Lieb-Thirring vs Blaschke for non-selfadjoint perturbations of certain model operators

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Plan of the talk

Recent results by :
A. Borichev, L. Golinskii, SK,
M. Demuth, M. Hansmann, G. Katriel,
C. Dubuisson, D. Sambou.

1. Introduction : classical Lieb-Thirring inequalities.

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2. Basic construction and results on zeros of holomorphic functions.
3. Applications to different models.

Introduction

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Theorem (Lieb-Thirring' 1975)

Let $p > 0, d \geq 3$. Then

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^p \leq C_{p,d} \int_{\mathbb{R}^d} V_-(x)^{p+d/2}\, dx = C_{p,d} ||V_-||^{p+d/2}_{L^{p+d/2}}$$

where $V_- = \max\{-V, 0\}$.

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Theorem (Frank-Laptev-Lieb-Seiringer' 2007)

Let $p \geq 1, d \geq 1$. *Then*

$$\begin{aligned}\sum_{\lambda \in \sigma_d(H):\ \mathrm{Re}\ \lambda < 0} |\,\mathrm{Re}\ \lambda|^p &\leq C_{p,d} \int_{\mathbb{R}^d} (\mathrm{Re}\ V)_-^{p+d/2}\, dx \\ &= C_{p,d}\, \|(\mathrm{Re}\ V)_-\|_{L^{p+d/2}}^{p+d/2}.\end{aligned}$$

BGK : a basic method

For simplicity, let $A_0 = A_0^*$ be a bounded operator (with a reasonably simple spectrum).

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Let $A = A_0 + B$, where $B \in \mathcal{S}_p$, for integer $p \geq 1$ (B is not necessarily self-adjoint).

For $\lambda \in \rho(A_0) = \bar{\mathbb{C}} \backslash \sigma(A_0)$, consider

$$f(\lambda) = \det{}_p (A - \lambda)(A_0 - \lambda)^{-1} = \det{}_p (I + (A - A_0)(A_0 - \lambda)^{-1}).$$

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One has :

- $\lambda \in \sigma_d(A) \;\Leftrightarrow\; \mathrm{Ker}\,(A - \lambda I)$ is non-trivial and of finite dimension $\Leftrightarrow \lambda \in Z_f$ (i.e., $f(\lambda) = 0$) counting multiplicities,

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- $f \in Hol(\rho(A_0))$ and the following bound holds

$$
\begin{aligned}
|f(\lambda)| &\leq \exp\left(||(A - A_0)(A_0 - \lambda)^{-1}||^p_{\mathcal{S}_p} \right) \\
&\leq \exp\left(\frac{\Gamma_p\, ||A - A_0||^p_{\mathcal{S}_p}}{d(\lambda, \sigma(A_0))^p} \right).
\end{aligned}
$$

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On the next stage, one uses the (conformal) uniformization

$$\rho(A_0) = \bar{\mathbb{C}} \backslash \sigma(A_0) \xrightarrow{\phi} \mathbb{D} = \{|z| < 1\}$$

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Rather often, one sees that $F(z) = d(z, E)$, where $E \subset \mathbb{T},\ \#E < \infty$.

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Theorem (Borichev-Golinskii-K' 2010)

Let $f \in Hol(\mathbb{D})$, $|f(0)| = 1$, *and*

$$\log |f(z)| \leq \frac{D}{d(z, E)^q},$$

with $q \geq 0$. *Then for any* $\varepsilon > 0$,

$$\sum_{z \in Z_f} (1 - |z|)\, d(z, E)^{(q-1+\varepsilon)_+} \leq C(q, \varepsilon)\, D.$$

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where $p, q \ge 0$. *Then for any* $\varepsilon > 0$,

$$\sum_{z \in Z_f} (1-|z|)^{p+1+\varepsilon} d(z,E)^{(q-1+\varepsilon)_+} \le C(p,q,\varepsilon)\, D.$$

Applications and extensions

- (bounded) Jacobi matrices : Let $J - J_0 \in \mathcal{S}_\infty$, where

$$J_0 = J(\{1\},\{0\},\{1\}) \begin{bmatrix} 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & \dots \\ 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

$$J = J(\{a_k\},\{b_k\},\{c_k\}) = \begin{bmatrix} b_1 & c_1 & 0 & \dots \\ a_1 & b_2 & c_2 & \dots \\ 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

and $\{a_k\},\{b_k\},\{c_k\} \subset \mathbb{C}$.

Applications and extensions

It is clear that $\sigma(J_0) = [-2, 2]$. Then **[BGK' 2010]** : for $p > 1$ and $\forall \varepsilon > 0$

$$\sum_{\lambda \in \sigma_d(J)} \frac{d(\lambda, [-2, 2])^{p+1+\varepsilon}}{|\lambda - 2|\,|\lambda + 2|} \leq C(p, \varepsilon) \|J - J_0\|_{\mathcal{S}_p}^p.$$

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- Similar results for d-dimensional Jacobi matrices.
- **Hansmann-Katriel' 2011** : the above relation is improved to

$$\sum_{\lambda \in \sigma_d(J)} \frac{d(\lambda, [-2,2])^{p+\varepsilon}}{(|\lambda - 2|\,|\lambda + 2|)^{1/2}} \leq C(p, \varepsilon) \|J - J_0\|_{\mathcal{S}_p}^p.$$

Applications and extensions

- Let $\widetilde{J}_0$ be periodic (or finite-zone) Jacobi matrix. In particular,

$$\sigma(\widetilde{J}_0) = \cup_{j=1}^{n} [\alpha_j, \beta_j].$$

Let $\widetilde{E} = \{\alpha_j, \beta_j\}_{j=1,\dots,n}$, and $J - \widetilde{J}_0 \in \mathcal{S}_\infty$.

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$$\sum_{\lambda \in \sigma_d(J)} \frac{d(\lambda, \sigma(\widetilde{J}_0))^{p+1+\varepsilon}}{d(\lambda, \widetilde{E})\,(1+|\lambda|)} \leq C(p,\varepsilon)\|J - \widetilde{J}_0\|_{\mathcal{S}_p}^p.$$

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$$\sum_{\lambda \in \sigma_d(A)} d(\lambda, \sigma(A_0))^p \leq C_p||B||_{\mathcal{S}_p}^p$$

Extensions to unbounded operators

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• **Dubuisson' 2013** : d-dimensional Dirac, Klein-Gordon operators and fractional Laplacian.

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For instance, for $m \geq 0$ consider operator

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Theorem (Dubuisson' 2013)

Let $V \in L^p(\mathbb{R}^d)$, $p > d$.

- $(m > 0)$ *For* $0 < \tau$ *small enough*

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m| \, (1 + |\lambda|)^{p+\max\{p/2,d\}+2\tau-1}} \leq C(p,d,\tau) \|V\|_{L^p}^p.$$

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- $(m = 0)$ *Then, for* $0 < \tau$ *small enough*

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Open problems

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THANK YOU