Famous sequences

Niveau(x) concerné(s):	Première (et/ou Terminale)
Notions abordées :	Suites, vocabulaire
Nombre de séances :	5
Séquence préparée par :	Julien Mougeot
Sources ***Séance 1***	http://creativityofmathematics.weebly.com/lazy-caterers-sequence.html https://en.wikipedia.org/wiki/Lazy_caterer%27s_sequence
Séance 2	https://en.wikipedia.org/wiki/Collatz_conjecture https://www.youtube.com/watch?v=m4CjXk_b8zo
Séance 3	https://en.wikipedia.org/wiki/Look-and-say_sequence https://www.youtube.com/watch?v=LpjX3kHXcR0
Séance 4	*The magic of maths*, Arthur Benjamin https://www.youtube.com/watch?v=SjSHVDfXHQ4
Séance 5	https://www.youtube.com/watch?v=t3d0Y-JpRRg

Descriptif des séances :

Séance 1 : The lazy caterer's sequence (découverte d'une suite, vocabulaire de base)

– Pair-work (worksheet #1 à compléter, voir page 2).
– Lors du bilan, faire écrire le vocabulaire de base nécessaire : sequence, term, term-to-term rule (or recursive formula), position-to-term rule (or explicit formula), nth term.
– S'il reste du temps (ou au début de la séance 2), donner d'autres exemples de suites célèbres : odd numbers, square numbers, prime numbers, triangular numbers...

Séance 2 : The Collatz conjecture

– Worksheet #2 à compléter (page 3).
– Visionner la vidéo "The Simplest Impossible Problem" (https://www.youtube.com/watch?v=m4CjXk_b8zo) afin de répondre aux questions 5 à 8.

Séance 3 : Look-and-say sequence

– En groupes, les élèves font les challenges (voir page 4).
– Visonner la vidéo (https://www.youtube.com/watch?v=LpjX3kHXcR0).
– Bilan : worksheet #3 à compléter (page 5).

Séance 4 : The Fibonacci sequence

– Travail en groupes (immortal rabbits). Les élèves cherchent à répondre au problème suivant : Baby rabbits are too young to produce in their first month, they take one month to mature. Suppose that each pair produces a new pair of baby rabbits every month thereafter, for the rest of their never-ending lives. If we start with one pair of baby rabbits, how many pairs of rabbits will there be twelve months later?
– Vidéo Arthur Benjamin (https://www.youtube.com/watch?v=SjSHVDfXHQ4)
– Worksheet #4 à compléter (page 6).

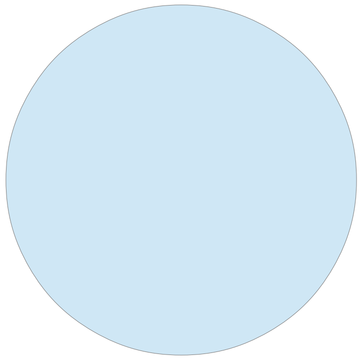
Séance 5 : The chessboard problem (suite géométrique)

– Vidéo (https://www.youtube.com/watch?v=t3d0Y-JpRRg)
– Worksheet #5 à compléter (page 7).

Worksheet #1 - Food for thought: the lazy caterer’s sequence

Imagine that you want to cut a delicious round pizza… But you don’t want to cut it into equal slices. Instead, you want to find the biggest number of pieces you can make with each cut. For example:

- the maximum number of pieces for 1 cut is 2;
- the maximum number of pieces for 2 cuts is 4.



1. What is the maximum number of pieces for:
  - **a.** 3 cuts?
  - **b.** 4 cuts?
  - **c.** 5 cuts?

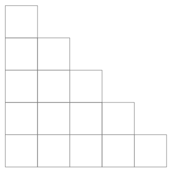
These numbers form **the lazy caterer’s sequence**.

