The constitution of mathematical documents:
Elementary pieces and arrangements

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TWO TYPES OF MATHEMATICAL SOURCES FROM EARLY IMPERIAL CHINA

First type: Sources yielded by tomb archeology (or the antiquities market)

Our text comes from
the first complete mathematics book ever found in a tomb from early imperial China

Book of mathematical procedures (筭數書)

A few elements of context

Tomb sealed *ca* 186BCE, at Zhangjiashan 張家山 (Hubei)
excavated in 1983-1984; Peng Hao 彭浩 published the whole text in 2001.

— The tomb of an official working at the administrative level of a county

— A book reflecting the duties of such an official: reference book? Textbook? Other?

— In the same tomb, among other books,

《二年律令》Han slips from Zhangjiashan: *Statutes and edicts of year 2 (of Empress Lü)*

For this book, we have a kind of social/cultural context, but no hint about readership/reading

Book on mathematical procedures 筭數書 (*Suanshushu*)

Recently excavated from a tomb sealed before *ca* 186 BCE

"石 a（率）
lü-ing on the basis of the *dan*

石a（率）之術曰：以所買〓（賣）為法，以得錢乘一石數以為實。其下有半者倍之，
少半者三之，有斗、升、斤、兩、朱（銖）者亦皆/74/破其上，令下從之為法。錢所
乘亦破如此。/75/

Procedure for *lü*-ing on the basis of the <u>*dan*</u>:

[1]   One takes what is exchanged as <u>divisor</u>.

[2]   One multiplies, by the cash obtained, the quantity of 1 *dan*,
which is taken as <u>dividend</u>.

[3]   Those for which, in their lower (rows),
          there is a half, one doubles them;
                    one third, one triples them.

[4]   Those for which there are *dou* and *sheng*, *jin, liang* and *zhu*,
          one ***likewise* breaks up** *all* their upper (rows),

[5]   one makes the (rows) below join them, (yielding a result) which is taken as **<u>divisor</u>**.

[6]   **What the cash** was **multiplying** is ***likewise* broken up like this**."

Problem "Trading salt" (slips 76-77)   **1 *dan* 4 *dou* 5 *sheng* 1/3 *sheng* salt** traded for **150 cash.**
If one wants to "*lü*" according to **the *dan***, how much cash?

[1-2]   以所買＿（賣）為法，以得錢乘一石數以為實。
One takes what is exchanged as divisor. One multiplies, by the cash obtained, the quantity of
1 dan, which is taken as dividend.

Dividend	1 *dan* multiplied by 150 cash	
Divisor	1 *dan* 4 *dou* 5 *sheng* 1 3 *sheng*	Upper Middle Lower

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If one wants to "*lü*" according to **the *dan***, how much cash?

Dividend	1 *dan* multiplied by 150 cash		
Divisor	1	*dan*	
	4	*dou*	Upper
	5	*sheng*	Middle
1	3	*sheng*	Lower

[3] 其下有半者倍之，少半者三之 Those for which, in their lower (rows), there is a half, one doubles them; one third, one triples them.

Dividend	1 *dan* multiplied by 150 cash		
Divisor	3	*dan*	
	12	*dou*	Upper
	15	*sheng*	Middle
1		*sheng*	Lower

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Units of weight in slip 47: "廿（二十）四 朱（銖）一兩」，三百八十四 朱（銖）一斤」，萬一千五百廿（二十）朱（銖）一鈞」，四萬六千八十朱（銖）一石。/47/ Twenty-four *zhu* is one *liang*」, Three hundred and eighty-four *zhu* is one *jin*」, eleven thousand five hundred and twenty *zhu* is one *jun*」, forty-six thousand and eighty *zhu* is one *dan*."

Units of capacity: 10 *sheng* for 1 *dou*, 100 *sheng* for 1 *dan*.

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If one wants to "*lü*" according to **the *dan***, how much cash?

Dividend	1 *dan* multiplied by 150 cash	
Divisor	3 *dan* 12 *dou* 15 *sheng* 1 *sheng*	Upper Middle Lower

[4]　有斗、升、斤、兩、朱（銖）者亦皆//破其上 Those for which there are *dou* and *sheng, jin, liang* and *zhu*, one *likewise* breaks up *all* their upper (rows)

Dividend	1 *dan* multiplied by 150 cash	
Divisor	300 120 15 1	Upper Middle Lower

Units of weight in slip 47: "廿（二十）四 朱（銖）一兩」，三百八十四 朱（銖）一斤」，萬一千五百廿（二十）朱（銖）一鈞」，四萬六千八十朱（銖）一石。/47/ Twenty-four *zhu* is one *liang*」, Three hundred and eighty-four *zhu* is one *jin*」, eleven thousand five hundred and twenty *zhu* is one *jun*」, forty-six thousand and eighty *zhu* is one *dan*."
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If one wants to "*lü*" according to **the** *dan*, how much cash?

Dividend	1 *dan* multiplied by 150 cash	
Divisor	300 120 15 1	Upper Middle Lower

[5] 令下從之為法。One makes the (rows) below join them, (yielding a result) which is taken as divisor.

