_\Omega\|\nabla f\|^p - \int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi(x_1))P(x_1)dx_1 \geq \int_{\Omega_{R,1}} W_R^{1-a}.\end{aligned}$$

where P_R, $\Omega_{R,1}$, and W_R are the usual definition of P, Ω, and W respectively, with function φ_R instead of φ. The constants are given by $A_R = C_R^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p$ and $C_R > 0$ is such that $\int_{\Omega_{R,1}} W_R^{-a} = 1$, i.e.

$$C_R = q^a\int_{\Omega_{R,1}}\|x\|^{-qa}dx.$$

It is now easy to verify that $\lim_{R\to+\infty} C_R = q^a\int_{\Omega_1}\|x\|^{-qa}dx = C$; A_R, and $\int_{\Omega_{1,R}} W_R^{1-a}$ also converge towards the right constants, so that equation (2.33) is still valid for the function φ, without any growth condition.

Optimality remains to be shown. Let f be the optimal function (which does not have compact support) given in theorem 2.19. First, note that

$$\lim_{R\to+\infty}\int_{\Omega_R} f^{p\frac{a-1}{a-p}} = \int_{\Omega} f^{p\frac{a-1}{a-p}}, \quad \lim_{R\to+\infty}\int_{\Omega_R}\|\nabla f\|^p = \int_{\Omega}\|\nabla f\|^p,$$

and also that for all R, f is an optimal function for inequality (2.33) on domain Ω_R. Then, by approximation by smooth functions with compact support, inequality (2.33) is true for f, so that, writing $A = \lim_{R\to+\infty} A_R$, and putting these facts together,

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi(x_1))P(x_1)dx_1 &\leq (a-d)\int_{\Omega} f^{p\frac{a-1}{a-p}} + A\int_{\Omega}\|\nabla f\|^p - \int_{\Omega_R} W^{1-a}\\ &= \lim_{R\to+\infty}\left((a-d)\int_{\Omega_R} f^{p\frac{a-1}{a-p}} + A_R\int_{\Omega_R}\|\nabla f\|^p - \int_{\Omega_{R,1}} W_R^{1-a}\right)\\ &= \lim_{R\to+\infty}\left(\int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi_R(x_1))P_R(x_1)dx_1\right).\end{aligned}$$

We can decompose that last integral as a sum of the two following integrals

$$\int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi_R(x_1))dx_1 - \int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi_R(x_1))\left(x_1\cdot\nabla\varphi_R(x_1) - \varphi_R(x_1)\right)dx_1.$$

By monotone convergence, the first term converges to the integral of the pointwise limit of its integrand. Furthermore, by convexity, for all $x_1 \in \mathbb{R}^{d-1}$, $x_1\cdot\nabla\varphi_R(x_1) - \varphi_R(x_1) \geq 0$, so Fatou's lemma applied to the second term yields

$$\lim_{R\to+\infty}\left(\int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi_R(x_1))P_R(x_1)dx_1\right) \leq \int_{\mathbb{R}^{d-1}} f^{p\frac{a-1}{a-p}}(x_1,\varphi(x_1))P(x_1)dx_1,$$

which finishes to prove equality in the previous inequalities, whence optimality.

Chapter 3

A family of Beckner inequalities under various curvature-dimension conditions

This chapter is a collaborative work with Ivan Gentil.

Abstract

In this chapter, we offer a proof for a family of functional inequalities interpolating between the Poincaré and the logarithmic Sobolev (standard and weighted) inequalities. The proofs rely both on entropy flows and on a $CD(\rho, n)$ condition, either with $\rho = 0$ and $n > 0$, or with $\rho > 0$ and $n \in \mathbb{R}$. As such, results are valid in the case of a Riemannian manifold, which constitutes a generalization to what was proved in [BGS18, Ngu18].

Contents

3.1 Introduction

The family of the Beckner inequalities interpolate between the Poincaré and the logarithmic Sobolev inequalities. For instance, let $d\mu = e^{-V}dx$ be a probability measure on $\mathbb{R}^d$, where $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth function satisfying $\nabla^2 V \geq \rho\,\mathrm{Id}$ for some $\rho > 0$. Then, Bakry-Émery's curvature-dimension condition implies that, for any $p \in (1, 2]$, and any nonnegative smooth function f,

$$\frac{p}{p-1}\left[\int f^2 d\mu - \left(\int f^{2/p} d\mu\right)^p\right] \leq \frac{2}{\rho}\int |\nabla f|^2 d\mu, \tag{3.1}$$

these results can be found in [BGL14, Sec. 7.6.2]. When $p = 2$, that is the usual Poincaré inequality for the measure μ and when $p \to 1$ the inequality becomes the logarithmic Sobolev inequality,

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu} d\mu \leq \frac{2}{\rho} \int |\nabla f|^2 d\mu,$$

both optimal when μ is the standard Gaussian measure.

Similar inequalities have first been proved by Bidaut-Véron-Véron in [BVV91] for the sphere, using the method proposed in [GS81]. In this chapter, we refer to them as Beckner inequalities, in reference to [Bec89], where Beckner proves inequallity (3.1) for the Gaussian measure. They are sometimes called convex inequalities.

These inequalities play a major role among functional inequalities for probability measures, being useful to understand both asymptotic behaviour of parabolic equations, and also the geometry of measured spaces. It is interesting to notice that for the Gaussian measure, although the Poincaré and the logarithmic Sobolev inequalities are optimal and have nonconstant extremal functions, whenever $p \in (1,2)$, Beckner inequalities do not have extremal functions, and even admit various improvements (see for instance [AD05, BG10, DEKL13]).

Other attractive examples appear for measures which satisfy a (weighted) Poincaré inequality but no logarithmic Sobolev inequality. More precisely, let $\varphi : \mathbb{R}^d \mapsto \mathbb{R}_+^*$ be a smooth and positive function such that for any $\beta > d/2$, $\int \varphi^{-\beta} dx < +\infty$. Now, define

$$d\mu_\beta = c_\beta \varphi^{-\beta} dx, \tag{3.2}$$

where c_β is a normalization constant such that μ_β is a probability measure on $\mathbb{R}^d$. In [BGS18], the authors prove that, when $\varphi = 1 + |x|^2$ then for any $\beta \geq d+1$ and $p \in [p^*, 2]$.

$$\frac{p}{p-1}\left[\int f^2 d\mu_\beta - \left(\int f^{2/p} d\mu_\beta\right)^p\right] \leq \frac{1}{\beta - 1}\int |\nabla f|^2 \varphi d\mu_\beta, \tag{3.3}$$

where $p^* = 1 + 1/(\beta - d)$. This inequality is rich enough to be equivalent to the Sobolev inequality on the sphere. This result has then been extended by N'Guyen in [Ngu18], where the author proves similar inequalities in $\mathbb{R}^d$ with a function φ satisfying the convex assumption $\nabla^2\varphi \geq c\,\mathrm{Id}$ for some constant $c > 0$. The only difference with inequality (3.3) is that the constant $(\beta-1)^{-1}$, in front of the right hand side, becomes $2(c(\beta-1))^{-1}$, which is consistant with previous results, since $c = 2$ for $\varphi : x \mapsto 1 + |x|^2$. In that regard, N'Guyen's result is more general, but there exists a limitation: the range of the parameter p for which the inequality remains valid is strictly smaller.

We would like to extend Beckner's inequality, and more precisely inequality (3.3), in the general context of curvature-dimension conditions. The goal is twofold. First, we extend some results of [BGS18, Ngu18] in the context of a Riemannian manifold. Then, under a curvature-dimension condition $CD(\rho, n)$ with n *negative*, we also prove Beckner inequalities. We recover for instance the weighted Poincaré inequality for generalized Cauchy distributions.

More precisely, we prove functional inequalities under two kinds of assumptions, the curvature-dimension conditions $CD(0, n)$ with $n > 0$ or $CD(\rho, n)$ with $\rho > 0$ and $n \in \mathbb{R}$. Let us give here a flavour of our results.

- In Theorem 3.6, we prove the following Poincaré inequality under $CD(0, n)$, $n > 0$. Let (M, g) be a smooth d-dimensional Riemannian manifold with a nonnegative Ricci curvature and let φ be a positive function such that $\mathrm{Hess}(\varphi) \geq cg$, $c > 0$. Then for any function f and $\beta \geq n+1$,

$$\mathrm{Var}_{\mu_{\varphi,\beta}}(f) \leq \frac{1}{c(\beta-1)} \int \Gamma(f)\varphi d\mu_{\varphi,\beta}, \tag{3.4}$$

where $\mu_{\varphi,\beta} = Z_{\varphi,\beta}\varphi^{-\beta}dx$ is a probability measure. This inequality generalizes previous results on the subject.

- In Theorem 3.16, we prove a family of Beckner inequalities under $CD(\rho, n)$ with $\rho > 0$ and $n \in \mathbb{R}$. On a d-dimensional Riemannian manifold, for any nonnegative function f,

$$\frac{p}{p-1}\left(\int f^2 d\mu - \left(\int f^{2/p} d\mu\right)^p\right) \leq 2\frac{n-1}{\rho n}\int \Gamma(f) d\mu.$$

  - for all $p \in (1, 2]$ if $n \geq d$,
  - for all $p \in [p^*, 2]$ if $n < -2$, where $p^* = 1 + \frac{1-4n}{2n^2+1}$.

  In this context, μ is the reversible measure and Γ is the carré du champ operator. The use of a negative dimension is new, up to our knowledge.

The chapter is organized as follows. In the next section, we state various definitions useful for the rest of the chapter. In Section 3.3 we prove weighted Beckner inequalities, like inequality (3.4) under $CD(0, n)$ conditions, $n > 0$. The Φ-entropy inequalities are also studied in this context. Section 3.4 is devoted to Beckner's inequality under the curvature-dimension condition $CD(\rho, n)$ conditions with $\rho > 0$ and $n \in \mathbb{R}$. Finally in Section 3.5, we apply our methods in the one dimensional case, giving another way to prove optimal weighted Poincaré inequality on $\mathbb{R}$.

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3.2 Settings and definitions

Consider a connected, C^∞ Riemannian manifold (M, g) of dimension d and the Laplace-Beltrami operator Δ_g given by, in a local chart

$$\Delta_g f = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right),$$

where g^{ij} are the components of the inverse metric tensor, g^{-1}. In this formula and in what follows, the Einstein notation was used, where the summation on indices is implied.

On this manifold, define the symmetric diffusion operator $L = \Delta_g + \Gamma^{\Delta_g}(V, .)$, where Γ^{Δ_g} is the carré du champ operator (the definition of which is recalled below) associated to Δ_g, and V is a C^∞ function. Most of the notions and results related to this operator Γ can be found in the quite thorough [BGL14].

Definition 3.1 (Carré du champ operator)**.** Given a differential operator L on a smooth manifold M, the carré du champ operator Γ^L is a symmetric bilinear map from $C^\infty(M) \times C^\infty(M)$ onto $C^\infty(M)$. It is defined by

$$\Gamma^L(a, b) = \frac{1}{2}(L(ab) - aLb - bLa).$$

An important example to keep in mind is the manifold $(\mathbb{R}^d, I_d)$, on which the Laplace-Beltrami operator is the usual Laplacian, and its carré du champ operator is, for any smooth functions a and b,

$$\Gamma^\Delta(a, b) = \nabla a \cdot \nabla b.$$

More generally, respectively using the Levici-Vita connection ∇, and in a local chart, operator Γ^{Δ_g} is given by

$$\Gamma^{\Delta_g}(a,b) = g(\nabla a, \nabla b) = g^{ij}\partial_i a \partial_j b.$$

We iterate this definition to get the second order operator, Γ_2.

Definition 3.2 (Iterated carré du champ operator)**.** Given a differential operator L on a smooth manifold M, the iterated carré du champ operator Γ_2^L is a symmetric bilinear map from $C^\infty(M)\times C^\infty(M)$ onto $C^\infty(M)$ defined by

$$\Gamma_2^L(a,b) = \frac{1}{2}\big(L(\Gamma^L(a,b)) - \Gamma^L(a, Lb) - \Gamma^L(b, La)\big).$$

We point out that if $L = \Delta_g + \Gamma^{\Delta_g}(V,.)$, then $\Gamma^L = \Gamma^{\Delta_g}$: the drift part of L, namely V, does not appear in this operator. For this reason, and for readability's sake, we shall simply use the notation Γ instead of Γ^{Δ_g} in what follows, as long as it is not ambiguous. We will also write $\Gamma(a,a) = \Gamma(a)$ for brievety. We will do the same for Γ_2, but one should keep in mind that the drift *does* play a role in Γ_2, and $\Gamma_2^L \neq \Gamma_2^{\Delta_g}$. The idea behind the carré du champ operator is that it contains all the information about the geometry of the space (M,g), and it thus proves worthwhile to stick to its use in the (sometimes heavy) calculations. Generally, a formula that is valid for the standard Laplacian on $\mathbb{R}^d$ involving only the Γ operator will remain valid on more general manifolds. For instance, if a, b, and c are smooth functions defined on $\mathbb{R}^d$, the Hessian of a may be expressed in the following fashion

$$\mathrm{Hess}(a)(\nabla b, \nabla c) = \frac{1}{2}(\nabla b \cdot \nabla(\nabla a \cdot \nabla c) + \nabla c \cdot \nabla(\nabla a \cdot \nabla b) - \nabla a \cdot \nabla(\nabla b \cdot \nabla c)),$$

hence the following lemma

Lemma 3.3 (Hessian of a function)**.** *Let a, b, c be smooth functions on (M,g). Then, the Hessian of a is given by*

$$\nabla^2 a(\nabla b, \nabla c) = \frac{1}{2}(\Gamma(b, \Gamma(a,c)) + \Gamma(c, \Gamma(a,b)) - \Gamma(a, \Gamma(b,c))),$$

where Γ is the carré du champ operator associated to the Laplace-Beltrami operator Δ_g.

Proof. Since the Hessian is symmetric and bilinear, it is sufficient to prove that

$$\nabla^2 a(\nabla b, \nabla b) = \Gamma(b, \Gamma(a,b)) - \frac{1}{2}\Gamma(a, \Gamma(b)).$$

By definition,

$$\begin{aligned}
\mathrm{Hess}(a)(\nabla b, \nabla b) &= g(\nabla_{\nabla b}\nabla a, \nabla b) \\
&= \nabla_{\nabla b} g(\nabla a, \nabla b) - g(\nabla a, \nabla_{\nabla b}\nabla b) \\
&= g(\nabla b, \nabla g(\nabla a, \nabla b)) - g(\nabla a, g(\nabla\nabla b, \nabla b)) \\
&= g(\nabla b, \nabla g(\nabla a, \nabla b)) - \frac{1}{2} g(\nabla a, \nabla g(\nabla b, \nabla b))
\end{aligned}$$

which is the claimed formula. □

In the development of this chapter, we shall assume so-called curvature-dimension conditions on the different diffusion operators:

Definition 3.4 (Curvature-dimension conditions $CD(\rho, n)$)**.** A diffusion operator L is said to satisfy a $CD(\rho, n)$ condition for $\rho \in \mathbb{R}$ and $n \neq 0$, if for every smooth function f,

$$\Gamma_2(f) \geq \rho\Gamma(f) + \frac{1}{n}(Lf)^2. \tag{3.5}$$

For example, the Bochner-Lichnerowicz formula implies that the Laplace-Beltrami operator Δ_g satisfies the $CD(\rho, n)$ condition with $n \geq d$ and $\rho \in \mathbb{R}$ whenever the Ricci curvature is uniformly bounded form below by ρg, [BGL14, Sec. C.6].

Denoting by dx the Riemannian measure associated to (M, g), we define μ to be the reversible measure associated to L, i.e. $d\mu = Z_V e^{-V} dx$, where Z_V a normalizing constant such that μ is a probability measure if finite, and $Z_V = 1$ otherwise. The triple (M, Γ, μ) is a Markov triple, as defined in [BGL14, Sec. 3.2]. Then, for any smooth functions a, b such that the integrals are well defined, we have the following integration by parts formula:

$$\int_M \Gamma(a, b)d\mu = -\int_M aLbd\mu = -\int_M bLad\mu,$$

a notable consequence of which is that $\int_M Lad\mu = 0$.

Remark 3.5. We could have chosen to study a general symmetric Markov semigroup, for instance a generic Markovian triple (E, Γ, μ) as proposed in [BGL14]. However, assuming L is a symmetric diffusion operator in a smooth Riemannian manifold, which is the case of interest for us, there exists a metric $\tilde{g}$ and a function V such that the operator can be rewritten $L = \Delta_{\tilde{g}} + \Gamma^{\Delta_{\tilde{g}}}(V, .)$, so we are not, in fact, losing any generality.

3.3 Weighted inequalities under nonnegative Ricci curvature

For a more pedestrian approach, we first explain the case of the Poincaré inequality, only to tackle more general inequalities later on.

3.3.1 Weighted Poincaré inequality

Let φ be a C^2 positive function on M such that

$$\nabla^2\varphi \geq cg \tag{3.6}$$

for some positive constant c. For $\beta \in \mathbb{R}$ such that $\varphi^{-\beta}$ is integrable with respect to μ, let $\mu_{\varphi,\beta} = Z_{\varphi,\beta}\varphi^{-\beta}\mu$, with the constant $Z_{\varphi,\beta}$ such that this new measure is a probability measure. The main result is the following:

Theorem 3.6 (Weighted Poincaré inequality)**.** *Assume that the diffusion operator L satisfies a $CD(0, n)$ condition with $n \geq d$, and fix a real number $\beta \geq n + 1$. Then for all smooth bounded functions f,*

$$\mathrm{Var}_{\mu_{\varphi,\beta}}(f) = \int f^2 d\mu_{\varphi,\beta} - \left(\int f d\mu_{\varphi,\beta}\right)^2 \leq \frac{1}{c(\beta - 1)} \int \Gamma(f)\varphi d\mu_{\varphi,\beta}. \tag{3.7}$$

Remark 3.7. As explained in Section 3.2, the main example to keep in mind is the Laplace-Beltrami operator on a Riemannian manifold with a nonnegative Ricci curvature, in which case $\mu_{\varphi,\beta} = Z_{\varphi,\beta}\varphi^{-\beta}dx$.

