On the structure of semigroups on L_p with a bounded H^∞-calculus

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Notation: $\Sigma_\varphi := \{z \in \mathbb{C} : |\arg(z)| < \varphi\}$.

Definition (Sectorial Operator)

$(A, D(A))$ densely defined operator with $\omega \in (0, \pi)$ such that

$$(S_\omega) \qquad \sigma(A) \subset \overline{\Sigma_\omega} \qquad \text{and} \qquad \sup_{\lambda \notin \Sigma_{\omega+\varepsilon}} \|\lambda R(\lambda, A)\| < \infty \quad \forall \varepsilon > 0.$$

Then $\omega(A) := \inf\{\omega : (S_\omega) \text{ holds}\}$.

Definition (Analytic C_0-semigroup)

Family of operators $(T(z))_{z\in\Sigma_\delta}$ $(\delta \in (0, \frac{\pi}{2}))$ satisfying

(i) $z \mapsto T(z)$ is analytic

(ii) $T(z_1 + z_2) = T(z_1)T(z_2) \qquad \forall z_1, z_2 \in \Sigma_\delta$

(iii) $\lim_{\substack{z\to 0 \\ z\in\Sigma_{\delta'}}} T(z)x = x \qquad \forall \delta' \in (0, \delta), \forall x \in X$

It is called bounded if $\sup_{z\in\Sigma_{\delta'}} \|T(z)\| < \infty$ for all $\delta' \in (0, \delta)$.

One has 1:1 correspondence

$$\text{bounded analytic } C_0\text{-semigroups} \leftrightarrow A \text{ sectorial with } \omega(A) < \frac{\pi}{2}$$

At least formally $T(z) = e^{-zA}$.

Given $f \in H_0^\infty(\Sigma_\sigma) := \left\{ f : \Sigma_\sigma \to \mathbb{C} \text{ analytic} : |f(z)| \le \frac{|\lambda|^\varepsilon}{(1+|\lambda|)^{2\varepsilon}} \right\}$ define

$$f(A) := \int_{\partial\Sigma_{\sigma'}} f(\lambda) R(\lambda, A)\, d\lambda \qquad (\omega(A) < \sigma' < \sigma).$$

Definition (Bounded H^∞-calculus)

$(A, D(A))$ sectorial has bounded $H^\infty(\Sigma_\sigma)$-calculus if for some $C \ge 0$

$$(H_\sigma) \qquad \|f(A)\| \le C \sup_{z \in \Sigma_\sigma} |f(z)| \qquad \forall f \in H_0^\infty(\Sigma_\sigma).$$

Then $\omega_{H^\infty}(A) := \inf\{\sigma : (H_\sigma) \text{ holds}\}$.

Theorem (C. Le Merdy)

$-A \sim (T(z))_{z\in\Sigma}$ bounded analytic C_0-semigroup on Hilbert space H.
Equivalent:

(i) *A has a bounded H^∞-calculus*

(ii) *there exists $S \in \mathcal{B}(H)$ invertible such that*

$$\left\|S^{-1}T(t)S\right\| \leq 1 \qquad \forall t \geq 0.$$

Put differently: contractive semigroups are generic for all semigroups with a bounded H^∞-calculus.

Can this be generalized to L_p $(1 < p < \infty)$? In one direction, one has

Theorem (L. Weis)

$-A \sim (T(z))_{z\in\Sigma}$ bounded analytic C_0-semigroup on L_p, positive and contractive on the real line.
Then A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

Can all semigroups on L_p with a bounded H^∞-calculus be obtained from such semigroups?

Theorem (S.F.)

$-A \sim (T(z))_{z\in\Sigma}$ bounded analytic C_0-semigroup on $L_p(\Omega)$ $(1 < p < \infty)$. Equivalent:

(i) *A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.*

Theorem (S.F.)

$-A \sim (T(z))_{z\in\Sigma}$ *bounded analytic* C_0*-semigroup on* $L_p(\Omega)$ $(1 < p < \infty)$. *Equivalent:*

(i) A *has a bounded* H^∞*-calculus with* $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

(ii) *There exists a bounded holomorphic* C_0*-semigroup* $(R(z))_{z\in\tilde{\Sigma}}$ *in some* $L_p(\tilde{\Omega})$*, positive and contractive on the real line with*
- $N \subset M \subset L_p(\tilde{\Omega})$ *closed subspaces invariant unter* $(R(z))$
- $S \in \mathcal{B}(L_p(\Omega), M/N)$ *isomorphism*

such that

$$T(z) = S^{-1} R_{M/N}(z) S \qquad \forall z \in \tilde{\Sigma}.$$

Theorem (S.F. (Reminder))

$-A \sim (T(z))_{z\in\Sigma}$ bounded analytic C_0-semigroup on $L_p(\Omega)$. Equivalent:

(i) *A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.*

(ii) $T(z) = S^{-1}R_{M/N}(z)S \qquad \forall z \in \tilde{\Sigma}$.

- On Hilbert spaces $\omega_{H^\infty}(A) < \frac{\pi}{2}$ holds automatically
- This seems to be open for L_p-spaces, but false for general subspaces of L_p-spaces (N.J. Kalton)
- (i) $\Rightarrow$ (ii) holds on every UMD-Banach lattice ($(R(z))$ lives on another UMD-Banach lattice)

Problem

Does the result hold without factorizing through a subspace-quotient as in the Hilbert space case?

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Problem

Does every positive contractive C_0-semigroup on a UMD Banach lattice have a bounded H^∞-calculus?

Main ideas of the proof (I):

- For $\alpha > 1$ give $H_0^\infty(\Sigma_{\frac{\pi}{2\alpha}+})$ a p-operator space structure as follows:

$$\begin{aligned} H^\infty(\Sigma_{\frac{\pi}{2\alpha}+}) &\hookrightarrow \mathcal{B}(L_p(\mathbb{R}; Y)) \\ f &\mapsto f(B^{\frac{1}{\alpha}}), \end{aligned}$$

  where $-B$ generates the shift semigroup $V(t)g(s) = g(s-t)$ on $L_p(\mathbb{R}; Y)$ for some vector-valued L_p-space Y.

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- For $\alpha > 1$ give $H_0^\infty(\Sigma_{\frac{\pi}{2\alpha}+})$ a *p*-operator space structure as follows:

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  where $-B$ generates the shift semigroup $V(t)g(s) = g(s-t)$ on $L_p(\mathbb{R}; Y)$ for some vector-valued L_p-space Y.
- *p*-complete boundedness of the functional calculus, e.g. mappings

$$\begin{aligned} \mathcal{B}(\ell_p^n(H^\infty(\Sigma))) \supset M_n(H^\infty) &\to M_n(\mathcal{B}(L_p(\mathbb{R}; Y))) \simeq \mathcal{B}(\ell_p^n(L_p(\mathbb{R}; Y))) \\ [f_{ij}] &\mapsto [f_{ij}(B)] \end{aligned}$$

  are uniformly bounded in *n*.

Main ideas of the proof (II):

- A factorization theorem of G. Pisier yields a semigroup as asserted, except for strong continuity (ultraproduct construction).
- Reduce to the strongly continuous part. □

Conclusions

Every bounded analytic C_0-semigroup on $L_p(\Omega)$ with generator $-A$ satisfying $\omega_{H^\infty}(A) < \frac{\pi}{2}$ can be obtained

- from a bounded analytic C_0-semigroup on $L_p(\tilde{\Omega})$, positive and contractive on the real line
- after passing to invariant subspace-quotients and similarity transforms

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Et bon appétit!