Dividend	1 *dan* multiplied by 150 cash	
Divisor	436	

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石a（率）之術曰：以所買＝（賣）為法，以得錢乘一石數以為實。其下有半者倍之，少半者三之，有斗、升、斤、兩、朱（銖）者亦皆/74/破其上，令下從之為法。錢所乘亦破如此。/75/

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If one wants to "*lü*" according to **the *dan***, how much cash?

Dividend	1 *dan* multiplied by 150 cash	
Divisor	436	

[6] 錢所乘亦破如此 "What the cash was multiplying is *likewise broken up like this*."

Dividend	300 multiplied by 150 cash	
Divisor	436	

Correlative ("likewise") transformation of 1 *dan*

— multiplied by 3 (breaking all units)

— its upper units are broken in *sheng*

ONLY now, execution of first multiplication!

$$\frac{cash\ multiplied\ by\ 1\ unit\ (dan)}{quantity\ bought} = \frac{cash\ multiplied\ by\ 1\ unit\ u_1}{q_1u_1 + q_2u_2 + \dfrac{m}{n}u_2}$$

$$= \frac{cash\ multiplied\ by\ n.1\ unit\ u_1}{nq_1u_1 + nq_2u_2 + mu_2}$$

and if $u_1 = k_1u_2$

$$= \frac{cash\ multiplied\ by\ n\,k_1u_2}{nq_1k_1u_2 + nq_2u_2 + mu_2}$$

Actions taken in Problem "Trading salt" 1 *dan* 4 *dou* 5 *sheng* 1/3 *sheng* salt traded for **150 cash**

If one wants to "*lü*" according to **the *dan***, how much cash?

Reminder: Units of capacity, forming a decimal system: 1 *dan* is 10 *dou,* 1 *dou* is 10 *sheng*

术(術)曰：三鹽之數以為法，亦三一石之升數，以錢乘之為實。/76-7/

Procedure:
[(4)-3-(5)-1] One triples the quantity of salt, which is taken as divisor.
[6]   One *likewise* **triples the quantity of *sheng* of** 1 *dan*, and
[2]   with the cash, one multiplies this, which is taken as dividend

General procedure
Procedure for *lü*-ing on the basis of the *dan*:
[1]   One takes what is exchanged as divisor.
[2]   One multiplies, by the cash obtained, the quantity of 1 *dan*,
       which is taken as dividend.
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            there is a half, one doubles them;
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[4]   Those for which there are *dou* and *sheng*, *jin, liang* and *zhu*,
            one *likewise* **breaks up** *all* their upper (rows),
[5]   one makes the (rows) below join them, (yielding a result) which is taken as **divisor**.
[6]   **What the cash was multiplying** is *likewise* **broken up like this**.

TWO TYPES OF MATHEMATICAL SOURCES FROM EARLY IMPERIAL CHINA

Second type: Sources handed down through the written tradition

Next example taken from
the first book handed down whose title (and content…) refers to, and only to, mathematics

The Nine Chapters on mathematical procedures (九章算術)

1st century CE (?)

— a book considered a "classic 經" from early on

Commentaries on *The Nine Chapters* to be handed down with the Canon

— a commentary selected (Liu Hui 劉徽 (263))/
— a subcommentary composed (by Li Chunfeng et al.)
in the context of the production of an anthology *The Ten Canons of mathematics*
under Li Chunfeng's supervision and presented to the throne in 656

For this book, we have no social/cultural context at the time, but we have hints about reading

The Nine Chapters on mathematical procedures (1st C CE)
Chapter 1

經分

術曰：以人數爲法，錢數爲實，實如法而一。有分者通之； 重有分者同而通之。

DIRECTLY SHARING

PROCEDURE:

[1] ONE TAKES THE QUANTITY OF PERSONS AS DIVISOR,

[2] THE QUANTITY OF CASH AS DIVIDEND,

[3] AND ONE DIVIDES THE DIVIDEND BY THE DIVISOR.

[4] IF THERE IS ONE TYPE OF PART, ONE *MAKES* THEM *COMMUNICATE*.

[5] IF THERE ARE SEVERAL TYPES OF PARTS,
     ONE *EQUALIZES* THEM AND HENCE *MAKES* THEM *COMMUNICATE*.

Euclid's algorithm : Finding the g. c. d. of *two integers*

```
FUNCTION NGCD(NA, NB)
     IA = NA
     IB = NB
  1  IF (IB.NE.0) THEN
      ITEMP = IA
      IA = IB
      IB = MOD(ITEMP, IB)
      GOTO 1
     END IF
     NGCD = IA
     RETURN
           END
```

```
while m is greater than zero:
   If n is greater than m, swap m and n.
   Subtract n from m.
n is the GCD
```

"經率   Directly *lü*-ing

術曰：以**所求率**乘錢數爲實，以**所買率**爲法，實如法得一。

Procedure:

     One multiplies by the ***lü* of what one seeks** the quantity of cash, which makes the <u>dividend</u>.

     One takes the ***lü* of what is bought** as <u>divisor</u>,

     dividing the dividend by the divisor yields the result."

