Twenty four tetrahedra for a cube

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1 The volume of a regular tetraedron?!

Do you know the volume of a cube of edge length a? Of course, it's almost the definition of the measure of a volume, it is a^3 and if a is measure in cm, the volume is in cm^3. If a is measured in inches, a^3 is in cubic inches. But do you know the volume of a regular tetrahedron of size a?

Say it loud, don't be afraid to admit it, the answer is more often **no** and besides *who cares* about the volume of the regular tetrahedron?!

And then what is a tetrahedron anyway? Tetrahedron is a Greek word which comes from tetra which means four (unicycle, bicycle, tricycle, it's easy but not everyone knows that most mammals are tetrapods, i.e. on all fours) and edron which means face. It is therefore a four-sided body. One flat body with many straight sides is called a polygon, of poly = many and gones = sides, in the same way, a body with many faces is called a polyhedron. The simplest of polygons is the triangle and a tetrahedron is the simplest of the polyhedra, because it is made up of four triangles that fit together. Try to construct a body with less flat faces that still has some volume! Among all the tetrahedra, the regular one has all edges of the same length. By the way, how many edges does it have?

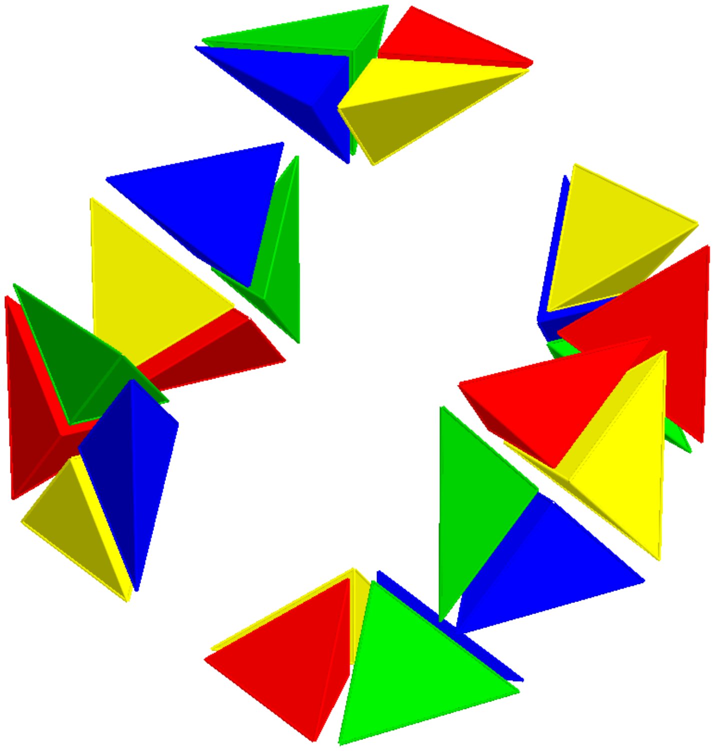
It is a small trip towards the calculation of this volume that we are going to work out. And I hope you find the walk pretty and interesting enough to motivate the magic formula at the end.

For that we will go back to the good old cube, that we are going to cut, in several different ways, into twenty-four tetrahedra, all of the same volume. The volume of a tetrahedron will therefore be a twenty-fourth of that of the cube. And *this* is the key.

2 $6 \times 4 = 24$

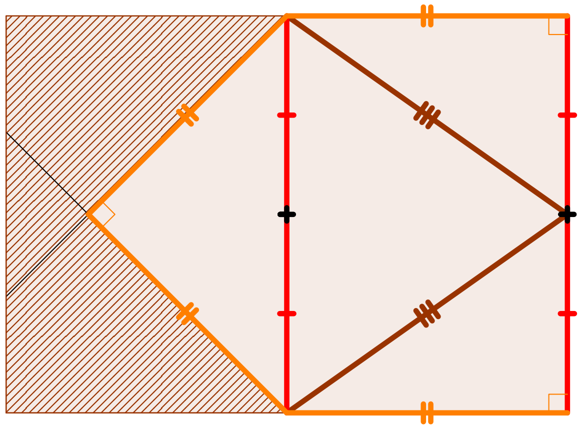
When you meet a cube in everyday life, especially during the holidays, it is often in the form of a six-sided die. A small, easy-to-imagine solid tangram is to decompose this cube into six pyramids with square bases by joining each of the eight vertices of the cube centered by an edge, which defines triangles based on the $4 \times 3 = 12$ edges of the cube.