2. Can you find a rule? Can you guess the maximum number of pieces for 6 cuts?

3. Describe the perfect cutting strategy to be sure that you get the maximum number of pieces.

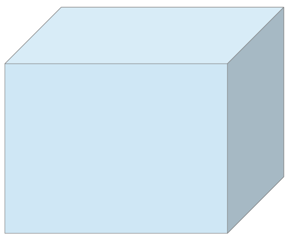
4. What is the maximum number of pieces for n cuts?
Check your formula using question **1.**

Fun fact: Fill the rows of a right-angled triangle with consecutive natural numbers, starting with a 1 in the top left corner (this triangle is called Floyd’s triangle, it is named after Robert Floyd, 1936-2001, an American computer scientist). The numbers along the left edge of the triangle form the lazy caterer's sequence.



3D challenge! Cut your cake and eat it…

5. Still hungry? Imagine that you have a big cake. It is so big and thick that you can cut it horizontally as well as vertically.
  - **a.** What is the maximum number of pieces for 2 cuts? 3 cuts? 4 cuts?
  - **b.** Can you find a rule? (It’s not a piece of cake!)



Worksheet #2 - The Collatz conjecture, aka the 3*n*+1 conjecture

Let us consider the **sequence** defined with this **term-to-term rule** (or **recurrence relation**):

Start with any positive integer. Then each term is obtained as follows:
- if the term is even, divide it by two;
- if the term is odd, triple it and add one to get the next term.

Repeat the process indefinitely.

1. Start with 10, apply the term-to-term rule and write down what you get after each step:

10 → … → … → … → … → … → … → … → … → … → … → … → …

2. Choose three integers from 1 to 25 and write the corresponding sequence each time:

… →

… →

… →

3. What do you notice? Can you guess what the Collatz conjecture is?

……………………………………………………………………………………………………

4. If you start with 10, the highest number in the sequence is 16, and you reach 1 after 6 steps.
We say that the height of 10 is 16 and the total stopping time of 10 is 6.

What is the total stopping time of 16? ……… of 15? ………

What is the height of 15? ………

What is the longest total stopping time you got from numbers you tried? …………

Could you find a number greater than 500 with a total stopping time less than 10? ……………

Watch the video 'The simplest impossible problem'. https://www.youtube.com/watch?v=m4CjXk_b8zo

5. What is the height of 27? ………

6. Is it true that, thanks to powerful computers, mathematicians have proved this conjecture?

……………………………………………………………………………………………………

7. The conjecture is named after Lothar Collatz (1910-1990), who introduced the idea in 1937. It is also known as the 3*x*+1 problem, the Ulam conjecture, Kakutani's problem, Hasse's problem or the Syracuse problem. With all this interest, what joke appeared in the 1960s?

……………………………………………………………………………………………………

……………………………………………………………………………………………………

8. If this conjecture is false, what could happen if a starting number doesn't reach the cycle 4-2-1?

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

"Mathematics is not yet ready for such problems" Paul Erdős (1913-1996)

Look-and-say sequence - Challenges

Les élèves travaillent par groupes de 4 et tentent de réussir les challenges suivants. Distribuer d'abord le challenge #1 à chaque groupe. Dès qu'un groupe réussit un challenge, lui distribuer le suivant.

Challenge #1: Try to figure out what comes next in this sequence: 1, 11, 21, 1211, 111221, ...	Challenge #2: True or false? No digits other than 1, 2 and 3 appear in the sequence.	Challenge #3: Start with a different number: Start with 5. Start with 100.	Challenge #4: True or false? If we start with any digit *d* from 0 to 9 then *d* will remain indefinitely as the last digit of the sequence.	Challenge #5: There is only one look-and-say sequence that doesn't grow indifinitely. It starts with a two-digit number. Could you find it?
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Worksheet #3 – Look-and-say sequence

Watch the video (https://www.youtube.com/watch?v=LpjX3kHXcR0)

Here are the first terms of a **look-and-say sequence**:

> 1 is read off as "one 1" or 11.
> 11 is read off as "two 1s" or 21.
> 21 is read off as "one 2, then one 1" or 1211.
> 1211 is read off as "one 1, one 2, then two 1s" or 111221.
> 111221 is read off as "three 1s, two 2s, then one 1" or 312211.

These kind of sequences were first analyzed by …………….……. a British mathematician born in 1937. He noted that they had interesting properties. For instance, starting with the number ……. yields an infinite loop. But when seeded with any other number, the sequence grows in some very specific ways. The ratio between the amount of digits in two consecutive terms converges to a number known as ……………………………
$\lambda \approx$ ………
This means that the amount of digits increases by about ……….. with every step in the sequence.