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PROCEDURE:
[1]   ONE TAKES THE QUANTITY OF PERSONS AS DIVISOR,
[2]   THE QUANTITY OF CASH AS DIVIDEND,
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[5]   IF THERE ARE SEVERAL TYPES OF PARTS,
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BM 13901 —Mesopotamian text
(Hoyrup, *Lengths, widths, surfaces*, Springer, 2002, p. 11-13**)**

Thureau-Dangin
« J'ai **additionné** la surface et (le côté de) mon carré : 45'.
Tu poseras 1°, l'*unité*. Tu **fractionneras** en deux 1° : 30'.
Tu **multiplieras** (entre eux) [30'] et 30' : 15'. Tu **ajouteras** 15' à 45' : 1°.
1° **est le carré de** 1°. 30', que tu as multiplié (avec lui-même), de 1° tu **soustrairas** :
30' est le (côté du) carré. »

Hoyrup
"1.   The surface **and** my confrontation: 45' is it. 1, the **projection**,
2.   you posit. The **moiety** of 1 you **break**, 30' and 30' you **make hold** each other.
3.   15' to 45' you **append**: by 1, 1 is the equalside. 30' which you have made hold
4.   in the inside of 1 you **tear out**: 30' the confrontation
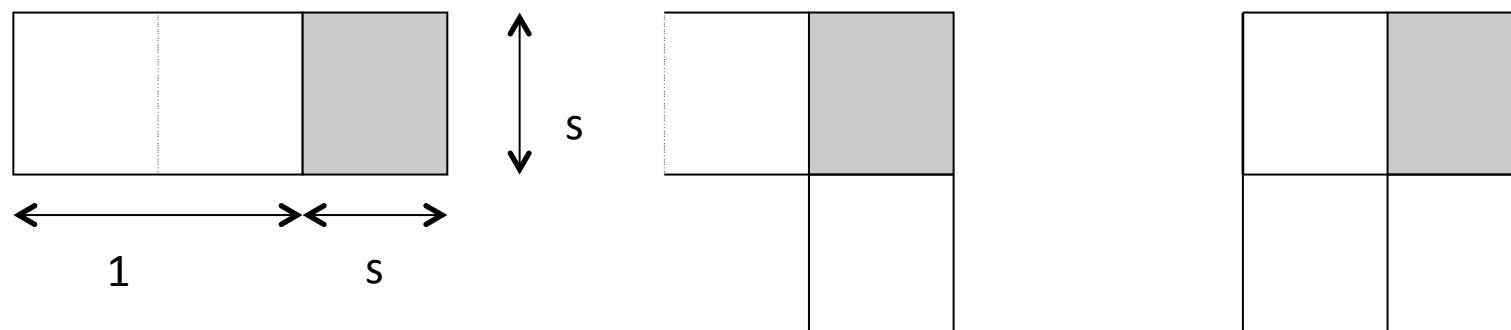
"*lü-ing by the dan*" or "Determining the unit (price) on the basis of the *dan*"

"One takes what is exchanged as divisor.

One multiplies, by the cash obtained, the quantity of one *dan*, which is taken as dividend.

Those for which, in their lower (rows),

there is a half, one doubles them; one third, one three-folds them.

Those for which there are   *dou* and *sheng*,   *jin, liang* and *zhu*,   **likewise** one **breaks up** *all* their upper (rows),

one makes the (rows) below join them, (yielding a result) which is taken as divisor.

What the cash was multiplying, **likewise, is broken up like this**.

石【行＋率】(率）石【行＋率】(率)之術曰：以所買＿（賣）為法，以得錢乘一石數以為實。其下有半者倍之，少半者三之，有斗、升、斤、兩、朱（銖）者亦皆//破其上，令下從之為法。錢所乘亦破如此。/74-75/

"Trading salt   Suppose one has one *dan* four *dou* five *sheng* a third *sheng* salt and that when trading it, one obtains one hundred and fifty cash. If one wants to **"*lü*" this on the basis of the *dan***, how much cash does this make? One says: one hundred and three cash ninety-five four hundred thirty-***ths cash. Procedure:"One **three-folds** the quantity of salt, which is taken as divisor. **Likewise**, one **three-folds** the quantity of *sheng* of one *dan*, and, with the cash, one multiplies this, which is taken as dividend.賈鹽　今有鹽一石四斗五升少半升，賈取錢百五十，欲石【行＋率】（率）之，為錢幾何」。曰：百三錢四百　卅(三十)□分錢九十五。术(術)/76/曰：三鹽之數以為法，亦三一石之升數，以錢乘之為實。/77/

$$\frac{cash \ \ times \ \ the \ \ triple \ \ of \ \ 1 \ \ dan \ \ expressed \ \ in \ \ sheng}{the \ \ triple \ \ of \ \ 1 \ \ dan \ \ 4 \ \ dou \ \ 5 \ \ sheng \ \ 1/3 \ \ sheng \ [in \ \ sheng]}$$