The net of this pyramid is not so clear to define: there is a base square, and the height of the pyramid is clearly the half of the edge side, but what are the lengths of the two legs of the isosceles triangles surrounding it? It's easier by cutting this pyramid in four parts (red, green, blue, yellow), like a cake, going through the corners. Animation



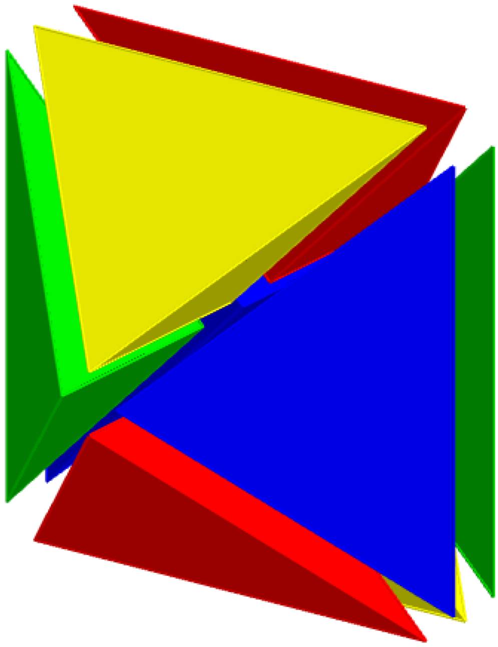
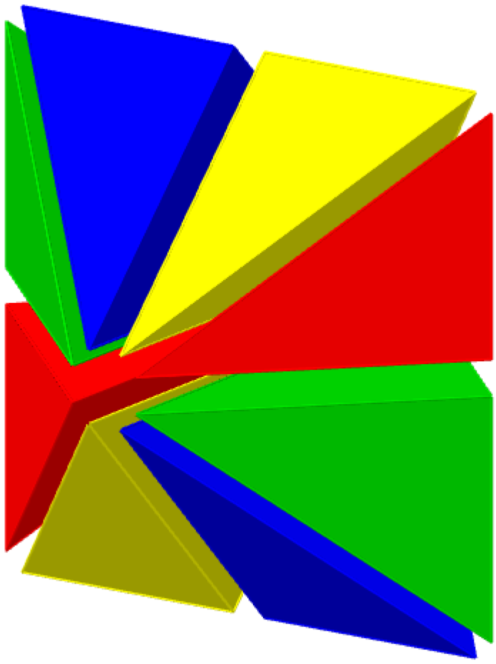
What are the lengths of the edges of each of the parts? There is a quarter of a square for the base horizontal face, it is an isosceles right triangle, there are then two vertical faces where the knife sliced along the height; they are easy to unfold since we know that this height is vertical and half the length of the side. There are finally the third side, of which we now know all the sides, it is an isosceles triangle. The two legs are the hypothenuses of vertical triangles and the short side belongs to the base square.

The following pattern is cut from an A4 sheet, which has exactly the desired format, and we get it by a few folds (cut out the hatched part then glue the outer segments having the same length):



This pattern contains four triangles; once glued, it is a tetrahedron. It is far from being regular: the base is a right triangle isosceles (half a square), the two vertical faces are the same rectangle triangles and the large isosceles face is composed of two of these right triangles. How many of these tetrahedra do we have for the cube? Four per side, or twenty-four! Here is the first decomposition of the cube of tetrahedra, into twenty-four tetrahedra, all equal. Animation

You can pave the entire space in many ways with this tetrahedron! With four of these tetrahedra (red, green, blue, yellow), we have seen that we can compose the pyramid with a square base, sixth of the cube. With two of these pyramids, glued by their square bases, we assemble an *octahedron*, a polyhedron with eight triangular faces and six vertices. But it is non-regular because it contains a median square between its two vertices sepratаed by a height totalling the edge side of the square. This octahedron may be composed in two different ways using the same parts, the way we defined it by gathering the small heights along the axis of symmetry with the sides of the square on the outside, or by taking as axis of symmetry these sides of the quarter squares, the segments half now adding to define the square. Animation Animation

When we consider the cube as paving the space, we see appearing these two types of decomposition of the same octahedron, the edges of the cube or the segment connecting two centers of cubes, serving as the axis of symmetry. And these octahedra can slide relative to each other others, that makes a very large variety of non-regular tilings of space...