This inequality happens to be optimal whenever $(M, g) = (\mathbb{R}^d, \mathrm{Id})$, and $\varphi(x) = 1 + |x|^2$, where the optimal constant is reached for projectors $x \mapsto x_i$, $1 \leq i \leq d$. This optimal case has been proved in [BBD+07] and also in [Ngu14, ABJ18] with different methods.

Proof. Fix $\beta \in \mathbb{R}\backslash\{2\}$, and define

$$\overline{L} := \varphi L - (\beta-1)\Gamma(\varphi, .).$$

Note that the operator $\overline{L}$ is, in fact, of the form $\overline{L} = \Delta_{\overline{g}} + \Gamma^{\Delta_{\overline{g}}}(\overline{V}, .)$, where $\overline{g} = \varphi^{-1}g$ and $\overline{V} = V + (d/2 - \beta)\log\varphi$, so the operator we are considering here is obtained from the first one through a conformal transformation. In what follows, everything written with an overline relates to objects associated to the operator $\overline{L}$ or the manifold $(M, \overline{g})$. For instance, the carré du champ operator is given by $\overline{\Gamma} = \varphi\Gamma$, its reversible measure is $\bar{\mu} = \mu_{\varphi,\beta}$, and

$$\overline{\Gamma}_2(f) = \Gamma_2^{\overline{L}}(f) = \varphi^2\Gamma_2(f) + (\beta-1)\varphi\nabla^2\varphi(\nabla f, \nabla f) + \frac{\Gamma(f)}{2}(\varphi L\varphi - (\beta-1)\Gamma(\varphi)) \\ + \varphi\Gamma(\varphi, \Gamma(f)) - \varphi Lf\Gamma(f, \varphi), \quad (3.8)$$

a proof of which can be stringed together with information from [BGL14, Sec. 6.9.2], for instance.

Now, fix f, a smooth and *bounded* function on M, and consider the Markov semigroup $(f_t)_{t\geq 0}$, solution of the initial-value system

$$\begin{cases} \partial_t f_t = \overline{L} f_t & \text{on } (0, +\infty) \times M, \\ f_0 = f & \text{on } M. \end{cases} \quad (3.9)$$

Consider the variance of f_t along the flow:

$$\Lambda(t) := \mathrm{Var}_{\bar{\mu}}(f_t) = \int f_t^2 d\bar{\mu} - \left(\int f_t d\bar{\mu}\right)^2 = \int f_t^2 d\bar{\mu} - \left(\int f d\bar{\mu}\right)^2,$$

because $\overline{L}$ is mass-preserving. Then, we use the following estimate:

Lemma 3.8. *If $\beta > 1$ and $\beta \neq 2$, then for all $t \geq 0$,*

$$\Lambda''(t) \geq -2c(\beta-1)\Lambda'(t) + \frac{4}{\beta-2}\int \varphi^2\left[(\beta-1)\Gamma_2(f_t) - (Lf_t)^2\right] d\bar{\mu}. \quad (3.10)$$

Furthermore, there is equality in (3.10) for all $t \geq 0$ whenever the Hessian of φ is a constant, i.e. when inequality (3.6) is an equality.

Assume that L satisfies the $CD(0, n)$ condition for some $n > 0$, and that $\beta - 1 \geq n$, and $\beta > 2$. Then, we deduce from equation (3.10) that

$$\Lambda''(t) \geq -2c(\beta-1)\Lambda'(t)$$

which we can integrate once betwen 0 and t to find that

$$-\Lambda'(t) \leq -\Lambda'(0)e^{-2c(\beta-1)t}$$

and then once again, between $t = 0$ and $t = +\infty$,

$$\Lambda(0) - \lim_{t\to+\infty}\Lambda(t) \leq \frac{-1}{2c(\beta-1)}\Lambda'(0).$$

The Markov semigroup studied is ergodic, in other words, it ensures the convergence of f_t towards its mean in $L^2(\bar{\mu})$, so that $\lim_{t\to+\infty}\Lambda(t) = 0$, and the previous inequality is simply the stated result, for smooth and bounded functions, and for $\beta > 2$. The general case is established by approximation. The particular case $\beta = 2$ (also implying $n = d = 1$) is proved by letting $\beta \to 2$ directly in inequality (3.7). $\square$

Proof of Lemma 3.8. By definition of $\bar{\mu}$, we may integrate by parts the derivative of Λ to find

$$\Lambda'(t) = 2\int f_t \overline{L} f_t d\bar{\mu} = -2\int \overline{\Gamma}(f_t)d\bar{\mu},$$

and, differentiating once again,

$$\Lambda''(t) = -4\int \overline{\Gamma}(f_t, \overline{L}f_t)d\bar{\mu} = -4\int \overline{\Gamma}(f_t, \overline{L}f_t)d\bar{\mu} + 2\int \overline{L}(\overline{\Gamma}(f_t)) = 4\int \overline{\Gamma}_2(f_t)d\bar{\mu}.$$

We may now use formula (3.8) to find that

$$\begin{aligned}\Lambda''(t) &= 4\int (\beta-1)\varphi\nabla^2\varphi(\nabla f_t, \nabla f_t)d\bar{\mu} \\ &+ 4\int \left[\varphi^2\Gamma_2(f_t) + \frac{\Gamma(f_t)}{2}(\varphi L\varphi - (\beta-1)\Gamma(\varphi)) + \varphi\Gamma(\varphi, \Gamma(f_t)) - \varphi L f_t \Gamma(f_t, \varphi)\right]d\bar{\mu}. \quad (3.11)\end{aligned}$$

First, the convexity assumption (3.6) on φ yields

$$4\int (\beta-1)\varphi\nabla^2\varphi(\nabla f_t, \nabla f_t) \geq 4c(\beta-1)\int \overline{\Gamma}(f_t)d\bar{\mu} = -2c(\beta-1)\Lambda'(t).$$

Now, in the second integral of equation (3.11), we may not directly use the integration by parts formula, because $\bar{\mu}$ is the invariant measure for $\overline{\Gamma}$, and not for Γ. We must thus rewrite it in terms of μ. First,

$$\begin{aligned}\int \frac{\Gamma(f_t)}{2}\varphi L\varphi d\bar{\mu} = Z_{\varphi,\beta}\int \frac{\Gamma(f_t)}{2}\varphi^{1-\beta}L\varphi d\mu &= -Z_{\varphi,\beta}\int \Gamma\left(\varphi, \varphi^{1-\beta}\frac{\Gamma(f_t)}{2}\right)d\mu \\ &= -\int \frac{\varphi}{2}\Gamma(\varphi, \Gamma(f_t))d\bar{\mu} - \frac{1-\beta}{2}\int \Gamma(f_t)\Gamma(\varphi)d\bar{\mu}.\end{aligned}$$

Then,

$$\begin{aligned}\int \varphi\Gamma(\varphi, \Gamma(f_t))d\bar{\mu} = Z_{\varphi,\beta}\int \varphi^{1-\beta}\Gamma(\varphi, \Gamma(f_t))d\mu = \frac{Z_{\varphi,\beta}}{2-\beta}\int \Gamma(\varphi^{2-\beta}, \Gamma(f_t))d\mu \\ = \frac{1}{\beta-2}\int \varphi^2 L\Gamma(f_t)d\bar{\mu},\end{aligned}$$

and likewise,

$$\begin{aligned}&-\int \varphi L f_t \Gamma(f_t, \varphi)d\bar{\mu} = \frac{Z_{\varphi,\beta}}{\beta-2}\int L f_t \Gamma(f_t, \varphi^{2-\beta})d\mu = \\ &\frac{Z_{\varphi,\beta}}{\beta-2}\int \left[\Gamma(f_t, \varphi^{2-\beta}L f_t) - \varphi^{2-\beta}\Gamma(f_t, L f_t)\right]d\mu = \frac{-1}{\beta-2}\int \varphi^2\left[(L f_t)^2 + \Gamma(f_t, L f_t)\right]d\bar{\mu}.\end{aligned}$$

We conclude putting these three identities together. □

3.3.2 Φ-entropy and weighted Beckner inequalities

Instead of the variance, we may consider a generic Φ-entropy along the flow. Choose a strictly convex real function $\Phi \in C^4(I)$, (where $I \subset \mathbb{R}_+^*$ is an open interval) and consider the Φ-entropy of any function $f : M \mapsto I$ such that the integrals below are well defined

$$\mathrm{Ent}^{\Phi}_{\mu_{\varphi,\beta}}(f) := \int \Phi(f)d\mu_{\varphi,\beta} - \Phi\left(\int f d\mu_{\varphi,\beta}\right),$$

so that $\mathrm{Var}_{\mu_{\varphi,\beta}} = \mathrm{Ent}^{x\mapsto x^2}_{\mu_{\varphi,\beta}}$. Generalizations of inequalities like (3.7) to Φ-entropies have already been studied under $CD(\rho,\infty)$ condition, $\rho > 0$ in [Cha04, BG10]. We find a generalized version of Theorem 3.6:

Theorem 3.9 (Φ-entropy inequalities)**.** *Assume that the diffusion operator L satisfies a $CD(0,n)$ condition with $n \geq 0$, and fix a real number $\beta \geq n+1$. Let $\Phi : I \mapsto \mathbb{R}$ be a strictly convex function such that*

$$\Phi^{(4)}\Phi'' \geq C_{n,\beta}(\Phi^{(3)})^2, \tag{3.12}$$

with

$$C_{n,\beta} = \frac{8(\beta-1-n)(2\beta-1)+9n}{8(\beta-1-n)(\beta-1)}.$$

Then, for all smooth bounded functions $f : M \mapsto I$, there holds

$$\mathrm{Ent}^{\Phi}_{\bar\mu}(f) \leq \frac{1}{2c(\beta-1)}\int \Phi''(f)\Gamma(f)\varphi d\bar\mu. \tag{3.13}$$

Proof. Let us first assume that $\beta > 2$, the case $\beta = 2$ can be proved by passing to the limit in (3.13). Assume also that f is a smooth and bounded function, the general case can be proved by approximations. Consider, just like before, the function f_t, solution of the initial-value system (3.9) starting from f, and define, for $t \geq 0$,

$$\Lambda(t) = \mathrm{Ent}^{\Phi}_{\bar\mu}(f_t) = \int \Phi(f_t)d\bar\mu - \Phi\left(\int f_t d\bar\mu\right) = \int \Phi(f_t)d\bar\mu - \Phi\left(\int f d\bar\mu\right),$$

where, again, $\bar\mu = \mu_{\varphi,\beta}$. Differentiating the entropy yields

$$\Lambda'(t) = -\int \frac{\overline{\Gamma}(\Phi'(f_t))}{\Phi''(f_t)}d\bar\mu,$$

and, from [BG10, Lem. 4],

$$\Lambda''(t) = \int \left[2\frac{\overline{\Gamma}_2(\Phi'(f_t))}{\Phi''(f_t)} + \left(\frac{-1}{\Phi''}\right)''(f_t)\left(\frac{\overline{\Gamma}(\Phi'(f_t))}{\Phi''(f_t)}\right)^2\right]d\bar\mu.$$

We follow exactly the same steps as in Section 3.3.1. For brievety, we write $h \coloneqq \Phi'(f_t)$. First, expanding $\overline{\Gamma}_2$ in terms of Γ_2, Γ and L, and also using the convexity hypothesis on φ (3.6), we find,

$$\begin{aligned}\Lambda''(t) \geq -2c(\beta-1)\Lambda'(t) + \int \Bigg[\frac{2}{\Phi''(f_t)}\Bigg(&\varphi^2\Gamma_2(h) + \frac{\Gamma(h)}{2}(\varphi L\varphi - (\beta-1)\Gamma(\varphi)) + \varphi\Gamma(\varphi,\Gamma(h)) \\ &- \varphi L h\Gamma(h,\varphi)\Bigg) + \varphi^2\left(\frac{-1}{\Phi''}\right)''(f_t)\left(\frac{\Gamma(h)}{\Phi''(f_t)}\right)^2\Bigg]d\bar\mu.\end{aligned}$$

We may now give these terms the same treatment as in the previous section: the goal is to remove all derivatives on φ. The calculations are not made explicit here; they involve the exact same ingredients we used before, only with more terms appearing. We finally find, using only integration by parts, that

$$\Lambda''(t) \geq -2c(\beta-1)\Lambda'(t) + \frac{1}{\beta-2}\int \frac{\varphi^2}{\Phi''(f_t)}\big(a_0\Gamma_2(h) + a_0'(Lh)^2 + a_2\Gamma(h)Lh + a_3\Gamma(h)^2\big)d\bar\mu,$$

where

$$\begin{cases} a_0 = 2(\beta - 1), \\ a_0' = -2, \\ a_2 = 3\dfrac{\Phi'''(f_t)}{(\Phi''(f_t))^2}, \\ a_3 = (\beta - 1)\dfrac{\Phi^{(4)}(f_t)}{(\Phi''(f_t))^3} + (1 - 2\beta)\dfrac{(\Phi'''(f_t))^2}{(\Phi''(f_t))^4}. \end{cases}$$

Just like for the Poincaré inequality proof, this inequality becomes an equality when $\mathrm{Hess}(\varphi) = cg$. Invoke the $CD(0, n)$ condition to find that

$$\Lambda''(t) \geq -2c(\beta - 1)\Lambda'(t) + \frac{1}{\beta - 2}\int \frac{\varphi^2}{\Phi''(f)}\big(a_1(Lh)^2 + a_2\Gamma(h)Lh + a_3\Gamma(h)^2\big)d\bar{\mu}, \tag{3.14}$$

where $a_1 = a_0/n + a_0' = \frac{2}{n}(\beta - 1 - n)$. In the same way as before, we want this integrated quantity to be nonnegative. Since Φ is strictly convex, it is sufficient to require the polynomial function $X \mapsto a_1X^2 + a_2X + a_3$ to be nonnegative, which itself is equivalent to

$$a_1 \geq 0 \quad \text{and} \quad a_2^2 - 4a_1a_3 \leq 0.$$

Straightforward computation yields that this, in turn, is equivalent, whenever $n \geq 0$, to

$$\beta \geq n + 1 \quad \text{and} \quad \Phi^{(4)}\Phi'' \geq C_{\beta,n}(\Phi''')^2,$$

where

$$C_{\beta,n} = \frac{8(\beta - 1 - n)(2\beta - 1) + 9n}{8(\beta - 1 - n)(\beta - 1)}.$$

If this condition is satisfied, the integrand is pointwise nonnegative, thus

$$\Lambda''(t) \geq -2c(\beta - 1)\Lambda'(t),$$

and integrating this twice yields the theorem. □

We can explicit this theorem in the particular case where $\Phi(X) = X^p$, $p > 1$. Condition (3.12) is then equivalent to

$$p \in [p^*, 2],$$

where

$$p^* = 1 + \frac{8(\beta - 1 - n) + 9n}{8\beta(\beta - 1 - n) + 9n} \in (1, 2]. \tag{3.15}$$

Thus, if this condition is satisfied,

$$\int f^p d\bar{\mu} - \left(\int f d\bar{\mu}\right)^p \leq \frac{p(p-1)}{2c(\beta - 1)}\int f^{p-2}\varphi\Gamma(f)d\bar{\mu}.$$

Rewriting this inequality with $\tilde{f} = f^{2/p}$ in place of f, we find

Corollary 3.10 (Weighted Beckner inequalities)**.** *Under the same assumptions as Theorem 3.9, for all $p \in [p^*, 2]$, and all smooth bounded functions f, we have*

$$\frac{p}{p-1}\left[\int f^2 d\mu_{\varphi,\beta} - \left(\int f^{2/p} d\mu_{\varphi,\beta}\right)^p\right] \leq \frac{2}{c(\beta - 1)}\int \varphi\Gamma(f)d\mu_{\varphi,\beta}, \tag{3.16}$$

where p^* is given by (3.15).

Corollary 3.10 is optimal (and thus so is theorem 3.9) in the sense that there exists no constant $0 < C < (c(\beta-1))^{-1}$ such that $\overline{L}$ satisfies a Beckner inequality $B_p(C)$ (see Definition 3.17 below), because $B_p(C)$ (for any $p \in (1,2)$) implies the Poincaré inequality with constant C. Indeed, testing inequality (3.16) with the function $1+\varepsilon f$, f bounded, we find that

$$\frac{p}{p-1}\left[\int(1+\varepsilon f)^2 d\bar{\mu} - \left(\int(1+\varepsilon f)^{2/p} d\bar{\mu}\right)^p\right] \leq \frac{2\varepsilon^2}{c(\beta-1)}\int \varphi\Gamma(f)d\bar{\mu},$$

which, after being expanded when ε is small, turns into

$$\int f^2 d\bar{\mu} - \left(\int f d\bar{\mu}\right)^2 + o(1) \leq \frac{1}{c(\beta-1)}\int \varphi\Gamma(f)d\bar{\mu},$$

which is exactly the optimal weighted Poincaré inequality 3.6.

Remark 3.11. In the case of interest, when $\Phi(X) = X^p$, we may explicit inequality (3.14)

$$\Lambda''(t) \geq -2c(\beta-1)\Lambda'(t) + \frac{1}{\beta-2}\int \varphi^2 h^{\frac{2-p}{p-1}}\left(a_1'(Lh)^2 + a_2'\frac{\Gamma(h)}{h}Lh + a_3'\frac{\Gamma(h)^2}{h^2}\right)d\bar{\mu},$$

where the a_i', $i \in \{1,2,3\}$ are real constants. The inequality can be then improved using the fact that

$$a_1'X^2 + a_2'X + a_3' = a_1'\left(X + \frac{a_2'}{2a_1'}\right)^2 - \frac{(a_2')^2 - 4a_1'a_3'}{4a_1'},$$

so that, whenever $\delta := \frac{-1}{4a_1'}((a_2')^2 - 4'a_1a_3') \geq 0$, we find that

$$\begin{aligned}\Lambda''(t) &\geq -2c(\beta-1)\Lambda'(t) + \frac{\delta}{\beta-2}\int \varphi^2 h^{\frac{2-p}{p-1}}\frac{\Gamma(h)^2}{h^2}d\bar{\mu} \\ &\geq -2c(\beta-1)\Lambda'(t) + \frac{\delta}{\beta-2}\frac{\left(\int \varphi h^{\frac{2-p}{p-1}}\Gamma(h)d\bar{\mu}\right)^2}{\int h^{\frac{p}{p-1}}d\bar{\mu}} \\ &= -2c(\beta-1)\Lambda'(t) + C_p\frac{\Lambda'(t)^2}{\Lambda(t)},\end{aligned}$$

by Jensen's inequality, where, for reference,

$$C_p = \frac{\delta}{\beta-2}(p-1)^2 p^{\frac{2-p}{p-1}} \geq 0.$$

This leads, when integrated, to a refined version of Beckner's inequality, which we will come back to in section 3.4, and specifically, corollary 3.18.