The Book of mathematical procedures
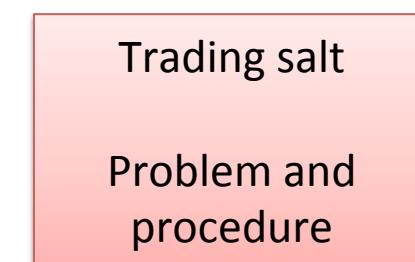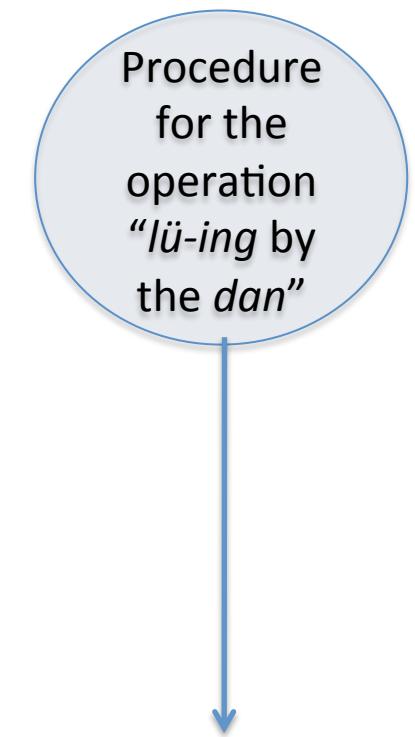
Procedure
for the
operation
"*lü-ing* by
the *dan*"

Trading salt

Problem and
procedure

"**Reducing the width**   **Procedure for looking for the reduced width**: One first puts the width, then if one says below there are a given amount of *bu*, **one takes one as a given amount**, one takes a half as a given amount⌐, one takes one third as a given amount, one **accumulates the partitionnings** so as to get rid of the parts of that which one looks for. One sums these (results), what makes the divisor. Then, **relying on this**, one puts the field, two hundred and forty *bu*, and one **likewise** takes one as a given amount, which is taken as the *bu* of the area (*jibu*). Eliminating from the *bu* of the area, what is like the divisor yields one *bu* of the length. What does not fill up the *bu*, one names its parts with the divisor⌐. One further says: If one restores , … /164-166/"   少廣 救(求)少廣之術曰：先直(置)廣，即曰：下有若干步，以一為若干，以半為若干，⌐以三分為若干，積分以盡所救(求)分,同之、以為法。即【耒昔】(藉)直(置)田二百【廿廿】（四十）步,亦以一為若干,以為積步,除積步,如法得從（縱）一步。不盈步者,以法命其分⌐。有(又)曰：復之，…

"Reducing the width   Reducing the width: If the width is one *bu* and a half *bu*, one takes one as two, ⌐a half as one; summing these (gives) three, which is taken as divisor. One hence puts two hundred and forty *bu* and likewise one takes one as two. Eliminating, when it is like the divisor, it yields, for the length, one *bu*, (i.e., dividing by the divisor yields the length in *bu*), ⌐which makes, for the length, one hundred and sixty *bu*. Accordingly, multiplying [this] by one *bu* and a half *bu* …. 少廣：廣一步、半步，以一為二，半為一，同之三以為法，即(置)二百四十步，亦以一為二,除，如法得從(縱)一步，為從(縱)百六十步。因以一步、半步乘. 167

$$\frac{240 \; bu}{1 \; bu + \dfrac{1}{2} \; bu} = \frac{240 \; bu \; times \; 2 \; (half \; bu)}{2 + 1 \; (half \; bu)}$$

Dividend				240		480	
Divisor	1			2		3	
		1	2	1	2		2

Section "Reducing the width" in the *Book of mathematical procedures*

Procedure for the operation "Reducing the width"