3 Cavalieri's principle of indivisibles

With the previous tetrahedron, we certainly pave the cube but we are far from the regular tetrahedron. Can we transform this tetrahedron all askew by making it more symmetrical?

What will help us is a fabulous idea due to a 17th century Italian, Cavalieri: *the method of indivisibles*. It is about understanding an object as a stack of very very thin parallel layers, so thin that we can no longer divide them. It is a daring attempt for its time to rub shoulders with infinity! Cut an object into 100 parallel layers. We can slide these layers relative to each other, the volume of this object does not vary. But each can still be divided into say 100 layers parallel, leading to 10,000 parallel layers that can slide relative to each other. They are the same slices as before so the total volume is always unchanged. The coup de Cavalieri is to *pass to the limit* and to decree that this property holds for an infinity of parallel layers that one makes slide continuously relative to each other.

By choosing a base and by continuous sliding all layers, we deduce properties of areas and volumes. For example we find that the area of a triangle depends only on the length of its base and its height. Likewise, the volume of a pyramid depends only on the area of its base and the length of its height (the distance from the top to the base). That is to say, in placing the base on the horizontal plane as it should be, that we can move at leisure the top (call it the *apex* to show off) of the pyramid in its horizontal plane provided we do not change its altitude.

However, the pattern of the regular tetrahedron contains only equilateral triangles while that of the preceding tetrahedron has no equilateral triangles, so it is not a simple application of Cavalieri's principle, keeping a base face fixed which will allow us to " regularize " our tetrahedron. We will see that it we will need two applications of the principle of Cavalieri to overcome it!

By duplicating an object and cutting it in the same way in slices, we see that we can rearrange these slices by alternating a layer of one and a layer of the other. We thus end up with an object which is twice as *tall* and twice as *bulky*. Passing to limit, this object has undergone an *affinity* of ratio two in the vertical direction. We could do the opposite, take every second layer and get an object half the size and half as high. By elaborating on this idea, we demonstrate that the volume depends *linearly* on its height: if we double the height, the volume also doubles.

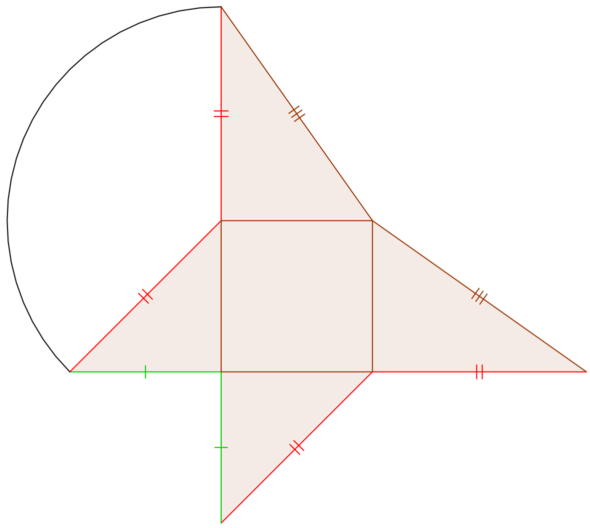
Note that we can revolve our solids: choose an oblique face as the new base and tilt the solid until the top drops back down to the base. That's a *revolution!* The base and heights are then not the same! With this principle, we see that the measurement of the volume of a *straight* block is the product of its three dimensions, the product of two constituting the area of the base and the third the height. But one paving stone, whether straight or not, has a volume which is the product of the area of a base by the height resulting from this base. It's not so obvious that it seems in the general case!

Well, a pyramid is not a paving stone, and its volume *only depends* on the area of its base and its height, but not quite as their simple product; what is it?