Remark 3.12. As explained in Section 3.1, we extend the result of Nguyen [Ngu18] in two aspects. First, Beckner inequalities (3.16) are proved in the more general context of Riemannian manifolds satisfying a $CD(0,n)$ condition, and secondly, the range of parameter $p \in [p^*, 2]$ given by (3.15) strictly contains the one proposed in [Ngu18].

On the other hand, in [BGS18], corollary 3.10 is proved in the special case of $M = \mathbb{R}^d$ and $\varphi(x) = 1+|x|^2$. Interestingly, the range found in their article is greater than what we can manage here. Indeed, it is valid for all $p \in [p^*_{BGS}, 2]$, where

$$1 < p^*_{BGS} = 1 + \frac{1}{\beta - d} < p^*.$$

It might be worth it to note that in our case, we used the fact that the second degree polynomial appearing in the integral is greater than 0, when this is in fact quite a gross lower bound. Indeed, the argument is that

$$\int \varphi^2 h^{\frac{2-p}{p-1}} \left(\frac{hLh}{\Gamma(h)} + \frac{a_2'}{2a_1'} \right)^2 \frac{\Gamma(h)^2}{h^2} d\bar{\mu} \geq 0,$$

but the squared term can probably be controlled in a way that leads to a wider range of p, since

$$\int \left[hLh + \frac{a_2'}{2a_1'} \Gamma(h) \right] d\mu = \left(\frac{a_2'}{2a_1'} - 1 \right) \int \Gamma(h) d\mu < 0$$

whenever h is not a constant function.

3.4 Spaces with positive curvature and real dimension

The family of inequalities considered in this chapter, especially for sections 3.4 and 3.5, is the following interpolation between the Poincaré inequality and the logarithmic Sobolev inequality

Definition 3.13 (Beckner inequalities)**.** The Markov triple (M, Γ, μ) is said to satisfy a Beckner inequality $B_p(C)$ with parameter $p \in (1, 2]$ and constant $C > 0$ if, for all nonnegative smooth bounded functions f,

$$\frac{p}{p-1} \left(\int f^2 d\mu - \left(\int f^{2/p} d\mu \right)^p \right) \leq 2C \int \Gamma(f) d\mu. \tag{3.17}$$

The constants in front of the integrals are chosen so that for $p = 2$, this is exactly the Poincaré inequality with constant C, and the limiting case $p \to 1$ corresponds to the logarithmic Sobolev inequality, again with constant C. Indeed,

$$\lim_{p \to 1} \frac{p}{p-1} \left(\int f^2 d\mu - \left(\int f^{2/p} d\mu \right)^p \right) = \int f^2 \log\left(\frac{f^2}{\int f^2 d\mu} \right) d\mu.$$

Remark 3.14. The weighted Beckner inequalities proved in the previous section can be seen as classical Beckner inequalities. Indeed, theorem 3.10 states that $B_p((c(\beta - 1))^{-1})$ is valid for the triple $(M, \varphi\Gamma, \mu_{\varphi,\beta})$.

In this section, we consider a diffusion operator L defined on (M, g), juste like in Section 3.3, with μ being its reversible measure. We assume that L satisfies a curvature-dimension condition $CD(\rho, n)$, with $\rho > 0$ and $n \in \mathbb{R}$. That means, specifically but not exclusively, that we will consider negative n. Such spaces, sometimes referred to as having negative *effective dimension*, have been studied in the past, with articles dating as far back as 2003, up to our knowledge [Sch03, Mil17a, Mil17b, BGS18]. Furthermore, one can easily construct examples of such operators with [Oht16, Cor. 4.13]. Writing down the $CD(\rho, n)$ inequality in good local coordinates like in [BGS18], one can check that a necessary criterion for the inequality to be true is that $d/n \leq 1$, which means that we are actually restricted to $n \in \mathbb{R} \backslash [0, d]$. The main difference, compared to the previous section, is that the curvature is *positive*, which is a stronger assumption, but n can be negative, which is a weaker assumption.

Let us first recall the well known result for the Poincaré inequality, the proof of which is given in [BGL14, Thm. 4.8.4].

Theorem 3.15 (Poincaré inequality under $CD(\rho, n)$)**.** *Assume that the diffusion operator L satisfies a $CD(\rho, n)$ condition, with $\rho > 0$ and $n \in \mathbb{R} \backslash [0, d)$. Then the following Poincaré inequality holds,*

$$\mathrm{Var}_\mu(f) \leq \frac{n-1}{\rho n} \int \Gamma(f) d\mu. \tag{3.18}$$

We extend this result to Beckner inequalities in the following theorem:

Theorem 3.16 (Beckner inequalities under $CD(\rho, n)$)**.** *Assume that the diffusion operator L satisfies a $CD(\rho, n)$ condition, with $\rho > 0$ and $n \in \mathbb{R}\backslash[-2, d)$. Then, $B_p\Big(\frac{n-1}{\rho n}\Big)$ is satisfied*

- *for all $p \in (1, 2]$ if $n \geq d$,*
- *for all $p \in [p^*, 2]$ if $n < -2$, where*

$$p^* = 1 + \frac{1-4n}{2n^2+1}.$$

Interestingly, we find nothing for $n \in [-2, 0)$, which corresponds to the weakest $CD(\rho, n)$ conditions possible, even though the Poincaré inequality remains valid in that range. It seems like there is not enough structure in that case.

Proof. In the same fashion as the previous section, fix f, a bounded smooth nonnegative function on M, and consider the function f_t, solution of the initial-value system

$$\begin{cases} \partial_t f_t = L f_t & \text{on } (0, +\infty) \times M, \\ f_0 = f & \text{on } M, \end{cases} \tag{3.19}$$

and consider its Φ-entropy along the flow, where Φ is assumed to be strictly convex.

$$\Lambda(t) = \mathrm{Ent}_\mu^\Phi(f_t) = \int \Phi(f_t) d\mu - \Phi\Big(\int f_t d\mu\Big),$$

Invoking once again,

$$\Lambda'(t) = -\int \frac{\Gamma(\Phi'(f_t))}{\Phi''(f_t)} d\mu, \quad \Lambda''(t) = \int \left[2\frac{\Gamma_2(\Phi'(f_t))}{\Phi''(f_t)} + \Big(\frac{-1}{\Phi''}\Big)''(f_t)\Big(\frac{\Gamma(\Phi'(f_t))}{\Phi''(f_t)}\Big)^2 \right] d\mu.$$

Classically, one can assume $-1/\Phi''$ to be convex, whence the $CD(\rho, n)$ condition yields

$$\Lambda''(t) \geq \int \frac{2}{\Phi''(f_t)} \left[\rho \Gamma(\Phi'(f_t)) + \frac{1}{n}(L\Phi'(f_t))^2 \right] d\mu,$$

and, when $n > 0$, this leads to

$$\Lambda''(t) \geq -2\rho \Lambda'(t),$$

which, ultimately, proves Poincaré-type inequalities, using arguments like the ones in section 3.3. The problem is that the constant appearing in the inequality is, in fact, not optimal, and furthermore, the argument fails whenever $n < 0$. For the sake of simplicity, we will now assume that $\Phi(X) = X^p$, with $p \in (1, 2]$, but the argument could very well be generalized to more general Φ. The derivatives of the entropy then become

$$\Lambda'(t) = -\frac{p}{p-1} \int f_t^{2-p} \Gamma(f_t^{p-1}) d\mu, \quad \Lambda''(t) = \frac{p}{p-1} \int \left[2 f_t^{2-p} \Gamma_2(f_t^{p-1}) + \frac{2-p}{p-1} f_t^{4-3p} \Gamma(f_t^{p-1})^2 \right] d\mu.$$

Rewriting these quantities with respect to $q := \frac{2-p}{p-1} \in [0, +\infty)$, and $h = f_t^{p-1}$,

$$\frac{p-1}{p}\Lambda'(t) = -\int h^q \Gamma(h) d\mu, \quad \frac{p-1}{p}\Lambda''(t) = \int \left[2h^q \Gamma_2(h) + q h^{q-2} \Gamma(h)^2 \right] d\mu.$$

To get the most information out of this, we shall apply the $CD(\rho,n)$ condition not to the function h, but to $\eta(h)$, where η is some function to be chosen later. Expanding every term in the $CD(\rho,n)$ inequality (3.5)

$$\Gamma_2(\eta(h)) \geq \rho\Gamma(\eta(h)) + \frac{1}{n}(L\eta(h))^2$$

yields

$$\eta'^2(h)\Gamma_2(h) + \eta'(h)\eta''(h)\Gamma(h,\Gamma(h)) + (\eta''(h)\Gamma(h))^2 \geq \rho\eta'^2(h)\Gamma(h) + \frac{1}{n}(\eta'(h)Lh + \eta''(h)\Gamma(h))^2.$$

This inequality is, in particular, true for the power function $\eta(x) = x^{\theta+1}$, with $\theta \in \mathbb{R}$, so that

$$\Gamma_2(h) \geq \underbrace{\rho\Gamma(h)}_{i} + \underbrace{2\theta\,\frac{1}{n}\frac{\Gamma(h)Lh}{h}}_{ii} \underbrace{-\theta\,\frac{\Gamma(h,\Gamma(h))}{h}}_{iii} + \underbrace{\theta^2\left(\frac{1-n}{n}\right)\frac{\Gamma(h)^2}{h^2}}_{iv} + \underbrace{\frac{1}{n}(Lh)^2}_{v}.$$

Multiplying this inequality by h^q and then integrating it, we are left with five terms to consider. The first term corresponds to the first derivative of the entropy, $\Lambda'(t)$, so we may leave it as it is. Terms ii and iii are trickier, because their sign is not known, so we must take care of them. Term iv is always negative since $n \notin [0,1]$, but it is compensated by another (positive) term of the same nature appearing naturally in $\Lambda''(t)$. The sign of the last term depends on the sign of n, so we must take care of it as well, at least for negative n.

Term ii: an integration by parts yields

$$\int h^{q-1}\Gamma(h)Lh d\mu = -\int \left[h^{q-1}\Gamma(h,\Gamma(h)) + (q-1)h^{q-2}\Gamma(h)^2\right]d\mu.$$

Term iii: we do not do anything for now with this term, and will adjust θ later so that it disappears.

Term v: we use the fact that, by definition,

$$\int \Gamma_2(h)d\mu = \int \left[\frac{1}{2}L(\Gamma(h)) - \Gamma(h,Lh)\right]d\mu = \int (Lh)^2 d\mu$$

to prove that, for any real function η,

$$\int \eta'^2(h)(Lh)^2 d\mu = \int \left[\eta'^2(h)\Gamma_2(h) + 3\eta(h)\eta'(h)\Gamma(h,\Gamma(h)) + 2(\eta''^2(h) + \eta'(h)\eta'''(h))\Gamma(h)^2\right]d\mu.$$

In particular,

$$\int h^q(Lh)^2 d\mu = \int \left[h^q\Gamma_2(h) + q(q-1)h^{q-2}\Gamma(h)^2 + \frac{3}{2}qh^{q-1}\Gamma(h,\Gamma(h))\right]d\mu.$$

Finally, we are left with an equality still involving the parameter θ:

$$\int h^q\Gamma_2(h)d\mu \geq \frac{\rho n}{n-1}\int h^q\Gamma(h)d\mu + \int \left[Ah^{q-1}\Gamma(h,\Gamma(h)) + Bh^{q-2}\Gamma(h)^2\right]d\mu$$

where

$$\begin{cases} A = \dfrac{1}{n-1}\left(\dfrac{3q}{2} - \theta(n+2)\right), \\ B = \dfrac{q(q-1)}{n-1} - \theta^2 - 2\theta\dfrac{q-1}{n-1}. \end{cases}$$

Choosing θ so that $A = 0$, i.e. $\theta = \frac{3q}{2(n+2)}$, we find that

$$\int \left[2f^q\Gamma_2(h) + qh^{q-2}\Gamma(h)^2\right]d\mu \geq \frac{2\rho n}{n-1}\int h^q\Gamma(h)d\mu + \alpha\int h^{q-2}\Gamma(h)^2 d\mu,$$

with

$$\begin{aligned}\alpha = 2B + q &= \frac{q}{2(n+2)^2}\left(q(4n-1) + 2n(n+2)\right)\\ &= \frac{1}{2(n+2)^2}\left(\frac{2-p}{p-1}\right)\left(\frac{2-p}{p-1}(4n-1) + 2n(n+2)\right)\end{aligned}$$

Remembering that $q \geq 0$, this constant α turns out to be nonnegative for the following range of parameters

$$\begin{cases} n \geq d \\ q \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} n < -2 \\ q \in [0, q^*] \end{cases} \quad \text{with } q^* = \frac{2n(n+2)}{1-4n},$$

or equivalently, in terms of the exponent p,

$$\begin{cases} n \geq d \\ p \in (1, 2] \end{cases} \quad \text{or} \quad \begin{cases} n < -2 \\ p \in [p^*, 2] \end{cases} \quad \text{with } p^* = 1 + \frac{1-4n}{2n^2+1},$$

Whenever $\alpha \geq 0$, we may, at last, compare Λ'' to Λ'. Indeed, we find that

$$\Lambda''(t) \geq -\frac{2\rho n}{n-1}\Lambda'(t),$$

which proves the claimed $B_p\left(\frac{n-1}{\rho n}\right)$ inequality when integrated twice. □

To prove this theorem, we used the nonnegativity of a specific term in a differential inequality, but in fact, we can do a little bit better and compare it to the other terms, in order to prove a refined version of the Beckner inequalities we are considering.

Definition 3.17. The Markov triple (M, Γ, μ) is said to satisfy a refined Beckner inequality $B_p^*(C, \theta)$ with parameter $p \in (1, 2]$ and constants $C > 0$ and $\theta \geq 0$ if, whenever $\theta \neq 1$,

$$\frac{p}{p-1}\left(\frac{1}{1-\theta}\right)\left(\int f^2 d\mu - \left(\int f^{2/p} d\mu\right)^{(1-\theta)p}\left(\int f^2 d\mu\right)^{\theta}\right) \leq 2C\int \Gamma(f)d\mu$$

for all smooth functions f, and, when $\theta = 1$,

$$\frac{p}{p-1}\left(\int f^2 d\mu\right)\log\left(\frac{\int f^2 d\mu}{\left(\int f^{2/p}d\mu\right)^p}\right) \leq 2C\int \Gamma(f)d\mu.$$

With this definition, $B_p(C)$ is the same as $B_p^*(C, 0)$. The inequality given for $\theta = 1$ is simply the limit of the other inequality when $\theta \to 1$. This indeed corresponds to an improved version of the Beckner inequality, because for all $x, y > 0$ and $\theta \in \mathbb{R}_+\backslash\{1\}$,

$$\frac{x - y^{1-\theta}x^{\theta}}{1-\theta} \geq x - y,$$

and more generally, $B_p^*(C, \theta)$ implies $B_p^*(C, \theta')$ for all $\theta' \in [0, \theta]$. Such improvements have been shown in [AD05, BG10] under the $CD(\rho, \infty)$ condition. The limit case, that is, for the usual entropy, is proposed in [BGL14, Thm. 6.8.1].