1st Problem and procedure

2nd Problem and procedure

3rd   ...   10th
Problems and procedures
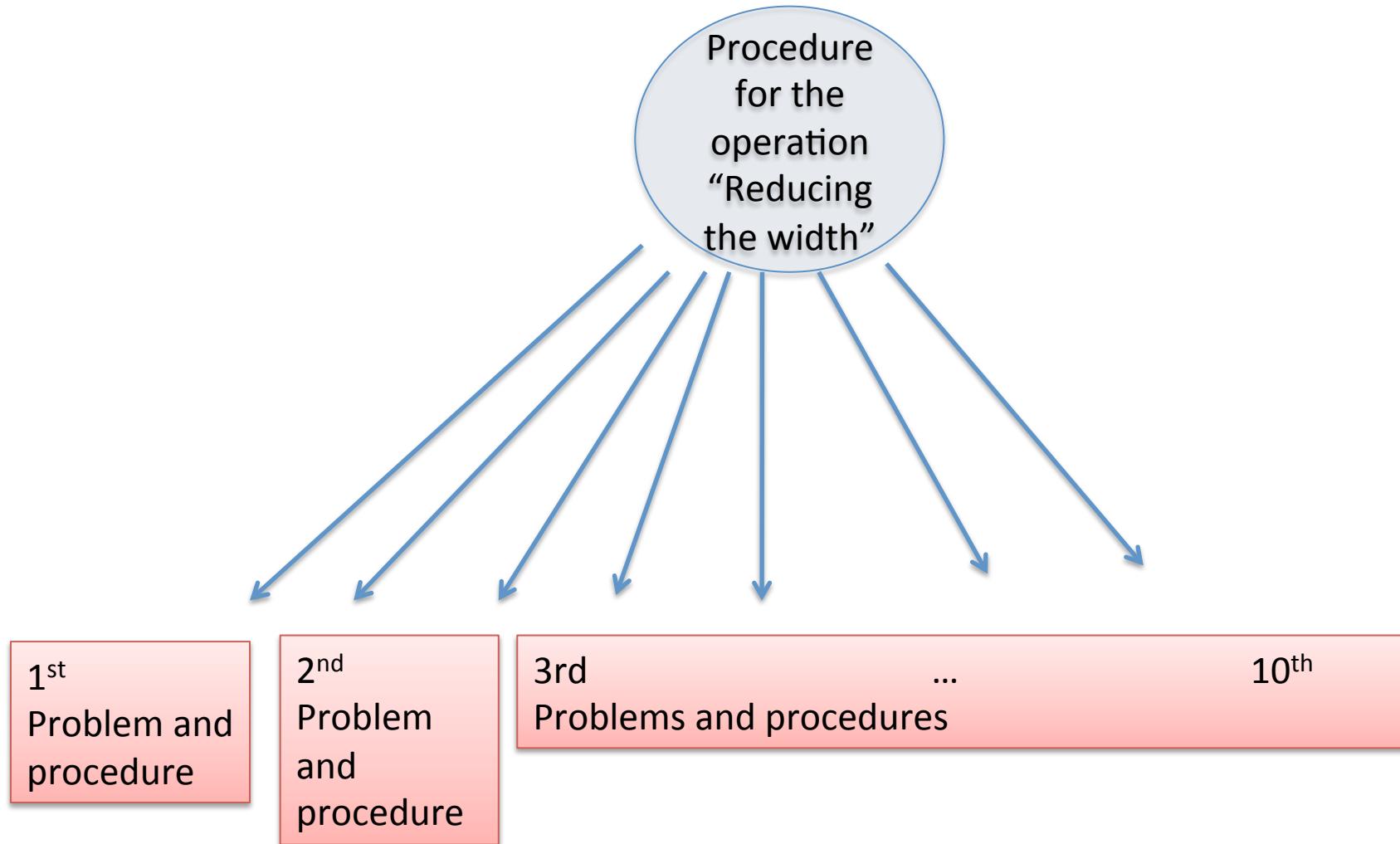
The Nine Chapters **FIRST EXAMPLE**：**The procedure for "SUPPOSE," followed by dozens of problems and procedures, which the commentators show are applications of it.**

(**2.0**) "SUPPOSE　今有
PROCEDURE: ONE MULTIPLIES, BY THE QUANTITY OF WHAT ONE HAS, THE _LÜ_ OF WHAT ONE SEEKS, WHAT MAKES THE DIVIDEND; ONE TAKES THE _LÜ_ OF WHAT ONE HAS AS DIVISOR. (…commentary) DIVIDING (…)術曰：以所有數乘所求率爲實。以所有率爲法(⋯)實如法而一

(**2.1**) "SUPPOSE THAT, HAVING ONE _DOU_ OF UNHUSKED GRAIN (OF FOXTAIL MILLET), ONE WANTS TO MAKE COARSELY HUSKED GRAIN. ONE ASKS HOW MUCH IT YIELDS.

　　ANSWER: IT MAKES SIX _SHENG_ OF COARSELY HUSKED GRAIN.

PROCEDURE：IF, HAVING UNHUSKED GRAIN, ONE SEEKS COARSELY HUSKED GRAIN, ONE THREE-FOLDS THIS, AND ONE DIVIDES BY FIVE (KC: literally: five then/thus one)."

今有粟一斗，欲爲糲米。問得幾何。
荅曰：爲糲米六升。
術曰：以粟求糲米，三之，五而一。

Li Chunfeng's commentary:

　quantity of what one has/unhusked millet —quantity of what one seeks/husked millet
　　　　　　　　　　1 _dou_—————???
　　　_lü_ of what one has—_lü_ of what one seeks
　　　　　　　　　　50—————30

— The "universal procedure" is applied
— In actual problems, two values picked from a global table (50, 30) and, since they are _lü_, they can be simplified, here into 5 and 3, integers prime with each other and without units.

Chapter 2 of *The Nine Chapters*

Procedure for the operation "suppose"