Conway proved that every sequence eventually splits ("decays") into a sequence of "atomic elements", which are finite subsequences that never again interact with their neighbors. There are 92 elements containing the digits 1, 2, and 3 only, which John Conway named after the chemical elements up to uranium. There are also two "transuranic" elements for each digit other than 1, 2, and 3.

Look-and-say sequences have some practical applications. For example, run-length encoding (RLE), a ……………………………… that was once used for …………………………………… and ………………………………….., is based on a similar concept.

Worksheet #4 – The Fibonacci sequence

Watch the video (https://www.youtube.com/watch?v=SjSHVDfXHQ4)

The **Fibonacci numbers** are the numbers in the following integer sequence, called the **Fibonacci sequence**, and characterized by the fact that every number after the first two is the sum of the two preceding ones: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

The sequence (F_n) of Fibonacci numbers is defined by the **recurrence relation**:
$F_n = F_{n-1} + F_{n-2}$ $(n \geq 3)$ with **seed values** $F_1 = 1$ and $F_2 = 1$.

The Fibonacci sequence is named after Italian mathematician ………………………… known as Fibonacci. His 1202 book *Liber Abaci* taught the Western world ………………………………….……… …………………………… that we use today.

Fibonacci numbers appear in nature surprisingly often. Give two examples:
…………………………………………………………………………………………………….
…………………………………………………………………………………………………….

Fibonacci numbers display beautiful number patterns.

F_n	1	1	2	3	5	8	13	21	34
F_n^2								441	1 156

What pattern can you see here? ……………………………………………………………………………

Now, let's add the squares of the first few Fibonacci numbers.

$+ \quad + \quad = \quad = \quad \times$

$+ \quad + \quad + \quad = \quad = \quad \times$

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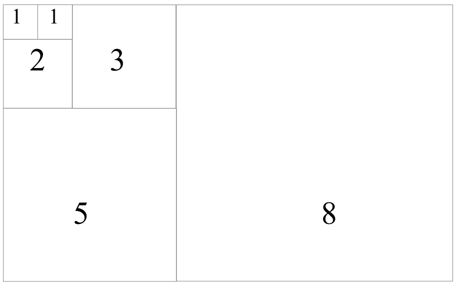

It's even more satisfying to understand why these patterns are true.
What is the area of the giant rectangle on the right?
On the one hand, it's the ……….. of the areas of the …………… inside it.
Area = …………………………………………
On the other hand, because it's a rectangle, the area is equal to …………………………………………
Area = …………………………………………
That's why $1^2 + \quad + \quad + \quad + \quad + \quad = \quad \times$

Now check this out: 13 ÷ 8 = ………………
÷ = ………………
÷ = ………………
÷ = ………………
÷ = ………………

These ratios get closer and closer to about …………………, known as the …………………………

Mathematics is not just solving for ***x***, it's also figuring out wh***y***. Arthur Benjamin.

Worksheet #5 – The chessboard problem

Watch the video (https://www.youtube.com/watch?v=t3d0Y-JpRRg)

Folklore suggests that when the creator of the game of chess, the Grand Vizier of India, showed his invention to the country's king, the ruler was highly impressed. He was so impressed, he told the inventor to name a prize of his choice.

1. What had the Grand Vizier asked for his reward?

2. Explain why the king was happy to comply.

3. The following table lists the first five assignments of grains of wheat to squares on the board. Represent the grains of wheat as exponential expressions.

Square #	Grains of wheat	Exponential expression
1	1	
2	2	
3	4	
4	8	
5	16	

Useful vocab:
This sequence is called a …………………………….………… , because the term to term rule is to ……………………………..
The factor between the terms is …… It is called …………………………
If the rule is to add or subtract a number each time, it is called an ………………… sequence.
The difference between the terms is called …………………………

4. How many grains of wheat are assigned to the 64^{th} square? ………………………

5. By the time that the fifth square is reached, the chessboard contains a total of …… grains of wheat. How would you determine the total amount of grains of wheat for all 64 squares?

6. Were you surprised at how quickly the numbers grew? Why or why not?

7. What would happen if we could lay these many grains of wheat end to end?