Theorem 3.18 (Improved Beckner inequalities)**.** *Under the same assumptions of in Theorem 3.16, the inequality $B_p^*\Big(\frac{n-1}{\rho n},\theta\Big)$ is satisfied for the same range of parameter p, and for θ given by*

$$\theta = \frac{1}{2(n+2)^2}\Big(\frac{p}{p-1}\Big)\Big(\frac{2-p}{p-1}\Big)\Big(\frac{2-p}{p-1}(4n-1)+2n(n+2)\Big).$$

Proof. Coming back to the proof of theorem 3.16, we have that

$$\int\big[2f^q\Gamma_2(h)+qh^{q-2}\Gamma(h)^2\big]d\mu \geq \frac{2\rho n}{n-1}\int h^q\Gamma(h)d\mu+\alpha\int h^{q-2}\Gamma(h)^2d\mu,$$

or, in terms of Λ,

$$\Lambda''(t)\geq -\frac{2\rho n}{n-1}\Lambda'(t)+\alpha\frac{p}{p-1}\int h^{q-2}\Gamma(h)^2d\mu.$$

Invoking Jensen's inequality, we find that

$$\begin{aligned}\int h^{q-2}\Gamma(h)^2d\mu &\geq \frac{\big(\int h^q\Gamma(h)d\mu\big)^2}{\int h^{q+2}d\mu}\\ &= \Big(\frac{p-1}{p}\Big)^2\frac{\Lambda'(t)^2}{\Lambda(t)},\end{aligned}$$

so that

$$\Lambda''(t)\geq -\frac{2\rho n}{n-1}\Lambda'(t)+\alpha\frac{p}{p-1}\frac{\Lambda'(t)^2}{\Lambda(t)},$$

just like we found in remark 3.11. Writing $\theta=\alpha\frac{p}{p-1}$, we may now integrate this inequality, to find that

$$-\frac{\Lambda'(t)}{\Lambda(t)^\theta}\leq -\frac{\Lambda'(0)}{\Lambda(0)^\theta}\exp\Big(-\frac{2\rho n}{n-1}t\Big),$$

which, integrated once more between 0 and $+\infty$, leads to the claimed inequality. □

Remark 3.19. As it turns out, the operator $\overline{L}$ from last section does not verify a good enough $CD(\rho,n)$ condition. The best constant $c(\beta-1)$ only arises using an integrated $CD(\rho,n)$ criterion. To see this, we can use the following result from [Bak94]: the operator $L=\Delta_g+\Gamma(V,.)$, defined on (M,g), satisfies a $CD(\rho,n)$ condition if, and only if,

$$\frac{n-d}{n}\big(\mathrm{Ric}_g-\nabla^2V-\rho g\big)\geq\frac{1}{n}\nabla V\otimes\nabla V,$$

this tensorial reformulation being valid for any $\rho\in\mathbb{R}$ and $n\notin[0,d]$. As stated in the proof to theorem 3.6, the operator $\overline{L}f=\varphi Lf-(\beta-1)\Gamma(\varphi,f)$ on (M,g) is related to a Laplace-Beltrami operator through the conformal transformation with conformal factor φ^{-1}. Thus, writing $\overline{L}=\Delta_{\overline{g}}+\Gamma^{\Delta_{\overline{g}}}(\overline{V},.)$, for some explicit function V, we find a somewhat easier to verify criterion for the $CD(\rho,n)$ condition in $(M,\overline{g})$:

$$\mathrm{Ric}_g+(\beta-1)\frac{\nabla^2\varphi}{\varphi}+\Big(2-d-\frac{(2\beta-1)^2}{n-d}\Big)\frac{\nabla\varphi\otimes\nabla\varphi}{4\varphi^2}+\Big(\frac{\Delta\varphi}{2\varphi}-2\beta\frac{\Gamma(\varphi)}{4\varphi^2}-\frac{\rho}{\varphi}\Big)g\geq 0. \qquad (3.20)$$

We leave it to the courageous to verify that indeed, even in the case where everything is nice and explicit, for instance for $\varphi(x)=1+|x|^2$, there exists no couple $(\rho,n)\in\mathbb{R}_+^*\times(\mathbb{R}\backslash[0,d])$ such that inequality (3.20) is verified and

$$\frac{\rho n}{n-1}=c(\beta-1),$$

making theorem 3.6 truly an *integrated* $CD(\rho,n)$ criterion result.

3.5 Results on the real line

In dimension $d = 1$, simplifications happen, so that we are able to do calculations directly. All manifolds of dimension 1 are conformal to $(\mathbb{R}, 1)$, and even though the study of a generic manifold $(\mathbb{R}, g)$ does not reduce exactly to the study of $(\mathbb{R}, 1)$, the calculations are similar, and so we will only consider the Lebesgue measure λ for the reference measure. Following the work in section 3.3, assume φ is a C^2, positive, convex function such that

$$\varphi'' \geq c$$

for some constant $c > 0$. Fix $\beta \in \mathbb{R}$ such that $\varphi^{-\beta}$ is in $L^1(\mathbb{R}, \lambda)$. Then, we find the following result version of the Poincaré inequality, for the probability measure $\mu_{\varphi,\beta} = Z_{\varphi,\beta}\varphi^{-\beta}\lambda$

Theorem 3.20. *Fix a real number $\beta > 1$. Then for all smooth bounded functions f,*

$$\int f^2 d\mu_{\varphi,\beta} - \left(\int f d\mu_{\varphi,\beta}\right)^2 \leq \frac{1}{c(\beta-1)} \int (f')^2 \varphi d\mu_{\varphi,\beta}. \tag{3.21}$$

For $\beta > 2$, this is in fact theorem 3.6, so this theorem is an extension to smaller exponants β.

Proof. The idea is to apply Theorem 3.16 to the special case of the operator $\overline{L}$ defined by

$$\overline{L}f = \varphi f'' - (\beta - 1)\varphi' f',$$

after proving it satisfies a $CD(\rho, n)$ condition with negative dimension. Since we are working in dimension 1, expliciting the $CD(\rho, n)$ is not too hard, and that is just what we do. Indeed, straightforward computations yield

$$\begin{aligned}
\overline{\Gamma}_2(f) &= \frac{1}{2}\overline{L}(\overline{\Gamma}(f)) - \overline{\Gamma}(f, \overline{L}f) \\
&= \frac{1}{2}\big((2\beta-1)\varphi\varphi'' + (1-\beta)(\varphi')^2\big)(f')^2 + \varphi\varphi' f' f'' + \varphi^2 (f'')^2,
\end{aligned}$$

so that, given $\rho \geq 0$ and $n \in \mathbb{R}\backslash[0, 1]$,

$$\overline{\Gamma}_2(f) \geq \rho\overline{\Gamma}(f) + \frac{1}{n}(\overline{L}f)^2$$

for all smooth functions f if, and only if,

$$A(f')^2 + Bf'f'' + C(f'')^2 \geq 0$$

for all f, where

$$\begin{cases}
A = \frac{1}{2}\big((2\beta-1)\varphi\varphi'' + (1-\beta)(\varphi')^2\big) - \rho\varphi - \frac{1}{n}(1-\beta)^2(\varphi')^2, \\
B = \frac{1}{n}(n + 2\beta - 2)\varphi\varphi', \\
C = \left(1 - \frac{1}{n}\right)\varphi^2.
\end{cases}$$

since $C \geq 0$, this is in turn equivalent to $B^2 - 4AC \leq 0$, which, after simplifications, boils down to the condition

$$-\left(\beta - \frac{1}{2}\right)\left(\frac{1}{n-1}\left(\beta - \frac{1}{2}\right) + \frac{1}{2}\right)\frac{(\varphi')^2}{\varphi} + \left(\beta - \frac{1}{2}\right)\varphi'' - \rho \geq 0. \tag{3.22}$$

For $n = 2(1-\beta)$, the first term in the above inequality disappears and the condition becomes

$$\left(\beta - \frac{1}{2}\right)\varphi'' - \rho \geq 0,$$

which is clearly true for $\rho = c\big(\beta - \frac{1}{2}\big)$, which proves that $\overline{L}$ satisfies the $CD(c\big(\beta - \frac{1}{2}\big), 2(1-\beta))$ condition. We may now apply the theorem 3.15 to conclude that the Poincaré inequality is valid with constant

$$\frac{n-1}{\rho n} = \frac{1-2\beta}{c(1-\beta)(2\beta-1)} = \frac{1}{c(\beta-1)},$$

which is the claimed result. □

This proof proves more than just the Poincaré inequality, since theorem 3.16 provides a range of exponants for which the Beckner inequality holds, but this is only true when $n < -2$, which corresponds to $\beta > 2$. As far as Beckner inequalities go, this result is exactly the same as the one in section 3.3, and as such only constitutes an example of application of the results in section 3.4. The interest, however, lies in the fact that we extend the range of β for which the Poincaré inequality is valid.

As it turns out, this inequality is optimal for $\beta \geq 3/2$, but it is not optimal anymore for $\beta \in [1, 3/2)$, as proved in [BJM16] for the function $\varphi(x) = 1 + x^2$. In fact, they find that inequality (3.21) is valid for $\beta \in (1/2, 3/2]$, and the optimal constant in that range changes from $(2(\beta-1))^{-1}$ to $(\beta - 1/2)^{-2}$. It might be worth noting that the method presented in theorem 3.20 actually works for the full range of β, as summed up in the following proposition

Proposition 3.21. *Let $\varphi : x \mapsto 1 + x^2$. Fix a real number $\beta > \frac{1}{2}$. Then for all smooth bounded functions f,*

$$\int f^2 d\mu_{\varphi,\beta} - \left(\int f d\mu_{\varphi,\beta}\right)^2 \leq \frac{1}{C_\beta}\int (f')^2 \varphi d\mu_{\varphi,\beta}, \tag{3.23}$$

where

$$C_\beta = \begin{cases} 2(\beta-1) \text{ if } \beta \geq \dfrac{3}{2}, \\ \left(\beta - \dfrac{1}{2}\right)^2 \text{ if } \beta \in \left(\dfrac{1}{2}, \dfrac{3}{2}\right). \end{cases} \tag{3.24}$$

Proof. The proof builds on the proof for theorem 3.20, using the explicit form of φ. The condition (3.22) becomes

$$-(2\beta-1)\left(\frac{1}{n-1}(2\beta-1) + 1\right)\frac{x^2}{1+x^2} + (2\beta-1) - \rho \geq 0,$$

for all $x \in \mathbb{R}$, or, equivalently,

$$\rho \leq 2\beta - 1 \quad \text{and} \quad -\frac{(2\beta-1)^2}{n-1} - \rho \geq 0.$$

Restricting our study to nonnegative curvatures, we thus prove that the condition $CD(\rho, n)$ (with $\rho > 0$) is satisfied if, and only if,

$$0 < \rho \leq 2\beta - 1 \quad \text{and} \quad 0 < \rho(1-n) \leq (2\beta-1)^2 \quad (\text{and } n \notin [0,1]).$$

We are looking, for a fixed $\beta > 1/2$, for the parameters (ρ, n) satisfying those criteria that lead to the best possible value of $\rho n/(n-1)$.

Assuming (ρ, n) is such a couple, since $n/(n-1)$ is decreasing on $\mathbb{R}_-^*$, it is necessary that $\rho(1-n) = (2\beta-1)^2$, which then implies that $\rho < 2(\beta-1)^2$. We thus reduce the problem to finding the maximum of the function

$$\frac{\rho n}{n-1} = \rho - \frac{\rho^2}{(2\beta-1)^2}$$

under the constraints $0 < \rho < \max(2\beta-1, (2\beta-1)^2)$. An easy study yields that this maximum is C_β as defined in equation (3.24). In other terms, the operator $Lf = (1+x^2)f'' + (1-\beta)2xf'$ satisfies

- $CD(2\beta-1, 2(1-\beta))$ when $\beta \geq 3/2$,
- $CD((2\beta-1)^2/2, -1)$ when $\beta \in (1/2, 3/2)$,

and this leads to the proposition. □

Remark 3.22. This method is applicable to other functions than just $x \mapsto 1+x^2$. For instance, for the function $\varphi : x \mapsto 1+x^2+x^4$, one finds that the condition $CD(2\beta-1, 4\beta-1)$ is satisfied, and it is the one that leads to the best possible $\rho n/(n-1)$ constant for the corresponding operator.

Chapter 4

Entropy flows and functional inequalities in convex sets

Abstract

We revisit entropy methods to prove new sharp trace logarithmic Sobolev and sharp Gagliardo-Nirenberg-Sobolev inequalities on the half space, with a focus on the entropy inequality itself and not the actual flow, allowing for somewhat robust and self-contained proofs.

Contents

4.1 Introduction

4.1.1 Brief introduction of the ideas

Sobolev inequalities have proved an important tool in the study of partial differential equations, notably in establishing existence results. More recently, they have been very fruitfully used in the study of the long term behavior of certain equations. For instance, the logarithmic Sobolev inequality can be used to establish a rate of convergence of the heat flow towards its mean on torus, or towards the self-similar profile on the Euclidean space.

These ideas and results fall in the general context of entropy methods [Jü16]. Boltzmann defined his entropy in 1872 by $\mathrm{Ent}(u) = \int u \log(u) dx$ for, say, positive functions defined on $\mathbb{R}^d$.

Now, if u is the solution of the heat equation, $\partial_t u = \Delta u$, differentiating the entropy yields

$$\frac{d}{dt}\operatorname{Ent}(u) = \int (1+\log(u))\Delta u dx = -\int \frac{\|\nabla u\|^2}{u} dx,$$

which implies two important things: the first one is that the entropy is nonincreasing along the flow, conveying the idea that the physical transformation u is describing is irreversible. The second one, maybe more profound, is that the logarithmic Sobolev inequality is exactly an equality relating the entropy and its derivative, which, after integration, implies exponential decay of the entropy with respect to time. This relationship between the heat equation and the entropy is no coincidence. It turns out that the space of probability measures, when equipped with the Wasserstein distance, can be formally seen as a Riemannian manifold, and in that setting, the heat equation is exactly the gradient flow of Boltzmann's entropy. This approach is mainly due to Felix Otto [Ott01], and the study of gradient flows on metric spaces has since been made rigorous [AGS08]. See also Filippo Santambrogio's survey on this topic [San17].

Somewhat recently, Manuel del Pino and Jean Dolbeault observed a similar result for the Gagliardo-Nirenberg-Sobolev (GNS) inequalities [dPD99, dPD02]. They may be rewritten in a way that involves both an entropy functional, different than Boltzmann's, and its derivative along a particular mass-preserving flow.

Theorem 4.1 ([dPD02, Corollary 13])**.** *Let $d \geq 3$, and $1 < p < \frac{d}{d-2}$. Then, the sharp Gagliardo-Nirenberg-Sobolev inequality*

$$\forall w \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \|w\|_{2p} \leq C\|\nabla w\|_2^\theta \|w\|_{1+p}^{1-\theta}, \tag{4.1}$$

where $\theta \in [0,1]$ is fixed by the parameters, is equivalent to

$$\forall u \in \mathcal{C}_c^\infty(\mathbb{R}^d)\ s.t.\ \|u\|_1 = \|v\|_1,\ \mathcal{F}(u) - \mathcal{F}(v) \leq \frac{1}{2}\int_{\mathbb{R}^d} u\left\|x - \nabla(u^{\alpha-1})\right\|^2 dx, \tag{4.2}$$

where $\alpha = (p+1)/(2p) < 1$, and

$$\mathcal{F}(u) = \int_{\mathbb{R}^d}\left[-\frac{u^\alpha}{\alpha} + u\left(1 + \frac{\|x\|^2}{2}\right)\right] dx \quad and \quad v(x) = \left(1 + \frac{\|x\|^2}{2}\right)^{1/(\alpha-1)}.$$

Again, the right-hand side in inequality (4.2) is exactly the derivative of the functional $\mathcal{F}$ along a flow obtained, depending on the value of α, from the porous medium equation or the fast-diffusion equation through a change of variables. The resulting exponential decay of the entropy $\mathcal{F}$ may be finally used to prove decay in various L^p norms using a general Csiszár-Kullback-Pinsker inequality [AMTU01]. Note even though $\mathcal{F}$ is not Boltzmann's original entropy, we still call it entropy, and we will use that word again throughout the chapter without a rigorous definition. Loosely speaking, entropies will be Lyapunov functional related to specific flows, but the important thing is that they are only a tool.

Logarithmic Sobolev and Gagliardo-Nirenberg-Sobolev inequalities are not the only ones that may be seen as an inequality between the entropy and its derivative along a flow. Indeed, Poincaré inequalities, and the so-called Beckner inequalities, interpolating inequalities between the Poincaré inequality and the logarithmic Sobolev inequality, are all natural examples of this [GZ19]. This motivates the study of the so-called generalized Sobolev inequalities.

While these generalized Sobolev inequalities may be used to study a particular flow, a striking fact is that they turn out to be contained in the flow itself [CJM+01, Jü16]. Indeed, in the good tradition of the Bakry-Émery method [BE85], differentiating the entropy along the flow twice instead of once, and then invoking geometric properties of the underlying space (the Bochner-Lichnerowicz inequality, or a curvature-dimension condition) as well as the convexity inherent of the entropy functional, allows to recover said generalized Sobolev inequality. This method was succesfully used by Toscani for the logarithmic Sobolev inequality [Tos97], and subsequently quite thoroughly investigated in [CJM+01].

In the case of linear flows, the existence of Markov semigroups makes the study simpler [BGL14], but in the general case, one has to resort to use tools from the realm of partial differential equations, possibly making the study quite convoluted. In this chapter, we wish to revisit the method for general entropies of the form

$$\mathcal{F}(u) = \int H(u) + uV, \tag{4.3}$$

where H is a convex function on $\mathbb{R}_+$, and V a strictly uniformly convex function on some subdomain of $\mathbb{R}^d$.

Remark 4.2. In equation (4.3) much like in the rest of the chapter, the integration against the standard Lebesgue measure, or the $(d-1)$-dimensional Hausdorff measure when integrating on boundaries, will always be implied.

We try for the proofs to be as self-contained as possible, while also keeping the calculations to a minimum. This is possible since we do not want to study the long-term behavior of the various flows considered, and are only interested in proving generalized Sobolev inequalities. Furthermore, we will, starting in section 4.2, use the same vocabulary that is used in [BGL14], i.e. we will make use of the *Carré du champ* operator Γ and its iterated version, Γ_2. This choice is motivated by two reasons: the first one is because the main results are very similar in nature with ones involving Markov semigroups. The second one is because it makes calculations systematic, and also makes the curvature-dimension hypotheses appear clearly, allowing for easy generalization of all the results to manifolds.