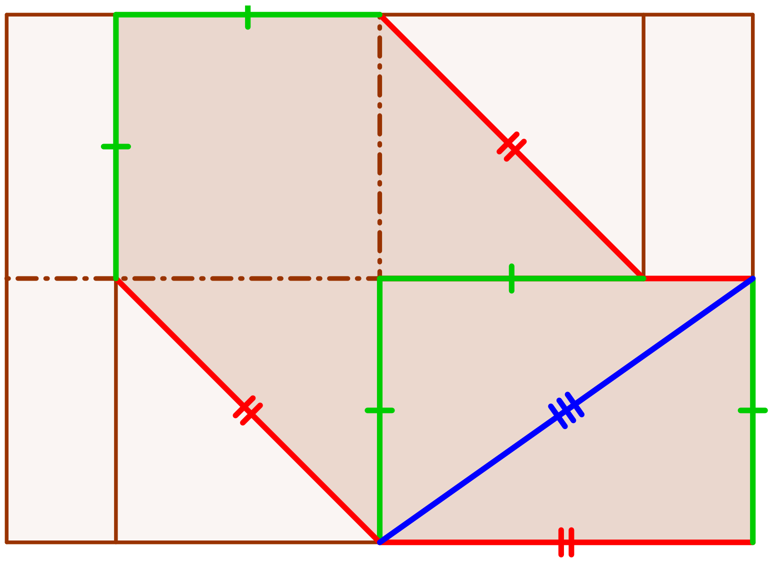
4 The third of a cube

First of all, there are a lot of different pyramids. The tetrahedron is one, it is a pyramid with a triangular base. The pyramids of Egypt have a square base and a well-aligned apex, it is a straight pyramid. But we can imagine pyramids with completely different bases and heights not square at all!

The *third of a cube* is one of them: its base is a square, its height is the same length as its base edge, but instead of being fine, wisely in the middle, it sits in a corner. The foot of its height is thus one of the four corners of the base. It therefore fits in a cube. We can notice that the vertical plane passing through the diagonal square and the vertex cuts the pyramid into a right triangle which is the same as that of the oblique faces: it has for sides an edge of the cube, a diagonal of a square and this large diagonal of the cube. We can therefore build the pattern of this pyramid: there is the square of base, two half-squares for the vertical sides, and transferring the diagonal of the square, complete with two large right triangles. Counting the side of the square as a unit, the lengths involved are 1, $\sqrt{2}$ and $\sqrt{3}$.



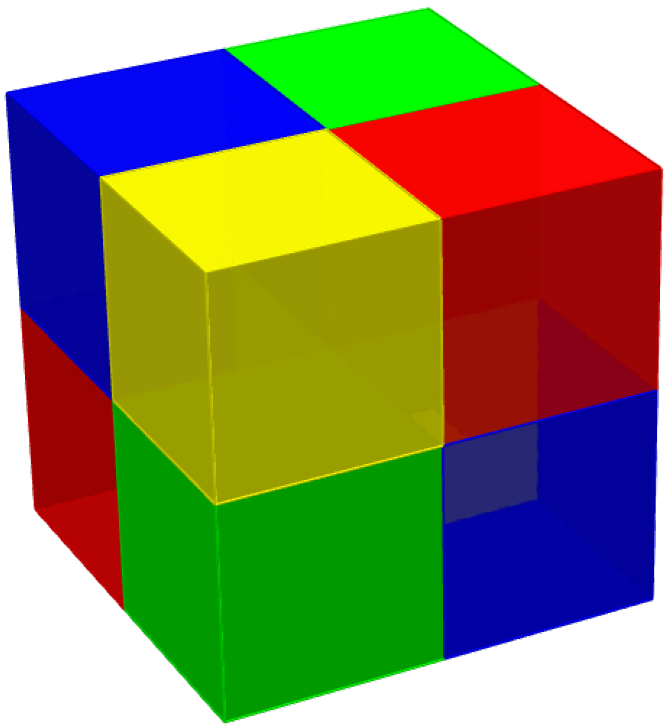
Again, if you don't have a compass, the A4 sheet can help us, at the cost of a little more duct tape. We fold it in four and we mark a small square, then we cut and glue:



If we build three of this shape, we can assemble them into one full cube! By playing on Cavalieri's principle and the linearity of the volume as a function of its height, we can apply affinities along any axis and get thirds of cuboids of all shapes. The moral of the story is that the volume of a pyramid is *the third* of the product of the area of its base by its height. This is true for any pyramid, not only leaning on a rectangle but as well on a triangle for the tetrahedron, or on a regular polygon with many sides, and even at the *continuous limit*, leaning on a *circle*: the king of volume calculations of antiquity, Archimedes, already knew that the volume of a cone is the third of the cylinder with the same base and the same height. Animation