2.1 Problem and procedure	2.2 Problem and procedure

2.3 …   Problems and procedures   …
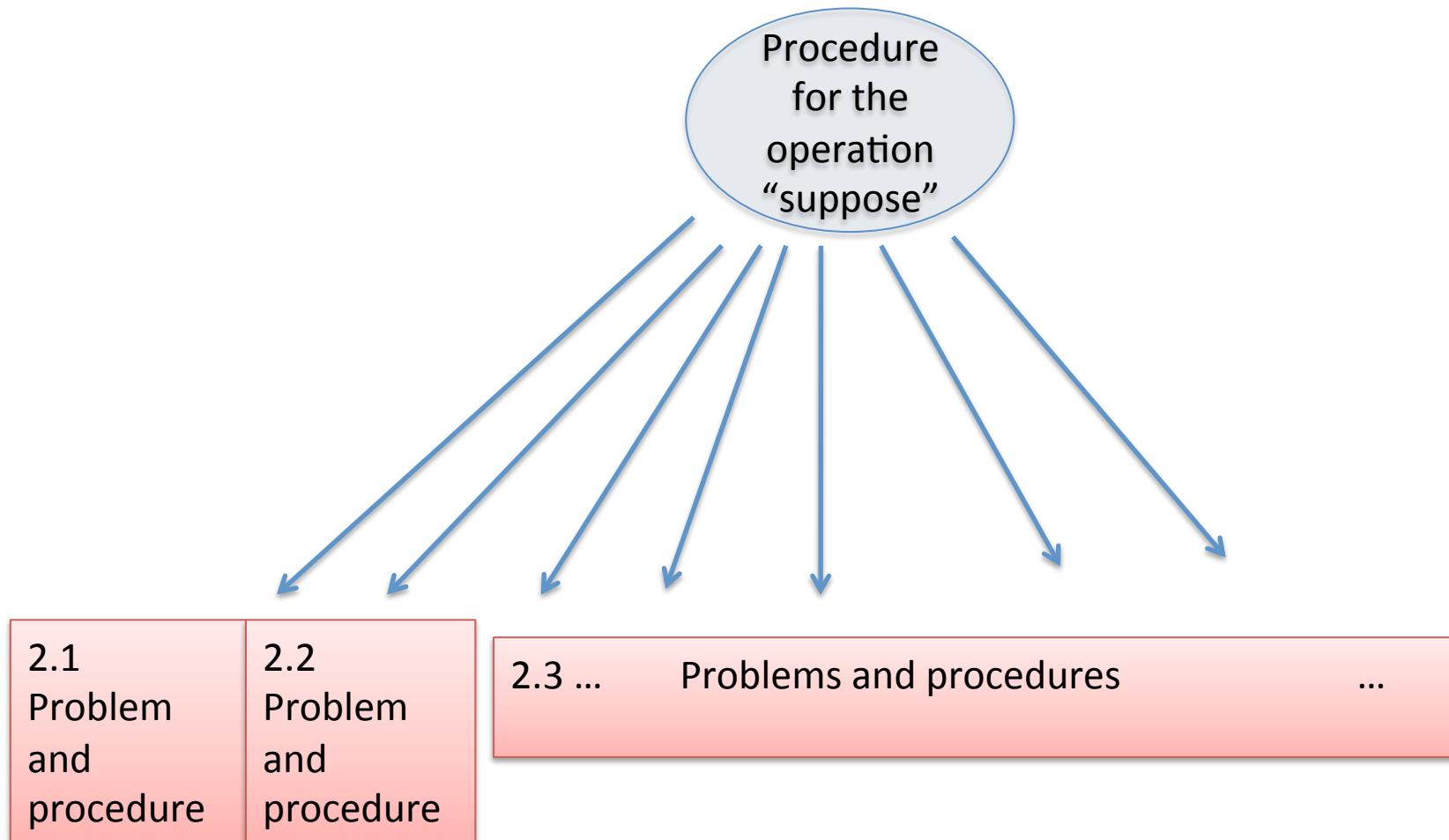
One can argue that this higher procedure and the whole dispositif is what Liu Hui has in mind when he speaks of "**abstract expressions** (空言 *kongyan*)" as something apparently desirable.

Abstraction as an actor's category

"REDUCING THE WIDTH (...)

PROCEDURE: ONE PUTS THE **INTEGRAL** (NUMBER OF) *BU* AS WELL AS **DENOMINATORS AND NUMERATORS OF THE PARTS.** ON MULTIPLIES, BY THE DENOMINATOR OF THE PARTS THAT IS IN THE LOWEST (POSITION)/BELOW, ALL THE NUMERATORS AS WELL AS THE INTEGRAL (NUMBER OF) *BU*. ONE DIVIDES, RESPECTIVELY, THESE NUMERATORS BY THE CORRESPONDING DENOMINATOR AND PUTS THESE (RESULTS, I.E., THE INTEGRAL PARTS OF THE RESULTS) ON THE LEFT HANDSIDE, NAMING THEM **THE PARTS THAT COMMUNICATE** (WITH THE INTEGERS). FURTHER, ONE MULTIPLIES BY THE DENOMINATORS ALL THE NUMERATORS AND THE (VALUES) THAT ALREADY COMMUNICATE. **WHEN** THE (VALUES) **ALL COMMUNICATE, THEN ONE HAS EQUALIZED THEM** AND ONE THUS ADDS THEM TO MAKE THE DIVISOR. ONE PUTS THE QUANTITY OF *BU* (OF THE AREA) THAT IS SOUGHT-FOR AND ONE MULTIPLIES IT BY THE **ACCUMULATED PARTS (*JIFEN,* OR PARTS OF THE PRODUCT) OF THE INTEGRAL (NUMBER OF)** *BU* TO MAKE THE DIVIDEND. DIVIDING THE DIVIDEND BY THE DIVISOR YIELDS THE *BU* OF THE LENGTH."

(4.1) SUPPOSE THERE IS A CROPLAND WHOSE WIDTH IS ONE *BU* AND HALF. IF ONE LOOKS FOR A CROPLAND OF ONE *MU*, ONE ASKS HOW MUCH THE LENGTH IS.