4.1.2 Model example: the Euclidean Sobolev inequality

Let us showcase the method with the study of a simple example which will serve as a guide in the next sections: the proof of the sharp Sobolev inequality on $\mathbb{R}^d$. Define the functions $H \in C^\infty(\mathbb{R}_+^*, \mathbb{R})$ and $v \in C^\infty(\mathbb{R}^d, \mathbb{R}_+^*)$ by

$$H(x) = -x^{1-1/d}, \quad \text{and} \quad v(x) = C(1+\|x\|^2)^{-d}, \tag{4.4}$$

where $C > 0$ has been chosen so that $-H'(v) = 1 + \|x\|^2$. For smooth positive functions u, we define the entropy

$$\mathcal{F}(u) = \int_{\mathbb{R}^d} H(u) - uH'(v). \tag{4.5}$$

Note that, since H is convex, $\mathcal{F}(u) \geq \mathcal{F}(v)$. Now choose a function $u_0 \in C^\infty(\mathbb{R}^d, \mathbb{R}_+^*)$ such that $\int_{\mathbb{R}^d} u_0 = \int_{\mathbb{R}^d} v$, and consider the relative entropy

$$\mathcal{F}(u \mid v) = \mathcal{F}(u) - \mathcal{F}(v)$$

along the flow

$$\begin{aligned} &\partial_t u = -\nabla \cdot \big(u \nabla (H'(v) - H'(u))\big) && \text{in } \mathbb{R}_+^* \times \mathbb{R}^d, \\ &u(0,.) = u_0 && \text{in } \mathbb{R}^d. \end{aligned} \tag{4.6}$$

The first derivative of the entropy is easily calculated using an integration by parts:

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(u) &= \int_{\mathbb{R}^d} \partial_t u(H'(u) - H'(v)) \\ &= \int_{\mathbb{R}^d} \nabla \cdot (u\nabla(H'(v) - H'(u)))(H'(v) - H'(u)) \\ &= -\int_{\mathbb{R}^d} u\|\nabla(H'(v) - H'(u))\|^2 \leq 0.\end{aligned}$$

Note that the flow (4.6) is the gradient flow of the entropy functional (4.5). The fact that the derivative of the entropy takes such a nice form is a general fact of gradient flows [San17].The calculations for the second derivative are slightly tricky, so we refer to the next section for the full details, but using both the fact that $\nabla^2 H'(v) = -2I_d$ and that $\left\|\nabla^2\phi\right\|_{HS}^2 \geq \frac{1}{d}(\Delta\phi)^2$, we find that

$$\frac{d^2}{dt^2}\mathcal{F}(u) \geq -4\frac{d}{dt}\mathcal{F}(u), \tag{4.7}$$

which, if one recalls that the first derivative is nonpositive, proves that the entropy along the flow has a strong convexity property which is really the core of the argument. Assuming that the function u converges, when t goes to infinity, to the stationnary solution v, it is quite clear that $\lim_{t\to+\infty}\mathcal{F}(u) = \mathcal{F}(v)$, and $\lim_{t\to+\infty}\frac{d}{dt}\mathcal{F}(u) = 0$. Now, integrating the second derivative of the entropy between 0 and $+\infty$ leads to

$$\mathcal{F}(u_0) - \lim_{t\to+\infty}\mathcal{F}(u) = \mathcal{F}(u_0 \mid v) \leq -\frac{1}{4}\left.\frac{d}{dt}\mathcal{F}(u)\right|_{t=0}. \tag{4.8}$$

Equation (4.8) is a special case of an entropy - entropy production inequality, to which we will come back later. It is quite obviously optimal, since equality happens for $u_0 = v$.

We may now rewrite equation (4.8) with the explicit quantities (4.4) to prove the sharp Sobolev inequality on $\mathbb{R}^d$: since $-H'(v) = 1 + \|x\|^2$,

$$\int_{\mathbb{R}^d} H(u_0) - H(v) - (u_0 - v)H'(v) \leq \frac{1}{4}\int_{\mathbb{R}^d} u_0\|\nabla H'(u_0) + 2x\|^2. \tag{4.9}$$

Expanding the right-hand side, we have to deal with three different terms. First, notice that

$$\frac{1}{4}\int_{\mathbb{R}^d} u_0\|\nabla H'(u_0)\|^2 = \frac{(d-1)^2(d-2)^2}{16d^2}\int_{\mathbb{R}^d}\left\|\nabla u_0^{1/2-1/d}\right\|^2$$

Next, the other square is

$$\int_{\mathbb{R}^d} u_0\|x\|^2 = \int_{\mathbb{R}^d} u_0(-H'(v) - 1),$$

which simplifies with the left-hand side. Finally, the double product can be integrated by parts once we notice, once again by homogeneity, that $u_0\nabla H'(u_0) = -\frac{1}{d}\nabla H(u_0)$:

$$\int_{\mathbb{R}^d} u_0\nabla H'(u_0)\cdot x = -\frac{1}{d}\int_{\mathbb{R}^d}\nabla H(u_0)\cdot x = \int_{\mathbb{R}^d} H(u_0),$$

and this also simplifies with the left-hand side. Since $\int u_0 = \int v$, the equation we are left with is

$$C \leq \int_{\mathbb{R}^d}\left\|\nabla u_0^{1/2-1/d}\right\|^2$$

for some explicit positive constant C. Replacing u_0 with $f = u_0^{1/2-1/d}$, we recover Sobolev's inequality.

4.1.3 Statement of the results

In this subsection, we state the main results of this chapter. Let us start by listing the hypotheses.

Let H be a strictly convex function from $\mathbb{R}_+$ to $\mathbb{R}$ such that $H(0) = 0$, and such that it is smooth on $\mathbb{R}_+^*$. On $\mathbb{R}_+^*$, define the functions $\psi = H'$, $U(x) = xH'(x) - H(x)$ and $U_2(x) = xU'(x) - U(x)$. In everything that follows, we shall make the following hypothesis:

Hypothesis A. Assume that $U_2 + \frac{1}{d}U \geq 0$.

Fix a closed convex set $\overline{\Omega} \subset \mathbb{R}^d$, and choose a positive integrable function $v \in C^\infty(\overline{\Omega})$ such that

Hypothesis B. $-\nabla^2\psi(v) \geq CI_d$ for some constant $C > 0$.

Theorem 4.3. *Let H be a strictly convex function from $\mathbb{R}_+$ to $\mathbb{R}$, such that H is smooth on $\mathbb{R}_+^*$ and $H(0) = 0$, and define $\psi = H'$. Fix a closed convex set $\overline{\Omega} \subset \mathbb{R}^d$, and a smooth positive function $v : \overline{\Omega} \to \mathbb{R}_+^*$. Under hypotheses A and B, for any positive function $u \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\int_\Omega u = \int_\Omega v$, the following inequality holds*

$$\int_\Omega H(u) - H(v) - (u-v)\psi(v) \leq \frac{1}{2C}\int_\Omega u\|\nabla(\psi(u) - \psi(v))\|^2. \tag{4.10}$$

This result may be seen as an immediate corollary of the (slightly) more general theorem that follows, where we allow v to take the value zero. However, we choose to present the two theorems separate, since theorem 4.3 feels a bit more natural, it being easy to relate to a gradient flow, as will be seen in section 4.2. To formulate this more general version, we first need to define the generalized inverse of a function.

Definition 4.4. Let $\psi : \mathbb{R}_+^* \to \mathbb{R}$ be a continuous strictly increasing function. Its generalized inverse ψ^{-1*} is given for $x \in \overline{\mathbb{R}}$ by

$$\psi^{-1*}(x) = \begin{cases} \psi^{-1}(x) & \text{if } x \in (\psi(0^+), \psi(+\infty)) \\ 0 & \text{if } x \leq \psi(0^+) \\ +\infty & \text{if } x \geq \psi(+\infty). \end{cases} \tag{4.11}$$

Instead of considering a smooth function v, we instead look at the generalized inverse of some convex function V, or, in other words, $v = \psi^{-1*}(-V)$. Note that v may very well be not differentiable, even if H and V are smooth. Also, since we do not want the function v to take the value $+\infty$, as nothing would be integrable anymore. We thus replace hypothesis B by the following

Hypothesis C. $-V < \psi(+\infty)$, and $\nabla^2 V \geq CI_d$ for some constant $C > 0$.

Theorem 4.5. *Let H be a strictly convex function from $\mathbb{R}_+$ to $\mathbb{R}$, such that H is smooth on $\mathbb{R}_+^*$ and $H(0) = 0$, and define $\psi = H'$. Fix a closed convex set $\overline{\Omega} \subset \mathbb{R}^d$, and a smooth function $V : \overline{\Omega} \to \mathbb{R}$. Define $v = \psi^{-1*}(-V)$, where ψ^{-1*} stands for the generalised inverse of ψ, as defined in (4.11).*

Under hypotheses A and C, for any positive function $u \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\int_\Omega u = \int_\Omega v$, the following inequality holds

$$\int_\Omega H(u) - H(v) + (u-v)V \leq \frac{1}{2C}\int_\Omega u\|\nabla\psi(u) + \nabla V\|^2. \tag{4.12}$$

Remark 4.6. In this whole chapter, we consider functions with compact support in a convex domain $\overline{\Omega}$, but $\overline{\Omega}$ will always be closed, which means that the functions are not necessarily equal to zero on $\partial\Omega$. This is of special importance, because we use theorems 4.3 and 4.5 to prove trace inequalities.

The proof to theorems 4.3 and 4.5 is rather long, and so will be split into two sections: section 4.2 contains the somewhat formal but accurate calculations, and section 4.3 addresses all the technicalities required to make the calculations rigorous. Among various Sobolev inequalities that may be proved using these results, two are, up to our knowledge, new and of particular interest.

Corollary 4.7 (Trace logarithmic Sobolev inequality)**.** *For all $h \in \mathbb{R}$, and for all positive function $u \in C^\infty(\mathbb{R}^d_+)$ such that $\int_{\mathbb{R}^d_+} u = 1$, the following inequality stands*

$$\int_{\mathbb{R}^d_+} u \log u \leq \frac{d}{2} \log\left(\frac{1}{2\pi d e} \int_{\mathbb{R}^d_+} \frac{\|\nabla u\|^2}{u}\right) - \log \gamma(\mathbb{R}^d_{+he}) - h\left(\int_{\partial \mathbb{R}^d_+} u\right)\left(\frac{1}{d}\int_{\mathbb{R}^d_+} \frac{\|\nabla u\|^2}{u}\right), \quad (4.13)$$

where γ stands for the standard Gaussian probability measure. Furthermore, there is equality when $u = C_h \exp\big(-\|x+he\|^2\big)$, where C_h is chosen such that $\int_{\mathbb{R}^d_+} u = 1$.

Note that for $h = 0$, this is the standard optimal logarithmic Sobolev inequality on the half space. Interestingly, the parameter h can be chosen either positive or negative, allowing the trace term to be used as an upper or a lower bound.

Corollary 4.8 (Concave GNS inequality)**.** *Let $\alpha > 1$. For all functions $f \in C^\infty(\mathbb{R}^d_+)$, the following inequality stands*

$$\|f\|_{2\alpha/(2\alpha-1)} \leq C_\alpha \|\nabla f\|_2^\theta \|f\|_{2/(2\alpha-1)}^{1-\theta}, \quad (4.14)$$

where

$$\theta = \left(1 - \frac{1}{d(\alpha-1)+1}\right)\left(1 - \frac{1}{2\alpha}\right).$$

Furthermore, there is equality when $f = \big(1 - \|x\|^2\big)_+^{1/(\alpha-1)}$, up to multiplication by a constant, rescaling, and translation by a vector in $\mathbb{R}^{d-1} \times \{0\}$.

This result is actually a special case of the more general trace inequality (4.29) that we will prove in section 4.2.3.

4.2 Formal proof

4.2.1 Some words on Γ-calculus

As stated in the introduction, we choose in this chapter to stick to the Gamma calculus formalism (see [BGL14]) even though we do not study Markov semigroups. Let us very briefly introduce some notions here, which, in this particular case, are tied to the standard Laplacian $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$, but may very well be used with other diffusion operators, such as the Laplace-Beltrami operator on manifolds.

Definition 4.9. The carré du champ operator is the symmetric bilinear map from $\mathcal{C}^\infty(\mathbb{R}^d) \times \mathcal{C}^\infty(\mathbb{R}^d)$ onto $\mathcal{C}^\infty(\mathbb{R}^d)$ defined by

$$\Gamma(a,b) = \frac{1}{2}(\Delta(ab) - a\Delta b - b\Delta a) = \nabla a \cdot \nabla b.$$

Its iterated version is defined by

$$\begin{aligned}\Gamma_2(a,b) &= \frac{1}{2}(\Delta(\Gamma(a,b)) - \Gamma(a,\Delta b) - \Gamma(b,\Delta a)) \\ &= \operatorname{tr}((\nabla^2 a)^t \nabla^2 b).\end{aligned}$$

Out of convenience, we will use the same notation for the bilinear maps and their respective quadratic maps, i.e. $\Gamma(a) = \Gamma(a,a)$ and $\Gamma_2(a) = \Gamma_2(a,a)$.

With this formalism, the Hessian may be written in the following way: if f, g, h are smooth functions, then

$$\nabla^2 f(\nabla g, \nabla h) = \frac{1}{2}(\Gamma(g,\Gamma(f,h)) + \Gamma(h,\Gamma(f,g)) - \Gamma(f,\Gamma(g,h))). \tag{4.15}$$

A quick proof of this fact on manifolds can be found in [GZ19, Lemma 2.3].

Remark 4.10. The standard Laplacian on $\mathbb{R}^d$ satisfies a $CD(0,d)$ condition, or in other words

$$\Gamma_2(a) \geq \frac{1}{d}(\Delta a)^2 \tag{4.16}$$

for all smooth functions a. This is nothing else than a Cauchy-Schwarz inequality, or a special case of the Bochner-Lichnerowicz inequality [BGL14, Theorem C.3.3]

Remark 4.11. In this chapter, we will consider functions defined on a closed convex subset $\Omega \subset \mathbb{R}^d$. The definition of Γ, Γ_2 trivially generalizes to such subsets. The major downside of using the Γ formalism is that the theory was not developped for functions taking nonzero values on the boundary of the domain, so instead of the usual neat integration by parts formula, we will have to use one adapted to our setting:

$$\begin{aligned}\int_\Omega \Gamma(a,b) &= -\int_\Omega b\Delta a + \int_{\partial\Omega} b\partial_\nu a \\ &= -\int_\Omega a\Delta b + \int_{\partial\Omega} a\partial_\nu b,\end{aligned}$$

where ∂_ν stands for the derivative along the outer normal vector.

4.2.2 Setting of the flow

In this subsection, we assume that every function we manipulate is nice and smooth, and we rigorously prove a generalized version of inequality (4.7), theorem 4.14, which is the key leading to theorem 4.3. We refer to section 4.3 for the technical study of the flow.

Fix some closed convex set $\overline{\Omega} \in \mathbb{R}^d$, and some strictly convex smooth function $H : \mathbb{R}_+^* \to \mathbb{R}$, and define $\psi = H'$. Let $v \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}_+^*)$ be a function such that

$$-\nabla^2 \psi(v) \geq C I_d \tag{4.17}$$

for some positive constant C.

Remark 4.12. Note carefully that we assume here v to be positive, which is only true in theorem 4.3. The rigorous proof of theorem 4.5 will wait until section 4.3.

Remark 4.13. Again, since Ω is closed, v is allowed to be nonzero at the boundary $\partial\Omega$. This is important, since the typical example for function v is, just like in subsection 4.1.2, $\psi^{-1}(a-\|x\|^2)$, for some $a \in \mathbb{R}$. Note that this inverse is not always well defined, and more generally, a positive function satisfying (4.17) might not exist. We will come back to this in section 4.3, as it will be of particular importance in the proof of theorem 4.5.

We consider the generalized entropy $\mathcal{F}$ defined by

$$\mathcal{F}(u) = \int_\Omega H(u) - u\psi(v), \tag{4.18}$$

for positive smooth functions u. Since H is convex, $\mathcal{F}(u) \geq \mathcal{F}(v)$ for any function u. The idea in this section is to consider the entropy along the flow of this very entropy, namely

$$\begin{aligned}
&\partial_t u = -\nabla \cdot (u\nabla\phi) && \text{in } \mathbb{R}_+^* \times \Omega && (4.19\text{a})\\
&\partial_\nu \phi = 0 && \text{in } \mathbb{R}_+^* \times \partial\Omega && (4.19\text{b})\\
&u(0,.) = u_0 && \text{in } \Omega. && (4.19\text{c})
\end{aligned}$$

where $u_0 \in \mathcal{C}_c^\infty(\overline{\Omega})$ is some positive initial data such that $\int_\Omega u_0 = \int_\Omega v$, and

$$\phi = \psi(v) - \psi(u). \tag{4.20}$$

Equation (4.19a) is a generalized Fokker-Planck equation: indeed, whenever $\psi = \log$ and v is the standard Gaussian, it is exactly a rewriting of the standard Fokker-Planck equation. We leave the technical study of this equation to section 4.3, and assume for now that the solution to this problem not only exists at all times, is unique, but also that it is positive and smooth (at least smooth enough to do the calculations we are about to do, say $\mathcal{C}^1$ with respect to time and $\mathcal{C}^3$ with respect to space). As a first remark, we see that the L^1 norm is preserved: using an integration by parts,

$$\begin{aligned}
\partial_t \int_\Omega u &= -\int_\Omega \nabla \cdot (u\nabla\phi)\\
&= -\int_{\partial\Omega} u\partial_\nu\phi = 0.
\end{aligned}$$

Now, consider the entropy along the flow, which we write for brievity $\Lambda(t) = \mathcal{F}(u(t,.))$ for $t \geq 0$. Differentiating the entropy with respect to time, we find, using an integration by parts,

$$\begin{aligned}
\Lambda'(t) &= \int_\Omega \partial_t u(\psi(u) - \psi(v))\\
&= \int_\Omega \nabla \cdot (u\nabla\phi)\phi = -\int_\Omega u\Gamma(\phi) =: -\mathcal{I}(u),
\end{aligned}$$

the boundary term being zero due to the Neumann boundary condition. The reason behind the choice of the flow should now appear more clearly: the derivative of the entropy is, up to the sign, what is sometimes called the *entropy creation* (or the generalized Fischer information) and written $\mathcal{I}(u)$. More importantly, this shows that Λ' is nonpositive. Now, since the entropy decreases along the flow, and since v is the only global minimum of $\mathcal{F}$ that has the same mass as u_0,it is reasonable to expect u to converge towards v in some sense as t goes to infinity, so we also assume that $\lim_{t\to+\infty} \mathcal{F}(u) = \mathcal{F}(v)$. In the good tradition of the Bakry-Émery method, we may differentiate the entropy once more to find the following proposition.