5 $8 \times 3 = 24$

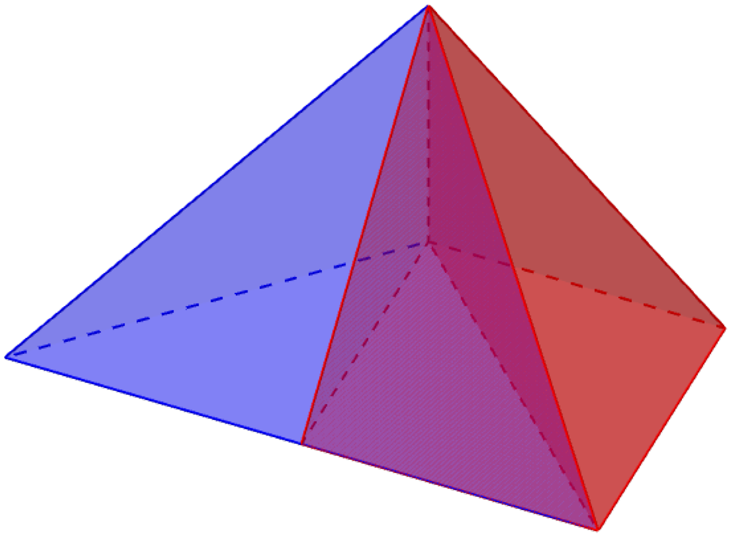
If we double the side of a cube, what is its volume in relation to the starting cube? Just count, there are four on the ground floor floor and four on the first floor, that is to say twice as much in width, depth and height, that's $2 \times 2 \times 2 = 2^3 = 8$. By the same reasoning, it will be the same for all the right cobblestones that build larger bodies. In fact, on closer inspection, this property is true for all volumes: the pyramid of Cheops is built from large stones, all looking like huge bricks. If the Pharaoh had wanted a pyramid twice as high, it would have taken his architect not two, not four, not six, but *eight* times more stones! Animation



On the contrary, a half side cube has an eighth of the volume of the original cube.

What does this have to do with our story? It is that by breaking down these eight small cubes of side half into third cube pyramids, we get $8\ times 3 = 24$ similar pyramids breaking down the original cube. This pyramid therefore has the *same volume* as the previous tetrahedron. However, by comparing their nets, we see that these polyhedra have no face in common. On the other hand, apart from the large side of the tetrahedron, their edges are of common lengths ($\frac{1}{2}$, $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$ times the large side).

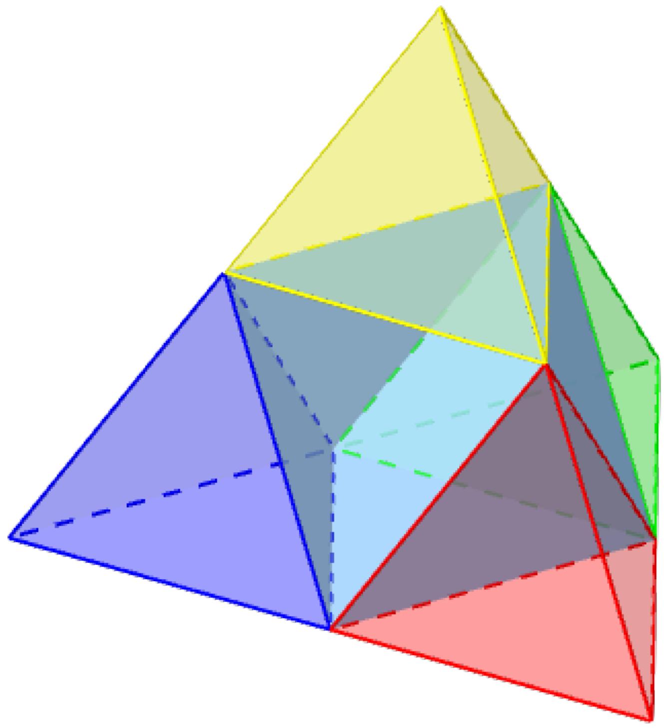
It is therefore not a simple application of Cavalieri's principle which will transform us into one other. In fact we cut the third of a cube along the vertical plane passing through the large diagonal and the height, around which it is rotated three-quarter turns, until we close it in this twenty-fourth cube tetrahedron Animation.



