$$e_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\xi_1(x) = x_1$$

- $\xi_2(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$

$$\begin{cases} \xi_2(e_1) = 0 \\ \xi_2(e_2) = 1 \\ \xi_2(e_3) = 0 \end{cases} \leftarrow \begin{cases} \alpha_1 - \alpha_3 = 0 \\ 2\alpha_2 + \alpha_3 = 1 \\ \alpha_3 = 0 \end{cases} \qquad \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 1/2 \\ \alpha_3 = 0 \end{cases}$$

$$\xi_2(x) = 1/2\, x_2$$

- $\xi_3(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$

$$\begin{cases} \xi_3(e_1) = 0 \\ \xi_3(e_2) = 0 \\ \xi_3(e_3) = 1 \end{cases} \qquad \begin{cases} \alpha_1 - \alpha_3 = 0 \\ 2\alpha_2 + \alpha_3 = 0 \\ \alpha_3 = 1 \end{cases} \qquad \begin{cases} \alpha_1 = 1 \\ \alpha_2 = -1/2 \\ \alpha_3 = 1 \end{cases}$$

$$\xi_3(x) = x_1 - \frac{1}{2} x_2 + x_3$$

$f : E \to \mathbb{K}$ forme linéaire

$\boxed{\mu_i = f(e_i)}$ pour $i = 1, \dots, n$

Alors $f = \mu_1 \xi_1 + \dots + \mu_n \xi_n$

On écrit $x = \xi_1(x) e_1 + \dots + \xi_n(x) e_n$

(cf. la définition des ξ_i)

$$f(x) = f(\xi_1(x) e_1 + \dots + \xi_n(x) e_n)$$

$$= \xi_1(x) f(e_1) + \dots + \xi_n(x) f(e_n)$$

car f linéaire

$$= \xi_1(x) \mu_1 + \dots + \xi_n(x) \mu_n$$

$$= (\mu_1 \xi_1 + \dots + \mu_n \xi_n)(x)$$

$$q(x) = x_1^2 + 2x_1x_2 + 3x_2x_3 - x_3^2 \qquad x = {}^t(x_1, \dots, x_n)$$

$n = 3$

$$\varphi(x,y) = x_1y_1 + x_1y_2 + x_2y_1 \qquad E = \mathbb{R}^3$$

$$+ \frac{3}{2}x_2y_3 + \frac{3}{2}x_3y_2 - x_3y_3$$

On veut :

$$q(x) = \lambda_1 f_1(x)^2 + \lambda_2 f_2(x)^2 + \lambda_3 f_3(x)^2$$

avec $f_1, f_2, f_3 \in E^*$

et (f_1, f_2, f_3) libre ($=$ base)

Ex : $f_1(x) = x_1 + x_3 \qquad f_3(x) = x_3 - x_1$

$f_2(x) = x_2 - x_3$

$$f_1(x)^2 - 2f_2(x)^2 + f_3(x)^2$$

$$= (x_1 + x_3)^2 - 2(x_2 - x_3)^2 + (x_3 - x_1)^2$$

$$= \dots \quad (\neq q(x))$$

Il faut trouver les bons f_1, f_2, f_3 et $\lambda_1, \lambda_2, \lambda_3$

Réduction de Gauss $\qquad x = {}^t(x_1, x_2, x_3)$

$n = 3$

$$q(x) = x_1^2 - 2x_1x_2 + 3x_1x_3 - x_2^2 + x_2x_3 - x_3^2$$

$$x_1^2 - 2x_1x_2 + 3x_1x_3 = x_1^2 + 2x_1\left(-x_2 + \frac{3}{2}x_3\right)$$

identité : $\boxed{(x_1+a)^2 - a^2 = x_1^2 + 2x_1 a}$ $\qquad a = -x_2 + \frac{3}{2}x_3$

$$x_1^2 - 2x_1x_2 + 3x_1x_3 = \left(x_1 - x_2 + \frac{3}{2}x_3\right)^2 - \left(-x_2 + \frac{3}{2}x_3\right)^2$$

$$= \left(x_1 - x_2 + \frac{3}{2}x_3\right)^2 - x_2^2 - \frac{9}{4}x_3^2 + 3x_2x_3$$

On remplace

$$q(x) = \left(x_1 - x_2 + \frac{3}{2}x_3\right)^2 \boxed{- 2x_2^2 + 4x_2x_3} - \frac{13}{4}x_3^2$$

$$-2x_2^2 + 4x_2x_3 = -2\left[x_2^2 - 2x_2x_3\right]$$

$$= -2\left[x_2^2 + 2x_2(-x_3)\right]$$

$$= -2\left[(x_2 - x_3)^2 - x_3^2\right]$$

On remplace

$$q(x) = \left(x_1 - x_2 + \frac{3}{2}x_3\right)^2 - 2(x_2 - x_3)^2 + 2x_3^2$$

$$q(x) = \left(x_1 - x_2 + \frac{3}{2}x_3\right)^2 - 2(x_2 - x_3)^2 \underbrace{- \frac{5}{4}x_3^2}_{-17/4\, x_3^2}$$

$$q(x) = f_1(x)^2 - 2f_2(x)^2 - \frac{5}{4}f_3(x)^2$$

avec $f_1(x) = x_1 - x_2 + \frac{3}{2}x_3 \quad f_2(x) = x_2 - x_3$

$$f_3(x) = x_3$$

$n = 3 \quad q(x) = x_1x_2 - 2x_2x_3$

$$x_1x_2 - 2x_2x_3 = x_1x_2 + x_1 \times 0 + x_2(-2x_3)$$

$$\boxed{(x_1 + a)(x_2 + b) - ab = x_1x_2 + x_1b + x_2a}$$ identité

$$b = 0 \quad a = -2x_3$$

$$q(x) = (x_1 - 2x_3)x_2$$

$$\boxed{ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right]}$$ identité

$a = x_1 - 2x_3 \quad b = x_2$

$$q(x) = \frac{1}{4}(x_1+x_2-2x_3)^2 - \frac{1}{4}(x_1-x_2-2x_3)^2$$

$$+ \underbrace{0 \times x_3^2}$$