Proposition 4.14. *The second derivative of the entropy along the flow of* (4.19) *is given by*

$$\Lambda''(t) = 2\int_\Omega -\nabla^2\psi(v)(\nabla\phi,\nabla\phi)u + (\Delta\phi)^2 U_2(u) + \Gamma_2(\phi)U(u) - \int_{\partial\Omega}\partial_\nu\Gamma(\phi)U(u), \tag{4.21}$$

where the functions U *and* U_2 *are given by*

$$U(x) = xH'(x) - H(x), \text{ and } U_2(x) = xU'(x) - U(x),\ x\in\mathbb{R}_+^* \tag{4.22}$$

Proof. Recall that $\Lambda'(t) = -\int_\Omega u\Gamma(\phi)$. Let us differentiate this expression once more

$$\begin{aligned}\Lambda''(t) &= -\int_\Omega (2u\Gamma(\phi,\partial_t\phi) + \partial_t u\Gamma(\phi))\\ &= -2\int_\Omega \Gamma(\phi,\partial_t\phi)u + \int_\Omega \nabla\cdot(u\nabla\phi)\Gamma(\phi)\\ &= -2\int_\Omega \Gamma\left(\phi,\partial_t\phi + \frac{1}{2}\Gamma(\phi)\right)u + \int_{\partial\Omega} u\Gamma(\phi)\partial_\nu\phi.\end{aligned}$$

The boundary term vanishes under the boundary condition (4.19b). Differentiating $\phi = \psi(v) - \psi(u)$ with respect to time,

$$\begin{aligned}\partial_t\phi + \frac{1}{2}\Gamma(\phi) &= \nabla\cdot(u\nabla\phi)\psi'(u) + \frac{1}{2}\Gamma(\phi)\\ &= \Gamma(\psi(u),\phi) + \frac{1}{2}\Gamma(\phi) + U'(u)\Delta\phi\\ &= \Gamma(\psi(v),\phi) - \frac{1}{2}\Gamma(\phi) + U'(u)\Delta\phi.\end{aligned}$$

Now, on the one hand, applying equation (4.15) with $f = \psi(v)$, $g = h = \phi$, we find

$$\begin{aligned}-2\Gamma\left(\phi,\Gamma(\psi(v),\phi) - \frac{1}{2}\Gamma(\phi)\right)u &= -2\nabla^2\psi(v)(\nabla\phi,\nabla\phi)u - \Gamma\big(\psi(v),\Gamma(\phi)\big)u + \Gamma\big(\phi,\Gamma(\phi)\big)u\\ &= -2\nabla^2\psi(v)(\nabla\phi,\nabla\phi)u - \Gamma\big(U(u),\Gamma(\phi)\big),\end{aligned}$$

On the other hand,

$$\begin{aligned}\Gamma(\phi,U'(u)\Delta\phi)u &= \Gamma(\phi,\Delta\phi)U'(u)u + \Delta\phi\Gamma(\phi,U'(u))u\\ &= \Gamma(\phi,\Delta\phi)U_2(u)) + \Gamma(\phi,\Delta\phi)U(u) + \Delta\phi\Gamma(\phi,U_2(u))\\ &= \Gamma(\phi,U_2(u)\Delta\phi) + \Gamma(\phi,\Delta\phi)U(u).\end{aligned}$$

We may now use integration by parts to find that

$$\begin{aligned}\Lambda''(t) = -2\int_\Omega \nabla^2\psi(v)(\nabla\phi,\nabla\phi)u + \int_\Omega U(u)\Delta\Gamma(\phi) - \int_{\partial\Omega}\partial_\nu\Gamma(\phi)U(u)\\ + 2\int_\Omega(\Delta\phi)^2U_2(u) - 2\int_{\partial\Omega}U_2(u)\Delta\phi\partial_\nu\phi - 2\int_\Omega\Gamma(\phi,\Delta\phi)U(u),\end{aligned} \tag{4.23}$$

which concludes the proof, because $\Gamma_2(\phi) = \frac{1}{2}\Delta\Gamma(\phi) - \Gamma(\phi,\Delta\phi)$, and because the second boundary term is zero. □

• Now, differentiating the boundary condition (4.19b), and multiplying by $\nabla\phi$, we find that

$$0 = \nabla^2\phi(\nabla\phi, \nu) + \nabla\nu(\nabla\phi, \nabla\phi) = \frac{1}{2}\partial_\nu\Gamma(\phi) + \nabla\nu(\nabla\phi, \nabla\phi), \text{ on } \partial\Omega$$

which, since Ω is convex, implies that $\partial_\nu\Gamma(\phi)$ is nonpositive.

• By convexity, and since $H(0) = 0$, we know that $U \geq 0$, so that the boundary term is nonpositive.

• Next, we may use the fact that the Laplacian on $\mathbb{R}_d$ satisfies the $CD(0, d)$ curvature-dimension condition (4.16), which implies that

$$(\Delta\phi)^2 U_2(u) + \Gamma_2(\phi)U(u) \geq (\Delta\phi)^2\Big(U_2(u) + \frac{1}{d}U(u)\Big).$$

Assume that this last term is nonnegative, and recall that we chose v so that $-\nabla^2\psi(v) \geq CI$, so we may now claim that

$$\Lambda''(t) \geq 2C\int u\Gamma(\phi) = -2C\Lambda'(t). \tag{4.24}$$

With this inequality, we are now able to prove the following theorem:

Theorem 4.15. *For all $u_0 \in \mathcal{C}_c^\infty(\overline{\Omega})$ such that $\int_\Omega u_0 = \int_\Omega v$, the following inequality stands:*

$$\mathcal{F}(u_0) - \mathcal{F}(v) \leq \frac{1}{2C}\mathcal{I}(u_0). \tag{4.25}$$

Proof. The work is essentially done with proposition 4.14, inequality (4.24) being the heart of the now classical Bakry-Émery method. Integrating inequality (4.24) between 0 and t, we find that

$$-\Lambda'(t) \leq -\Lambda'(0)e^{-2Ct}$$

and then once again, between $t = 0$ and $t = +\infty$, yields

$$\Lambda(0) - \lim_{t\to+\infty}\Lambda(t) \leq -\frac{1}{2C}\Lambda'(0).$$

Now, recall that $\Lambda'(0) = -\mathcal{I}(u_0)$, and further assume that $\lim_{t\to+\infty}\mathcal{F}(u) = \mathcal{F}(v)$ to conclude. □

The assumption of convergence we made on the entropy will be rigorously proved in section 4.3. Nevertheless, we insist that it is a behavior naturally expected: indeed, the derivative of the entropy is strictly negative whenever $\nabla\phi \neq 0$, and $u = v$ is the only function verifying both $\nabla\phi = 0$ and $\int u = \int v$, so mass preservation must imply this convergence.

Remark 4.16. Even though we fixed the value of H at 0, theorem 4.15 is invariant under summation of H with a constant: if it is true for H, it remains true for $H + C$, where $C \in \mathbb{R}$. However, while ψ is invariant under this operation, U *is not*, and becomes $U - C$. This invariance property is recovered in equation (4.21) with the help formula

$$2\Big(\int_\Omega \Gamma_2(\phi) - (\Delta\phi)^2\Big) - \int_{\partial\Omega}\partial_\nu\Gamma(\phi) = 0. \tag{4.26}$$

4.2.3 Equivalent formulations of the entropy inequality

As has already been seen in subsection 4.1.2, inequality (4.25) is completely equivalent to Sobolev's inequality when $\Omega = \mathbb{R}^d$, and with

$$H(x) = -x^{1-1/d}, \qquad V(x) = 1 + \|x\|^2,$$
$$\psi(x) = H'(x) = -\frac{d-1}{d} x^{-1/d}, \qquad v(x) = \psi^{-1}(-V) = \left(\frac{d-1}{d}\left(1+\|x\|^2\right)\right)^{-d}.$$

The Sobolev inequality being a limit case of the GNS inequality, it turns out that just changing the exponent in the definition of H leads to the whole family. Indeed, inequality (4.25) with $H(x) = -x^\alpha/\alpha$ for some $\alpha \in \left(1 - \frac{1}{d}, 1\right)$ readily implies the GNS inequality family mentioned in [dPD02],with the help of proposition 4.1. The case $H(x) = x^\alpha$ for $\alpha > 1$ is also considered in [dPD02], and may be proved just the same with theorem 4.5.

It is worth noting that the choice $\overline{\Omega} = \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $V(x) = a + \|x\|^2$ implies that the normal derivative of V on $\partial\Omega$ is $\partial_\nu V(x) = 2x \cdot \nu = 0$. This simple but important fact, as will be made clearer in the proof of theorem 4.19, may then be used to prove sharp GNS or logarithmic Sobolev inequalities on $\mathbb{R}^d_+$, following the exact same calculations as for the whole Euclidean space case. See for instance [BCEF+17].

Following an idea in [Naz06], we may choose V to be $V(x) = a + \|x+e\|^2$, where e is a constant vector in $\mathbb{R}^d$. Bruno Nazaret succesfully used this idea to recover the sharp Sobolev inequality on the half space $\mathbb{R}^d_+$, and has later been used to prove trace GNS inequalities on the half-space in [BCEF+17], and on convex domains in [Zug19]. Again, it proves fruitful here, where theorem 4.3 leads to the same inequalities as those found in those articles in the $p = 2$ case. We will not prove them here, as the purpose of this chapter is not to be exhaustive, but will instead focus on two new inequalities, which proofs can be adapted for other inequalities.

We first turn to the proof of the trace logarithmic Sobolev inequality.

Proof of corollary 4.7. Fix $h \in \mathbb{R}$, $\overline{\Omega} = \mathbb{R}^d_+$, and let e be the dth unit vector, which is orthogonal to $\partial\mathbb{R}^d_+$. Let

$$H(x) = x\log(x) - x, \qquad V(x) = \frac{1}{2}\|x+he\|^2,$$
$$\psi = H' = \log, \qquad v(x) = \psi^{-1}(\beta_h - V) = \frac{1}{C_h} e^{-\frac{1}{2}\|x+he\|^2},$$

where $C_h =: \exp(\beta_h)$ has been chosen so that $\int_{\mathbb{R}^d_+} v = 1$, or in other words, $C_h = (2\pi)^{d/2}\gamma(\mathbb{R}^d_{+he})$, with γ being the standard Gaussian measure. With those choices, $U(x) = x$, $U_2(x) = 0$, so that theorem 4.3 applies with constant $C = 1$. For any nonnegative $u \in C^\infty_c(\mathbb{R}^d_+)$ such that $\int u = \int v = 1$, the following inequality stands

$$\int_{\mathbb{R}^d_+} H(u) - H(v) - (u-v)\psi(v) \le \frac{1}{2}\int_{\mathbb{R}^d_+} u\|\nabla\psi(u) + \nabla V\|^2.$$

Notice first that $v\psi(v) - H(v) = U(v) = v$, so that we are left with

$$\int_\Omega u\log u - u\log(v) \le \frac{1}{2}\int_\Omega \frac{\|\nabla u\|^2}{u} + \frac{1}{2}\int_\Omega u\|\nabla V\|^2 + \int_\Omega \nabla V \cdot \nabla u.$$

Now, noticing that $\frac{1}{2}\|\nabla V\|^2 = V = -\log(C_h v)$, the respective second terms on the right and left-hand side simplify. We integrate by parts the last term to find

$$\begin{aligned}\int_{\mathbb{R}^d_+} u\log u &\leq \frac{1}{2}\int_{\mathbb{R}^d_+} \frac{\|\nabla u\|^2}{u} - \log(C_h)\int_{\mathbb{R}^d_+} u - \int_{\mathbb{R}^d_+} u \operatorname{div}(x+he) + h\int_{\partial\mathbb{R}^d_+} u\partial_\nu(x+he) \\ &= -d - \log(C_h) + \frac{1}{2}\int_{\mathbb{R}^d_+}\frac{\|\nabla u\|^2}{u} - h\int_{\partial\mathbb{R}^d_+} u.\end{aligned} \tag{4.27}$$

The inequality we thus get is already a form of logarithmic Sobolev inequality, but we may go a little bit further to find a version that is similar to the standard inequalities. To do this, we rescale the function u and optimize with respect to the parameter. Indeed, inequality (4.27) stays true when replacing u by $u_\lambda = \lambda^d u(\lambda\,.)$, so, for all $\lambda > 0$, we find that

$$\int_{\mathbb{R}^d_+} u\log u \leq -d - \log(C_h) - d\log(\lambda) + \frac{\lambda^2}{2}\int_{\mathbb{R}^d_+}\frac{\|\nabla u\|^2}{u} - h\lambda\int_{\partial\mathbb{R}^d_+} u. \tag{4.28}$$

Now, we may choose for λ the value that minimizes the right-hand side of the inequality, but the resulting inequality is not pretty. Instead, we choose the λ that we would choose if $h = 0$, or, in other words, if there was no trace term and we were trying to prove the standard inequality. Hence, for

$$\lambda = \left(\frac{1}{d}\int_{\mathbb{R}^d_+}\frac{\|\nabla u\|^2}{u}\right)^{-\frac{1}{2}},$$

inequality (4.28) turns into inequality (4.13) and corollary 4.7 is proved. Note that for $u = v$ all the inequalities are, in fact, equalities, which proves optimality. □

Remark 4.17. Another version of a trace logarithmic Sobolev inequality has been found independantly in [BCEF+17] using optimal transport and an improved Borell-Brascamp-Lieb inequality.

Remark 4.18. Note that while we studied the case of $\overline{\Omega} = \mathbb{R}^d_+$, the proof can immediately be extended to convex cones, much like in [Zug19]. Writing $\overline{\Omega}$ as the epigraph of the convex function φ, the trace term would then become $\int_{\mathbb{R}^{d-1}} u(x,\varphi(x))dx$.

Instead of proving corollary 4.8, we instead showcase the method in a slightly more general case. In particular, the result showcases, just like for the logarithmic Sobolev inequality, the ease with which trace inequalities may be recovered.

Theorem 4.19. *Let $\alpha > 1$. For all $h \in \mathbb{R}$, and for all positive $u \in C^\infty(\mathbb{R}^d_+)$ such that $\int_{\mathbb{R}^d_+} u = 1$, the following inequality stands*

$$\int_{\mathbb{R}^d_+} u^\alpha \leq \left(a_h\left\|\nabla u^{\alpha-1/2}\right\|_{L^2(\mathbb{R}^d_+)} - hb_h\int_{\mathbb{R}^d_+} u^\alpha\right)\left\|\nabla u^{\alpha-1/2}\right\|_{L^2(\mathbb{R}^d_+)}^{-1/2\delta}, \tag{4.29}$$

where $\delta = d(\alpha-1)+1$, and a_h and b_h are explicit positive constants. Furthermore, there is equality when $u(x) = \left((\alpha-1)(\beta_h - \|x+he\|^2)\right)_+^{1/(\alpha-1)}$, where β_h is a normalizing constant.

Proof. Fix $\alpha > 1$, $h \in \mathbb{R}$, $\Omega = \mathbb{R}^d_+$, let e be the dth unit vector. Then, consider

$$H(x) = \frac{x^\alpha}{\alpha(\alpha-1)}, \qquad V(x) = \|x+he\|^2,$$

$$\psi(x) = H'(x) = \frac{x^{\alpha-1}}{\alpha-1}, \qquad v(x) = \psi^{-1*}(\beta_h - V) = \left((\alpha-1)(\beta_h - \|x+he\|^2)\right)_+^{1/(\alpha-1)},$$

where, again, β_h has been chosen so that $\int_{\mathbb{R}^d_+} v = 1$. In that case, $U(x) = (\alpha - 1)H(x)$ and $U_2(x) = (\alpha - 1)^2 H(x) \geq 0$, so that, again, theorem 4.5 applies: for all nonnegative $u \in \mathcal{C}^\infty(\mathbb{R}^d_+)$ such that $\int u = 1$,

$$\int_{\mathbb{R}^d_+} H(u) - H(v) + (u - v)V \leq \frac{1}{4} \int_{\mathbb{R}^d_+} u \|\nabla \psi(u) + \nabla V\|^2.$$

Expanding both sides, then doing an integration by parts and simplifying, yields

$$A \int_{\mathbb{R}^d_+} u^\alpha \leq B - h \int_{\partial \mathbb{R}^d_+} u^\alpha + D \int_{\mathbb{R}^d_+} \left\| \nabla u^{\alpha - 1/2} \right\|^2,$$

where A, B and D are positive constants given by

$$A = \frac{1}{(\alpha - 1)} + d, \qquad B = \int_{\mathbb{R}^d_+} v\big(v^{\alpha - 1} - \alpha \beta_h\big), \qquad D = \frac{\alpha}{(2\alpha - 1)^2}.$$

This inequality holding for any function of unit mass, we may, just like in the proof of theorem 4.7, rescale it with a certain parameter. Replacing u by $u_\lambda = \lambda^d u(\lambda\, .)$, we find that

$$A \int_{\mathbb{R}^d_+} u^\alpha \leq B\lambda^{-\delta + 1} - \frac{h}{\lambda} \int_{\partial \mathbb{R}^d_+} u^\alpha + D\lambda^{\delta + 1} \int_{\mathbb{R}^d_+} \left\| \nabla u^{\alpha - 1/2} \right\|^2,$$

where $\delta = d(\alpha - 1) + 1$. Taking the minimum of that expression with respect to $\lambda > 0$ does not result in an expression very legible, so we instead evaluate it in the argument of the minimum corresponding to the case $h = 0$, which is

$$\lambda = \left(\frac{B(\delta - 1)}{D(\delta + 1) \int \left\| \nabla u^{\alpha - 1/2} \right\|^2} \right)^{\frac{1}{2\delta}},$$

thus leading to theorem 4.19. Writing $f = u^{\alpha - 1/2}$ and letting $h = 0$ proves corollary 4.8. □

Remark 4.20. Interestingly, trace GNS inequalities in the convex case (i.e. when $\alpha \in (1 - 1/d, 1)$) admit a nicer formulation. This is made possible in the calculations because the constant B changes sign, and can then be absorbed by the gradient term using Young's inequality, which just so happens to maintain optimality [BCEF+17].

4.3 Study of the degenerate parabolic PDE

In this section, we fix some convex domain $\overline{\Omega} \Subset \mathbb{R}^d$. Our goal is to show that the calculations we did in section 4.2 are valid. In this context, we are only interested in proving the entropy inequality (4.25), allowing us to make use of solutions to an approximated problem rather than the nontrivial system (4.19). We propose a quick and (almost) self-contained proof of the entropy inequality (4.25). However, the study of solutions to the full problem is both relevant and delicate, and many open questions remain. We refer for instance to the work of [CJM+01].