   ANSWER: ONE HUNDRED AND SIXTY *BU*.

PROCEDURE: (SINCE) BELOW THERE IS A HALF, THAT IS ONE OF TWO PARTS, ONE TAKES ONE AS TWO, A HALF AS ONE; SUMMING THESE YIELDS THREE, WHICH IS TAKEN AS DIVISOR. ONE PUTS TWO HUNDRED AND FORTY *BU* AND LIKEWISE, TAKING ONE AS TWO, ONE MULTIPLIES IT, WHICH MAKES THE DIVIDEND. DIVIDING THE DIVIDEND BY THE DIVISOR YIELDS THE LENGTH IN *BU*

今有田廣一步半。求田一畝，問從幾何。

荅曰：一百六十步。

術曰：下有半，是二分之一。以一爲二，半爲一，并之得三，爲法。置田二百四十步，亦以一爲二乘之，爲實。實如法得從步。

Idem for 4.2...till 4.11

Chapter 4 of *The Nine Chapters*

Procedure for the operation "reducing width"

| 4.1 Problem and procedure | 4.2 Problem and procedure | 4.3 Problems and procedures ... 4.11 |
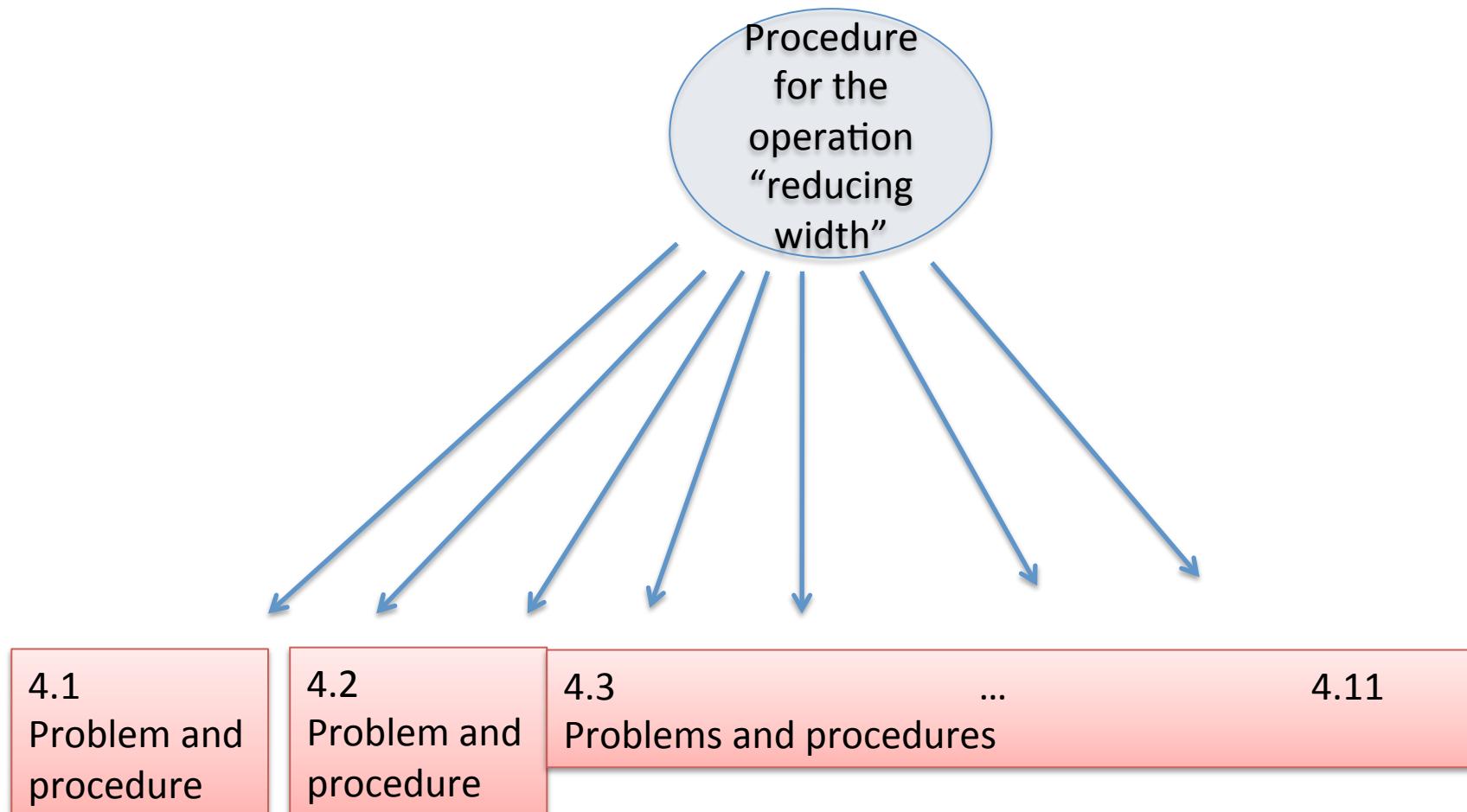
For "Reducing the width", the higher-level procedures present important contrasts