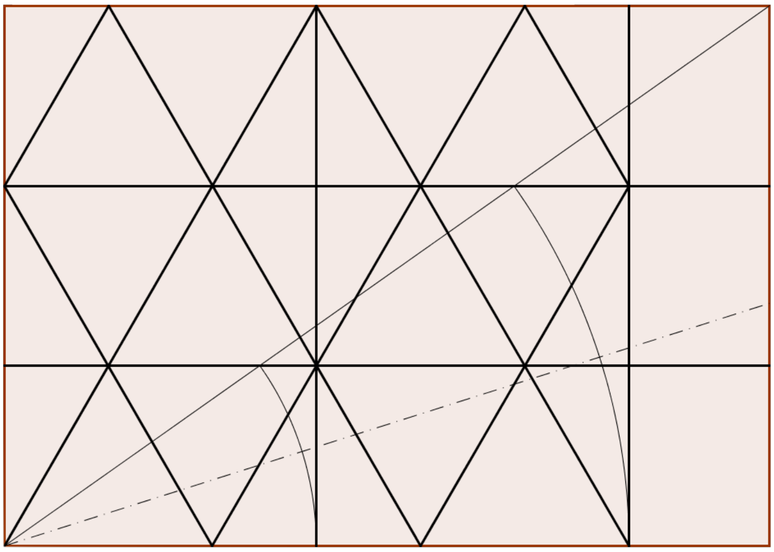
6 Double tetrahedron

The fact that there are eight small cubes in a double-sided cube can be seen using magnification transformations. Such a transformation doesn't just need an enlargement ratio but as well a point, it is the eye of the artist in the machines to draw in perspective by Albrecht Dürer, it is the objective of the camera, it is the *center of the homothety*. And in considering a cube of double edge length, from the point of view of any of its eight summits, we can imagine that we can attach a single sided cube. This argument may apply also in the case of other polyhedra, for example the tetrahedron, that is a pyramid with a triangular base and four vertices. In a double sized tetrahedron, whose volume is therefore eight times that of the basic one, there are four basic tetrahedra and an equivalent remaining volume.

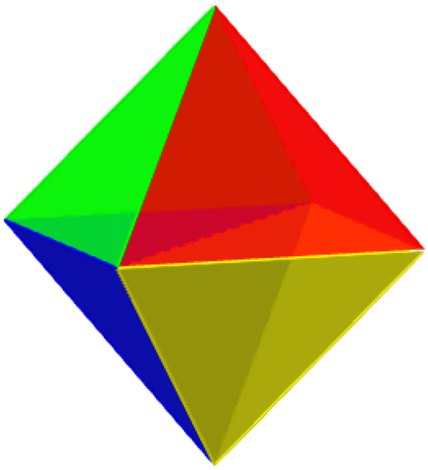
What is the form of this hole? Let's count its faces: there are four triangles common to the four basic tetrahedra associated with each vertex, and four more triangles on all four triangular faces, of double size, of the great tetrahedron, which makes eight triangles. It is therefore an *octahedron*. If the tetrahedron is composed of equilateral triangles, this octahedron is regular. Animation.



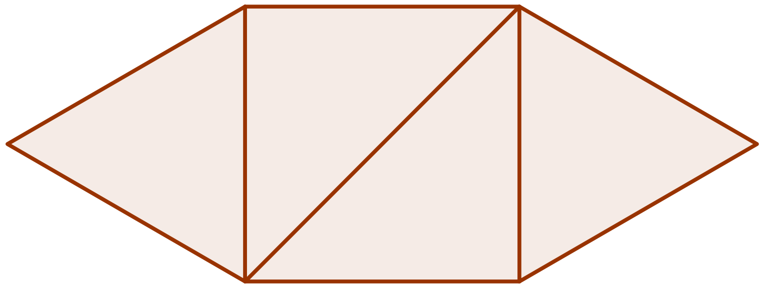
Less known than the tetrahedron and the cube, the *octahedron* is a Platonic solid as well. It contains three squares, and when we take it into hand, we put one of these three squares horizontally, highlighting the vertical axis of symmetry. It is then not so easy to imagine the other two squares and see that all the vertices are actually equivalent. If we put the octahedron on a table, it stabilizes on one face, and its symmetries are even less apparent. They are however the same symmetries than those of the cube. Its net is made up of eight equilateral triangles. Folding an A4 sheet in three in the direction of its length, the intersections with the diagonal shown on the side can help us, but it's easier to look for a compass or a protractor.