Equations (4.19a) and (4.19b) are not only nonlinear, but also degenerate. Equation (4.19a) may be written

$$\partial_t u = \Delta U(u) + \text{l.o.t},$$

where the function U is given by $U(x) = x\psi(x) - H(x)$, as introduced in section 4.2. We want to modify the function U in order to have both a lower and an upper bound on the parabolicity, so that the system falls in the scope of standard parabolic theory.

To that effect, for $\varepsilon > 0$, we choose an approximation of U, written U_ε, that coincides with U in the range $[\varepsilon, 1/\varepsilon]$. To regain parabolicity, we want U_ε to be strictly increasing and affine outside of that range, but we also want it smooth, so we impose that U_ε is affine in the range $\mathbb{R}\backslash\big[\varepsilon/2, \varepsilon^{-1}+\varepsilon\big]$ instead, as pictured on figure 4.1.

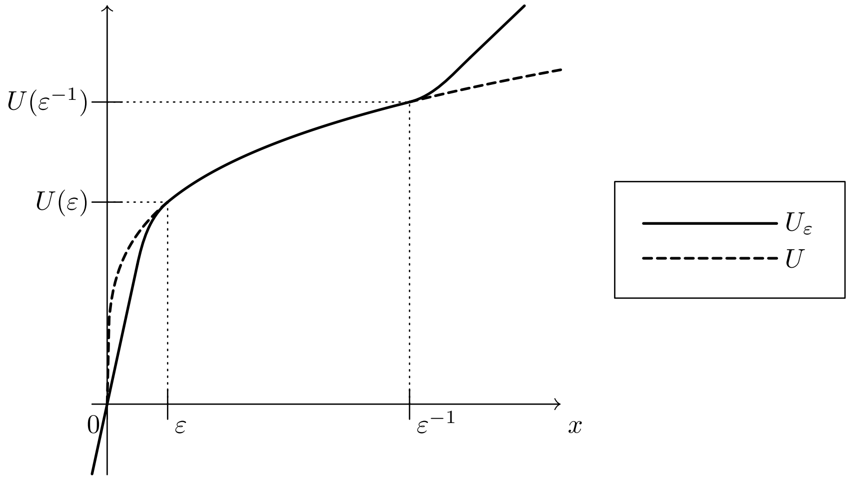

Figure 4.1: U_ε, an approximation of U.

From this choice of U_ε, and from the fact that $U'(x) = x\psi'(x)$, we may also define ψ_ε and H_ε on $\mathbb{R}_+^*$, by $\psi_\varepsilon(x) = \int_\varepsilon^x \frac{U_\varepsilon'(t)}{t}dt + \psi(\varepsilon)$, $H_\varepsilon(x) = x\psi_\varepsilon(x) - U_\varepsilon(x)$, so that they coincide respectively with ψ and H on the interval $[\varepsilon, 1/\varepsilon]$. With this definition, ψ_ε is equivalent to a log on $(0, \varepsilon/2)$; for this reason, we use the function U_ε in the formulation of the desingularized problem, because it is well-defined and smooth on the whole of $\mathbb{R}$, which is needed if we want to directly apply the classical parabolic theory.

Thus, consider the problem (4.19) in which we replace U with U_ε

$$\partial_t u = \Delta U_\varepsilon(u) - \nabla\cdot(u\nabla\psi_\varepsilon(v)) \quad \text{in } \mathbb{R}_+^*\times\Omega, \tag{4.30a}$$
$$-\partial_\nu U_\varepsilon(u) + u\partial_\nu\psi_\varepsilon(v) = 0 \quad \text{in } \mathbb{R}_+^*\times\partial\Omega, \tag{4.30b}$$
$$u(0,.) = u_0 \quad \text{in } \Omega. \tag{4.30c}$$

4.3.1 Study of the desingularized problem

Theorem 4.21. *Assume that $\overline{\Omega}$ is smooth and bounded, and that u_0 and v are smooth functions on $\overline{\Omega}$ such that u_0 verifies the compatibility condition (4.30b). Then system (4.30) admits a unique smooth solution on $\overline{\Omega}\times\mathbb{R}$.*

This is the only classical result we invoke, and we will not prove it. Its proof can be found in [LSU68, Theorem 7.4, p. 491]. Even though there exists versions of comparison principles in [LSU68], we formulate our own here. Let us first define subsolutions and supersolutions.

Definition 4.22. Let u_1 (resp. u_2) be a smooth function defined on $\mathbb{R}_+\times\overline{\Omega}$. We say that u_1 is a subsolution (u_2 is a supersolution) of (4.30) if for all time $t\geq 0$,

$$\begin{cases}\partial_t u_1 \leq \Delta U_\varepsilon(u_1) - \nabla\cdot(u_1\nabla\psi_\varepsilon(v)) & \text{in } \Omega\\ -\partial_\nu U_\varepsilon(u_1) + u_1\partial_\nu\psi_\varepsilon(v)\geq 0 & \text{in } \partial\Omega,\end{cases} \quad\text{and}\quad \begin{cases}\partial_t u_2 \geq \Delta U_\varepsilon(u_2) - \nabla\cdot(u_2\nabla\psi_\varepsilon(v)) & \text{in } \Omega\\ -\partial_\nu U_\varepsilon(u_2) + u_2\partial_\nu\psi_\varepsilon(v)\leq 0 & \text{in } \partial\Omega.\end{cases} \tag{4.31}$$

Remark 4.23. This definition and the following proposition are more general than needed, since we will only consider actual solutions of the system, but it doesn't require any additional work, so we might as well prove it.

Proposition 4.24 (Comparison principle)**.** *If u_1 is a subsolution and u_2 is a supersolution to* (4.30) *such that $u_1 \leq u_2$ at time $t = 0$, then $u_1 \leq u_2$ for all times $t \geq 0$.*

Proof. Let u_1 and u_2 be as in (4.31). Their time derivatives $\partial_t u_1, \partial_t u_2$ are continuous functions on a compact with respect to the space variable, and thus bounded at all times, hence, by domination, the following quantities are well-defined and equal:

$$\partial_t \int_\Omega (u_1 - u_2)_+ = \int \partial_t (u_1 - u_2)_+.$$

Next, for $m \in \mathbb{N}^*$, choose ρ_m to be a (non decreasing) C^1 function approximating $\chi_{\mathbb{R}_+^*}$. For example, consider $\rho_m(x) = \rho(mx)$, where

$$\rho(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ -2x^3 + 3x^2 & \text{if } x \in (0,1) \\ 1 & \text{if } x \geq 1, \end{cases}$$

so that $\|\rho_m'\|_\infty = \frac{3}{2}m$. Using this approximation, we may write that

$$\partial_t \int_\Omega (u_1 - u_2)_+ = \lim_{m \to +\infty} \int_\Omega \partial_t (u_1 - u_2) \rho_m(Z), \tag{4.32}$$

where Z can be any function such that $Z(x) > 0 \iff u_1(x) > u_2(x)$. We fix $Z = U_\varepsilon(u_1) - U_\varepsilon(u_2)$. Since the function U_ε is strictly increasing on $\mathbb{R}$, such a Z constitutes a valid choice for equation (4.32). Using (4.31) and integrating by parts, we find

$$\begin{aligned} \int_\Omega \partial_t (u_1 - u_2) \rho_m(Z) &\leq \int_\Omega \big(\Delta U_\varepsilon(u_1) - \nabla \cdot (u_1 \nabla \psi_\varepsilon(v)) - \Delta U_\varepsilon(u_2) + \nabla \cdot (u_2 \nabla \psi_\varepsilon(v))\big) \rho_m(Z) \\ &\leq \int_\Omega (-\nabla U_\varepsilon(u_1) + \nabla U_\varepsilon(u_2) + (u_1 - u_2)\nabla \psi_\varepsilon(v)) \cdot (\rho_m'(Z) \nabla Z) \\ &= \int_\Omega ((u_1 - u_2) \nabla Z \cdot \nabla \psi_\varepsilon(v) - \Gamma(Z)) \rho_m'(Z) \\ &\leq \frac{3}{2} m \int_{\{0 < Z < 1/m\}} |u_1 - u_2| \|\nabla Z\| \|\nabla \psi_\varepsilon(v)\|, \end{aligned}$$

since $0 \leq \rho_m' \leq 3m/2$, and $\Gamma(Z) \geq 0$. Finally, the mean value theorem applied to U_ε yields

$$|u_1 - u_2| \leq \left\| \frac{1}{U_\varepsilon'} \right\|_\infty |U_\varepsilon(u_1) - U_\varepsilon(u_2)|,$$

which, applied to $x \in \{Z < 1/m\}$, is enough to take the limit and conclude that

$$\lim_{m \to +\infty} \int_\Omega \partial_t (u_1 - u_2) \rho_m(Z) \leq 0,$$

thereby concluding the proof. ☐

Let us now look into positive functions. If $u > 0$, we then write $\phi_\varepsilon = \psi_\varepsilon(v) - \psi_\varepsilon(u)$, and the equation (4.30a) takes the form

$$\partial_t u = -\nabla \cdot (u \nabla \phi_\varepsilon),$$

allowing us to determine the positive stationary solutions. It is clear that v is one of them, and, more generally, all functions u such that $\phi_\varepsilon = \mathrm{cst}$, are such solutions, and, as it turns out, they are the only ones. Indeed, if u is such a solution, testing equation (4.30a) against ϕ_ε, and then integrating by parts and using (4.30b), we find

$$\begin{aligned} 0 &= -\int_\Omega \phi_\varepsilon \nabla \cdot (u \nabla \phi_\varepsilon) \\ &= \int_\Omega u \Gamma(\phi_\varepsilon). \end{aligned}$$

Furthermore, notice that, by definition, $\psi_\varepsilon(x) = a \log(x) + b$ for all $x \in (0, \varepsilon/2)$, and also for all $x > \varepsilon^{-1} + \varepsilon$, but with different constants. Therefore, ψ_ε is actually a bijection between $\mathbb{R}_+^*$ and $\mathbb{R}$, and we may define, for any $\alpha \in \mathbb{R}$, the positive stationary solution

$$v_\alpha = \psi_\varepsilon^{-1}(\psi_\varepsilon(v) + \alpha). \tag{4.33}$$

These functions, being solutions, are both super- and subsolutions; and for any constant $C > 0$, we can find $\alpha_1 < \alpha_2$ such that $0 < v_{\alpha_1} < C < v_{\alpha_2}$ everywhere in $\overline{\Omega}$, thus giving a priori L^∞ bounds on positive solutions, as well as $L^{-\infty}$ bounds, both uniform in time.

4.3.2 Proof of the entropy inequality

We will now prove the entropy inequality (4.25) for the approximated entropy $\mathcal{F}_\varepsilon$. To that effect, owing to theorem 4.21 we now know that the system (4.19) has a smooth solution, so that proposition 4.14 is valid for the desingularized entropy flow. From there, three facts remain to be shown to conclude the proof of theorem 4.15: we will prove that

1. $-\nabla^2 \psi_\varepsilon(v) \geq CI$;
2. everywhere in $\overline{\Omega}$,
$$U_{\varepsilon,2}(u) + \frac{1}{d} U_\varepsilon(u) = \left(\frac{1}{d} - 1\right) U_\varepsilon(u) + u U_\varepsilon'(u) \geq 0; \tag{4.34}$$
3. the entropy $\mathcal{F}_\varepsilon(u)$ converges to $\mathcal{F}_\varepsilon(v)$ when $t \to +\infty$.

For the first point, we may assume that ε has been chosen so that $\varepsilon \leq v \leq \varepsilon^{-1}$ everywhere in $\overline{\Omega}$. This implies that $\psi_\varepsilon(v) = \psi(v)$, and trivially, $\nabla^2 \psi_\varepsilon(v) \leq -CI$.

The second point boils down to the construction of U_ε. We have assumed that $U_2(u) + \frac{1}{d} U(u) \geq 0$, so inequality (4.34) is of course satisfied whenever $\varepsilon < u < \varepsilon^{-1}$. We also made it so that for all $r < \varepsilon/2$, $U_\varepsilon(r) = ar$ for some $a > 0$. Then $U_{\varepsilon,2}(r) = 0$, which directly implies that inequality (4.34) is satisfied in that range, and the same argument works for the range $r > \varepsilon + \varepsilon^{-1}$. It thus suffices to show that inequality (4.34) is satisfied in the ranges $(\varepsilon/2, \varepsilon)$ and $\left(\varepsilon^{-1}, \varepsilon^{-1} + \varepsilon\right)$. It turns out that the choice of the smooth connections can be made so that it is true: to convince oneself of this fact, notice that it suffices to choose a smooth nonnegative connection for the quantity $U_{\varepsilon,2}(u) + \frac{1}{d} U_\varepsilon(u)$ on the interval $(\varepsilon/2, \varepsilon)$ (and also on the interval $(\varepsilon^{-1}, \varepsilon^{-1} + \varepsilon)$) and then use the following identity to recover U_ε

$$\left(\frac{1}{d} - 1\right) U_\varepsilon(x) + x U_\varepsilon'(x) = x^{2-1/d} \left(x^{-1+1/d} U_\varepsilon(x)\right)',$$

which also guarantees that $U_\varepsilon' > 0$. Finally, we prove the following lemma:

Lemma 4.25. *If $\int_\Omega u_0 = \int_\Omega v$, then u converges towards v almost everywhere, and*

$$\lim_{t\to+\infty} \mathcal{F}_\varepsilon(u) = \mathcal{F}_\varepsilon(v).$$

Proof. The comparison principle 4.24 ensures that there exists constants $0 < m < M$ such that $m \le u \le M$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}_+$. Recall the proof of theorem 4.15, we showed that

$$0 \le \mathcal{I}_\varepsilon(u) \le e^{-2Ct}\mathcal{I}_\varepsilon(u_0),$$

so it is clear that $\lim_{t\to+\infty} I_\varepsilon(u) = 0$, which readily implies that $\lim_{t\to+\infty}\|\nabla\phi_\varepsilon\|_2 = 0$, since $\mathcal{I}_\varepsilon(u) = \int_\Omega u\Gamma(\phi_\varepsilon) \ge m\int_\Omega \Gamma(\phi_\varepsilon) = m\|\nabla\phi_\varepsilon\|_2^2$. The fact that u is uniformly bounded on $\Omega\times\mathbb{R}_+$ implies that ϕ_ε is, too. Thus, ϕ_ε is uniformly bounded in $H^1(\Omega)$, and we may extract a sequence of real numbers $(t_k)_{k\in\mathbb{N}}$ such that $\phi_\varepsilon|_{t=t_k} \rightharpoonup \phi_*$ weakly in $H^1(\Omega)$. By weak lower semicontinuity, $\|\nabla\phi_*\|_2 \le \liminf_{k\to+\infty}\|\phi_\varepsilon|_{t=t_k}\|_2 = 0$, so that ϕ_* is in fact a constant.

Now, since u is also, in fact, bounded in $H^1(\Omega)$, we may, without loss of generality, assume that $u|_{t=t_k}$ converges almost everywhere to some function u_*. By uniqueness of the limit,

$$\phi_* = \psi(v) - \psi(u_*),$$

so that u_* is actually one of the positive stationary solutions of (4.19) defined in equation (4.33). But the fact that the flow is mass-preserving, combined with the dominated convergence theorem, implies that

$$\int_\Omega u_* = \int_\Omega u_0 = \int_\Omega v,$$

but the only stationary solution that has the same mass as v is v itself, so that $u_* = v$. Indeed,

$$\frac{d}{d\alpha}v_\alpha = \frac{1}{\psi_\varepsilon' \circ \psi_\varepsilon^{-1}(\psi_\varepsilon(v)+\alpha)} > 0.$$

Finally, we may conclude that u converges almost everywhere to v as $t\to+\infty$, and invoking, once again, dominated convergence, $\lim_{t\to+\infty}\mathcal{F}_\varepsilon(u) = \mathcal{F}_\varepsilon(v)$. $\square$

At this stage, we have proved the following: there exists $\varepsilon_0 > 0$, depending only on v and Ω, such that for all $\varepsilon \in (0,\varepsilon_0)$,

$$\mathcal{F}_\varepsilon(u_0) - \mathcal{F}_\varepsilon(v) \le \frac{1}{2C}\mathcal{I}_\varepsilon(u_0) \tag{4.35}$$

for all $u_0 \in \mathcal{C}_c^\infty(\overline{\Omega})$, provided that $\int_\Omega u_0 = \int_\Omega v$ and that u_0 satisfies the approximated compatibility condition (4.30b). Now, fix some positive smooth function u_0 satisfying the regular compatibility condition (4.19b), and that has the same mass as v, then fix any $0 < \varepsilon < \min(\varepsilon_0, \min(u_0))$. By construction, the approximated entropy of u_0 is then the same as the regular entropy of u_0, and the same goes for the entropy production, so inequality (4.35) is valid, and is identical to inequality (4.25).

4.3.3 Extension to convex domains and generic smooth positive functions

We have now proved that the entropy inequality

$$\mathcal{F}(u_0) - \mathcal{F}(v) \le \frac{1}{2C}\mathcal{I}(u_0) \tag{4.36}$$

holds true for smooth and positive functions u_0 defined on a compact, convex and smooth set, as long as they verify the compatibility condition (4.19b), and that they have the same mass as the function v. Working on the inequality (4.36) rather than the partial differential equation (4.19), we may generalize this result by lifting the constraints.

Let us first extend the class of functions for which inequality (4.36) holds true. Let $\overline{\Omega} \subset \mathbb{R}^d$ be compact, convex and smooth, and let u be a positive and smooth function defined on $\overline{\Omega}$. We want to construct $\tilde{u} = u + g$, an approximation of u that verifies the compatibility condition (4.19b): on $\partial\Omega$,

$$\begin{aligned} &\partial_\nu(\psi(v) - \psi(\tilde{u})) = 0 \\ \iff\ &\partial_\nu(g+u)\psi'(g+u) = \partial_\nu\psi(v), \end{aligned}$$

and it is thus sufficient to find a function g, small in some sense, such that, on $\partial\Omega$,

$$g = 0, \tag{4.37}$$

$$\partial_\nu g = \frac{\partial_\nu \psi(v)}{\psi'(u)} - \partial_\nu u = \frac{\nabla(\psi(v) - \psi(u))}{\psi'(u)} \cdot \nu. \tag{4.38}$$

In dimension 1, the construction is somewhat straightforward. Assume just for now that $\overline{\Omega} = \mathbb{R}_+$. The problem reduces to finding a reasonably small function that is zero on $\partial\mathbb{R}_+ = \{0\}$, and that has an assigned slope at that same point. We thus construct a function that looks like a small ridge: choose η, defined on $\mathbb{R}_+$, such that it is smooth, has compact support in $[0, 3]$, and is equal to identity on $[0, 1]$, like pictured on figure 4.2. Then, the function $x \mapsto C\delta\eta(x/\delta)$, where C is the desired slope at zero, satisfies everything we need: its L^∞ norm tends towards zero when $\delta \to 0$, the L^∞ norm of its derivative is bounded, and its support is included in $[0, 3\delta]$.