For the *Book of mathematical procedures*, the higher-level procedure reads:
"**Reducing the width   Procedure for looking for the reduced width**: One first puts the width, then if one says below there are a given amount of *bu*, **one takes one as a given amount**, one takes a half as a given amount⌐, one takes one third as a given amount, one **accumulates the partitionnings** (*jifen*) so as to get rid of the parts of that which one looks for. One sums these (results), what makes the divisor. Then, **relying on this**, one puts the field, two hundred and forty *bu*, and one **likewise** takes one as a given amount, which is taken as the *bu* of the area (*jibu*). Eliminating from the *bu* of the area, what is like the divisor yields one *bu* of the length. What does not fill up the *bu*, one names its parts with the divisor⌐. … … /164-166/" "

As for *The Nine chapters*
"REDUCING THE WIDTH (…) PROCEDURE: ONE PUTS THE **INTEGRAL** (NUMBER OF) *BU* AS WELL AS **DENOMINATORS AND NUMERATORS OF THE PARTS**. ON MULTIPLIES, BY THE DENOMINATOR OF THE PARTS THAT IS IN THE LOWEST (POSITION)/BELOW, ALL THE NUMERATORS AS WELL AS THE INTEGRAL (NUMBER OF) *BU*. ONE DIVIDES, RESPECTIVELY, THESE NUMERATORS BY THE CORRESPONDING DENOMINATOR AND PUTS THESE (RESULTS, I.E., THE INTEGRAL PARTS OF THE RESULTS) ON THE LEFT HANDSIDE, NAMING THEM **THE PARTS THAT COMMUNICATE** (WITH THE INTEGERS). FURTHER, ONE MULTIPLIES BY THE DENOMINATORS ALL THE NUMERATORS AND THE (VALUES) THAT ALREADY COMMUNICATE. **WHEN** THE (VALUES) **ALL COMMUNICATE, THEN ONE HAS EQUALIZED THEM** AND ONE THUS ADDS THEM TO MAKE THE DIVISOR. ONE PUTS THE QUANTITY OF *BU* (OF THE AREA) THAT IS SOUGHT-FOR AND ONE MULTIPLIES IT BY THE **ACCUMULATED PARTS (*JIFEN*, OR PARTS OF THE PRODUCT) OF THE INTEGRAL (NUMBER OF)** *BU* TO MAKE THE DIVIDEND. DIVIDING THE DIVIDEND BY THE DIVISOR YIELDS THE *BU* OF THE LENGTH

Another case of the rewriting of the higher-level procedure

Book of mathematical procedures
"lü-ing by the dan" or "Determining the unit (price) on the basis of the *dan*"
"One takes what is exchanged as divisor.
One multiplies, by the cash obtained, the quantity of one *dan*, which is taken as dividend.
Those for which, in their lower (rows),
there is a half, one doubles them; one third, one three-folds them.
Those for which there are *dou* and *sheng*, *jin, liang* and *zhu*, **likewise** one breaks up *all* their upper (rows),
one makes the (rows) below join them, (yielding a result) which is taken as divisor.
What the cash was multiplying, **likewise, is broken up like this**."

The Nine Chapters, **Problems 2.34---2.37**
"Determining the unit price on the basis of various units: the *sheng*, the *pi,* the *zhang* and the *dan*"

經率 DIRECTLY LÜ-ING
術曰：以所求率乘錢數爲實，以所買率爲法，實如法得一。
"PROCEDURE: ONE MULTIPLIES BY THE *LÜ OF WHAT ONE SEEKS* THE QUANTITY OF CASH, WHICH MAKES THE DIVIDEND. ONE TAKES THE *LÜ OF WHAT IS BOUGHT* AS DIVISOR, DIVIDING THE DIVIDEND BY THE DIVISOR YIELDS THE RESULT."

Second remark

(4.1) SUPPOSE THERE IS A CROPLAND WHOSE WIDTH IS ONE *BU* AND HALF. IF ONE LOOKS FOR A CROPLAND OF ONE *MU*, ONE ASKS HOW MUCH THE LENGTH IS.
　　ANSWER: ONE HUNDRED AND SIXTY *BU*.
PROCEDURE: (SINCE) BELOW THERE IS A HALF, THAT IS ONE OF TWO PARTS, ONE TAKES ONE AS TWO, A HALF AS ONE; SUMMING THESE YIELDS THREE, WHICH IS TAKEN AS DIVISOR. ONE PUTS TWO HUNDRED AND FORTY *BU* AND LIKEWISE, TAKING ONE AS TWO, ONE MULTIPLIES IT, WHICH MAKES THE DIVIDEND. DIVIDING THE DIVIDEND BY THE DIVISOR YIELDS THE LENGTH IN *BU*

Dividend			
Divisor	1		
		1	2

2		
1		2

240		
2		
1		2

480		
3		
		2

$$\frac{240\ bu}{1\ bu + \frac{1}{2}\ bu} = \frac{240\ bu\ times\ 2\ (half\ bu)}{2 + 1\ (half\ bu)}$$

Dividend			
Divisor	1		
		1	2

2		
	2	2

2		
	1	2

240		
2		
	1	2

480		
3		
		2

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