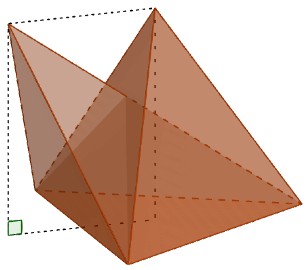
The conclusion is, that for a same edge length, a regular octahedron has a volume *quadruple* of the regular tetrahedron. Moreover, we can decompose this octahedron into four non-regular tetrahedra: we cut along two of the three squares that the octahedron contains. Animation.



The net of this tetrahedron therefore contains two half-squares and two equilateral triangles. Note that to assemble it, we fold the square in two-half squares according to orthogonal planes, that is to say with a dihedral angle (angle between two faces which intersect along an edge in a plane perpendicular to this edge) which is a right angle.



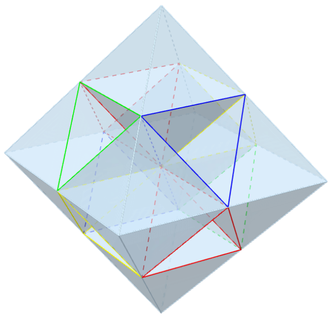
As a quarter octahedron, this tetrahedron has the same volume as the regular tetrahedron but we can see this fact in a more direct manner: by taking as base one of the equilateral triangles, we can apply the principle of Cavalieri: the two tetrahedra have the same base and the same height. Animation



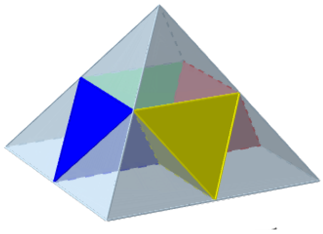
Notice that a half octahedron, made up of two of these tetrahedra, forms a square-based pyramid, which is convenient to handle.

7 Double octahedron

How about playing this game with an octahedron? In the same way as for a tetrahedron, each of the six vertices of the octahedron can be seen as the center of a dilation of ratio two, including six small octahedra in a large double octahedron. What shape are the holes? By observing carefully, we convince ourselves that there are *eight* regular tetrahedra Animation.



Or rather, and easier to manipulate, with a half-octahedron, we picture the pyramid with a square base, you then need five per vertex, plus one "head down", and four regular tetrahedra remain Animation.



The volume of a pyramid is therefore worth two regular tetrahedra. We already knew this because this pyramid is composed of two quarter octahedron tetrahedra, which have the same volume as the regular ones.

8 $24 = 4 \times 3 + 8 + 4$

These two tetrahedral pieces, the regular tetrahedron and the quarter octahedron, can also compose an entire cube:

We first compose an octahedron, on which we add a regular tetrahedron on each of the eight faces, it is the Kepler *stella octangula*. Then, according to each of the three free faces of the tetrahedra, add a new quarter of an octahedron, i.e. four in each direction, their hypotenuse coinciding with the four edges of the cubes parallel to this direction. So there are $4 \times 3 = 12$ quarters of octahedron to complete the cube. In all, there are therefore $4\, times 3 + 8 + 4 = 24$ tetrahedra all of the same volume for a cube. Animation

Therefore we now have three tetrahedra and a pyramid with a square base, all of the same volume, the twenty-fourth of the cube. Animation

9 The magical formula!

We are coming to the end of our journey and are able to give the formula for the volume of the regular tetrahedron. We see that if the cube is of edge length c, the half octahedron has for longer edge c, diagonal of a square broken into two half-squares, isosceles right triangles of sides $\frac{c}{\sqrt{2}}$, which is also the side of the equilateral triangle which composes its two other faces. The regular tetrahedron included in the cube has therefore for edge length $a = \frac{c}{\sqrt{2}}$ and its volume is a twenty-fourth of the cube of side $c = \sqrt{2}\, a$. By consequent

$$\mathrm{Vol}(\mathcal{T}_a) = \frac{(\sqrt{2}\, a)^3}{24}.$$

Animation