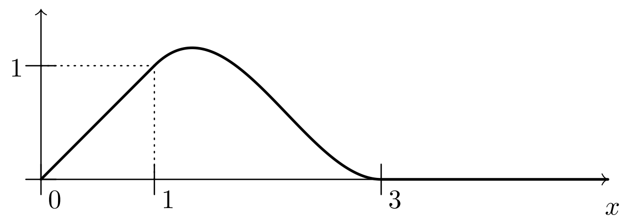

Figure 4.2: A ridge function η.

Let us now return to more general Ω, and extend this construction. Note that since Ω is smooth, its boundary admits a neighbordhood verifying the *unique nearest point* property (for a short reference, see for instance [Foo84]). In other words, there exists an open neighbordhood U of $\partial\Omega$ and a smooth function $P : U \to \partial\Omega$ such that for all $x \in U$, $d(x, \partial\Omega) = \|x - P(x)\|$. This function P is called the projection onto $\partial\Omega$, it is smooth, and its gradient at x is orthogonal to the tangent space at $P(x)$. Thus, for all $\delta \in (0, \delta_0)$, with δ_0 sufficiently small, the function

$$g(x) = \delta\eta(d(x, \partial\Omega)/\delta)\left(\frac{\partial_\nu\psi(v)}{\psi'(u)} - \partial_\nu u\right)(P(x))$$

is well-defined, smooth, and satisfies both assumptions (4.37) and (4.38). Furthermore, writing $C_g = \left(\int_\Omega u\right)\left(\int_\Omega u + g\right)^{-1}$ and invoking the dominated convergence theorem, it is quite clear that $\lim_{\delta\to 0} \mathcal{F}(C_g(u+g)) = \mathcal{F}(u)$, and $\lim_{\delta\to 0} \mathcal{I}(C_g(u+g)) = \mathcal{I}(u)$, and thus the compatibility condition is lifted. Next, we want to further extend the result to more general domains. Let $\overline{\Omega}$ be convex and compact and $u \in C_c^\infty(\overline{\Omega})$. The domain $\overline{\Omega}$ may be approximated from within by smooth convex sets:

Lemma 4.26. *For each $\varepsilon > 0$, there exists $\overline{\Omega}_\varepsilon \subset \overline{\Omega}$ such that $\overline{\Omega}_\varepsilon$ is smooth and convex, and $|\Omega \backslash \Omega_\varepsilon| < \varepsilon$.*

Using this fact, for any $\varepsilon > 0$, inequality (4.36) holds for the restriction of $C_\varepsilon u|_{\Omega_\varepsilon}$, where C_ε is the normalisation constant $C_\varepsilon = \left(\int_\Omega u\right)\left(\int_{\Omega_\varepsilon} u\right)^{-1}$. The dominated convergence theorem allows to take the limit when $\varepsilon \to 0$, proving inequality (4.36) for compact domains. We may finally extend the result for unbounded domains by considering $\overline{\Omega} \cap B(0, R)$, where $\overline{\Omega}$ is assumed to be closed and convex. Again, dominated convergence allows to take the limit $R \to +\infty$, whence we proved theorem 4.3 in its full generality.

Proof of the lemma. Let $\overline{\Omega} \subset \mathbb{R}^d$ be compact and convex. Fix the distance function $d_\Omega : x \mapsto d(x, \Omega)$, and choose some smoothing kernel $\rho : \mathbb{R}^d \to \mathbb{R}_+$, such that $\rho \in C_c^\infty(\mathbb{R}_d)$, and satisfiying $\int \rho = 1$ and $B_1 \subset \operatorname{supp}(\rho) \subset B_2$. Define, for $\delta > 0$, $\rho_\delta = \delta^{-d}\rho(./\delta)$. The function $d_\Omega * \rho_\delta$ is smooth, and also convex since $\rho \geq 0$ and Ω is convex. Now, notice that

$$\{x \in \Omega, d(x, \partial\Omega) > 2\delta\} \subset \{d_\Omega * \rho_\delta = 0\} \subset \{x \in \Omega, d(x, \partial\Omega) > \delta\}.$$

We now claim that there exists $t_0 > 0$ such that $\{d_\Omega * \rho_\delta < t_0\} \subset \Omega$. This is due to the continuity of $d_\Omega * \rho_\delta$, and also the fact that $\overline{\Omega}$ is compact. Now, by Sard's theorem, there exists a $t \in (0, t_0)$ such that $\{d_\Omega * \rho_\delta < t\}$ is smooth, and convex since it is a sublevel set of a convex function, and

$$\{x \in \Omega, d(x, \partial\Omega) > 2\delta\} \subset \{d_\Omega * \rho_\delta < t\} \subset \Omega.$$

Finally, notice that $\{x \in \Omega, d(x, \partial\Omega) > 2\delta\} + B_{2\delta} = \mathring{\Omega}$, and thus, Brunn-Minkowski's inequality allows us to conclude that we may have chosen δ small enough so that $|\Omega \backslash \{d_\Omega * \rho_\delta < t\}| < \varepsilon$, which concludes the proof. □

4.3.4 Generalized inverse and concave inequalities

In this subsection, we prove theorem 4.5. As mentioned in remark 4.13, positive functions v satisfying $-\nabla^2\psi(v) \geq CI_d$ might not alway exist. The natural example of when this is a problem is the flow related to the porous medium equation: when $H(x) = \frac{x^\alpha}{\alpha(\alpha-1)}$, with $\alpha > 1$, then ψ is a one to one map from $\mathbb{R}_+$ onto itself, and for any choice of $a \in \mathbb{R}$, the function $x \mapsto a - \|x\|^2$ takes negative values. We would like to still make sense of this computation in that case.

Instead of fixing the function v, choose a function $V \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ such that its Hessian is bounded below, $\nabla^2 V \geq CI_d$. While $\psi^{-1}(-V)$ might not be well defined, we may consider, for $\varepsilon > 0$, the function $v_\varepsilon = \psi_\varepsilon^{-1}(-V)$. Recall that ψ_ε behaves like a natural logarithm on a neighbourhood of zero, as well as towards infinity, so that v_ε is well defined for all $\varepsilon > 0$. Furthermore, $v_\varepsilon \in C^\infty(\Omega, \mathbb{R}_+^*)$. As ε goes to 0, v_ε converges to the so-called *generalized inverse* of ψ, applied to $-V$, which we will write ψ^{-1*}:

- if $\psi(0^+) < -V < \psi(+\infty)$, then it is clear that $\lim_{\varepsilon\to 0} \psi_\varepsilon^{-1}(-V) = \psi^{-1}(-V)$;
- if $-V \leq \psi(0^+)$, then, in particular, $-V < \psi_\varepsilon(\varepsilon)$, and so $0 < \psi_\varepsilon^{-1}(-V) < \varepsilon$, so that $\lim_{\varepsilon\to 0} \psi_\varepsilon^{-1}(-V) = 0$;
- finally, if $-V \geq \psi(+\infty)$, $\psi_\varepsilon^{-1}(-V) > 1/\varepsilon$, proving that $\lim_{\varepsilon\to 0} \psi_\varepsilon^{-1}(-V) = +\infty$.

Let $v = \psi^{-1*}(-V) = \lim_{\varepsilon\to 0} v_\varepsilon$. Note that the function v is, in general, not even differentiable. For example, in the case where $H(x) = x^2/2$ and $V(x) = 1 - \|x\|^2$, the generalized inverse of $\psi(x) = x$ and the limit function v are given by

$$\psi^{-1*}(x) = x_+, \quad v(x) = \left(1 - \|x\|^2\right)_+.$$

We do not really know how to make sense of the case where v is not finite, so we further assume that $-V < \psi(+\infty)$ everywhere.

We may now fix an $\varepsilon > 0$ and return to the previous subsections, where we replace the function v by the function v_ε. The study of the partial differential equation, subsection 4.3.1 remains unchanged, and the conclusions are the same. As far as subsection 4.3.2, points 2. and 3. are unchanged too, and point 1. is trivial: $-\nabla^2\psi_\varepsilon(v_\varepsilon) = \nabla^2 V \geq C I_d$ by hypothesis, and the conclusion is still valid. If $\overline{\Omega}$ is compact, convex and smooth, for all smooth positive functions u_0 satisfying both $\int_\Omega u_0 = \int_\Omega v_\varepsilon$ and the compatibility condition (4.30b),

$$\mathcal{F}_\varepsilon(u_0) - \mathcal{F}_\varepsilon(v_\varepsilon) \leq \frac{1}{2C}\mathcal{I}_\varepsilon(u_0).$$

Again, following section 4.3.3 we may lift the compatibility condition, as well as the smoothness condition for Ω. We will tackle the boundedness only later, out of convenience. Let us write the entropy inequality fully.

$$\int_\Omega H_\varepsilon(u_0) - H_\varepsilon(v_\varepsilon) - (u - v_\varepsilon)\psi_\varepsilon(v_\varepsilon) \leq \frac{1}{2C}\int_\Omega u_0 \|\nabla\psi_\varepsilon(u_0) - \nabla\psi_\varepsilon(v_\varepsilon)\|^2. \tag{4.39}$$

By construction, $\psi_\varepsilon(v_\varepsilon) = -V$. Furthermore, for a fixed positive $u_0 \in \mathcal{C}^\infty(\Omega)$, we may choose $\varepsilon_0 > 0$ so that $\varepsilon_0 < u_0 < 1/\varepsilon_0$, and equation (4.39) rewrites

$$\int_\Omega H(u_0) - H_\varepsilon(v_\varepsilon) + (u - v_\varepsilon)V \leq \frac{1}{2C}\int_\Omega u_0\|\nabla\psi(u_0) + \nabla V\|^2. \tag{4.40}$$

for any $0 < \varepsilon < \varepsilon_0$. We just need to pass to the limit to prove theorem 4.5. By Fatou's lemma,

$$\begin{aligned}\int_\Omega H(u_0) - \liminf_{\varepsilon\to 0} H_\varepsilon(v_\varepsilon) + (u - v)V &\leq \liminf_{\varepsilon\to 0}\left(\int_\Omega H(u_0) - H_\varepsilon(v_\varepsilon) + (u - v_\varepsilon)V\right)\\ &\leq \frac{1}{2C}\int_\Omega u_0\|\nabla\psi(u_0) + \nabla V\|^2,\end{aligned}$$

so it suffices to show that $\liminf_{\varepsilon\to 0} H_\varepsilon(v_\varepsilon) \leq H(v)$. Let $x \in \overline{\Omega}$, we are faced with two cases, since we assumed that v is finite everywhere.

- If $-V(x) \in (\psi(0^+), \psi(+\infty))$, then $0 < v(x) < +\infty$ and, since H_ε coincides with H on the interval $[\varepsilon, 1/\varepsilon]$, it is clear that $\lim_{\varepsilon\to 0} H_\varepsilon(v_\varepsilon(x)) = H(v(x))$.
- If $-V(x) \leq \psi(0^+)$, then $v_\varepsilon(x) \to 0$, and since $H_\varepsilon(0) = 0$,

  $$H_\varepsilon(v_\varepsilon(x)) = \int_0^{v_\varepsilon(x)} \psi_\varepsilon(t)dt \leq v_\varepsilon(x)\psi_\varepsilon(v_\varepsilon(x)) = -v_\varepsilon(x)V(x),$$

  so $\lim_{\varepsilon\to 0} H_\varepsilon(v_\varepsilon(x)) \leq 0 = H(v(x))$.

This concludes the proof of theorem 4.5. Notice that while we proved it for $\mathcal{C}^\infty$ functions, it makes sense for the function $u_0 = v$ even though it is not necessarily differentiable. Indeed, on the interior of $\operatorname{supp}(v)$, v is smooth and $\nabla\psi(v) = -V$. On the other hand, on the interior of $\Omega\setminus\operatorname{supp}(v)$, $\psi(v)$ is still well defined, because v can only be zero when $-V \leq \psi(0^+)$, which means that $\psi(0^+) \in \mathbb{R}$, and we may conclude that $v\|\nabla\psi(v) + V\|^2 = 0$ on that set. In this sense, inequality (4.12) is optimal, because both sides are equal to 0 when $u_0 = v$.

Remark 4.27. In the particular case of $H(x) = \frac{x^\alpha}{\alpha(\alpha-1)}$, we use generalized inverses only when $\alpha > 1$. It just so happens that in that case, the function $U(x) = \frac{x^\alpha}{\alpha}$ is convex, and thus $U_2 \geq 0$. This implies that to get inequality (4.24) and ultimately to the entropy inequality theorem 4.3, we only need a $CD(0,\infty)$ assumption, and not any more the stronger $CD(0,d)$ assumption. This does not matter so much in our case because we are only considering $\mathbb{R}^d$, but it might prove useful on manifolds.

Notations

$\|\,.\,\|$, $\|\,.\,\|_*$	norm, dual norm; (2.16)
$\|u\|_{L^p(\Omega)}$, $\|u\|_p$	norm of the Lebesgue space $L^p(\Omega)$
$\mathcal{C}^\infty(\Omega)$	smooth functions on Ω
$\mathcal{C}_c^\infty(\Omega)$	smooth functions with compact support in Ω
$\mathcal{C}_b^\infty(\Omega)$	smooth bounded functions on Ω
$\mathcal{C}^{k,\beta}$	space of the k times differentiable functions of which the kth derivative is β-Hölder continuous
$W^{1,p}(\Omega)$	Sobolev space of L^p functions with L^p distributional gradient
$\mathcal{H}^s$, $\|\,.\,\|$	s-dimensional Hausdorff measure, Lebesgue measure
$\{u > t\}$	(strict) superlevel set of the function u
$\mathcal{P}(\mathbb{R}^d)$	probability measures on $\mathbb{R}^d$
$\mathcal{P}_2(\mathbb{R}^d)$	probability measures with finite second order moment
$\mathcal{P}_{ac}(\mathbb{R}^d)$	absolutely continuous probability measures w.r.t the Lebesgue measure
$\square$	infimal convolution; 2.5
epi, epi_s, dom	epigraph, strict epigraph, essential domain; 2.6
supp(u)	support of function u
$B(x_0, r)$	open ball of center x_0 and radius r
W^*	Legendre transform of W; 2.9
$\mathring{\Omega}$, $\overline{\Omega}$	interior, closure of the set Ω
Γ, Γ_2	Carré du champ operator, iterated Carré du champ operator; 3.1, 3.2
∇, ∇^2, Δ	gradient, Hessian, Laplacian
Ric	Ricci curvature
$CD(\rho, n)$	curvature-dimension condition
$(\,.\,)_+$	positive part
ψ^{-1*}	generalized inverse; 4.4
χ_A	indicator function of the set A

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Inégalités de Sobolev optimales : approches dynamiques

Résumé. Dans cette thèse, on propose des outils dynamiques pour démontrer des inégalités de type Sobolev. Dans un premier temps, dans le cadre de la théorie de Brunn-Minkowski, une inégalité de Borell-Brascamp-Lieb étendue avec le transport optimal permet de retrouver non seulement l'inégalité de Gagliardo-Nirenberg-Sobolev optimale sur $\mathbb{R}^d$, mais aussi une version à trace sur les épigraphes convexes. La clé est l'étude d'une convolution infimale donnant lieu à des équations d'Hamilton-Jacobi.

Dans un deuxième temps, la méthode de Bakry-Émery est utilisée à plein dans des variétés riemanniennes. Sous hypothèse de courbure-dimension $CD(0, n)$, avec $n > 0$, on généralise une inégalité de Poincaré à poids, et puis on étend aussi l'inégalité de Beckner aux variétés vérifiant $CD(\rho, n)$, cette fois pour $\rho > 0$ et $n < 0$. Finalement, on revisite les méthodes d'entropie dans des domaines du plan euclidien pour démontrer des inégalités de Sobolev généralisées, menant par exemple à une inégalité de Sobolev logarithmique à trace.

Mots-clés : Brunn-Minkowski, équation de Hamilton-Jacobi, inégalité de Sobolev, inégalité de Poincaré, inégalité de Gagliardo-Nirenberg-Sobolev, condition de courbure-dimension, entropie, flot gradient.

Dynamical approaches to sharp Sobolev inequalities

Abstract. In the present document, we propose dynamical tools to study Sobolev-type inequalities. First, within the Brunn-Minkowski theory, an extended Borell-Brascamp-Lieb inequality, proved with optimal transport, is used to recover not only the sharp Gagliardo-Nirenberg-Sobolev inequality on the Euclidean space, but also prove a trace version of it on convex epigraphs. The study of an infimal convolution is crucial, leading to Hamilton-Jacobi equations.
We then make use of the Barky-Émery method, both in manifolds and on $\mathbb{R}^d$. Under curvature-dimension hypothesis $CD(0, n)$, with $n > 0$, we generalize previous work on a weighted Poincaré inequality, and we also extend the Beckner inequality to manifolds satisfying $CD(\rho, n)$, this time with $\rho > 0$ and $n < 0$. Lastly, we revisit entropy methods and gradient flows in domains of the Euclidean space to prove so-called generalized Sobolev inequalities, allowing us to prove, for example, a sharp trace logarithmic Sobolev inequality.

Keywords: Brunn-Minkowski theory, Hamilton-Jacobi equation, Sobolev inequality, Poincaré inequality, Gagliardo-Nirenberg-Sobolev inequality, curvature-dimension condition, entropy, gradient flow.

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