New results about the link between entropic and displacement interpolations

Nicola Gigli

from a joint work with L. Tamanini

Content

- A curious convergence: informal statement
- The problem: need of an approximation procedure
- Statement of the main results and bits of proofs

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We can then define

$$\rho_t := P_t(f)\, P_{1-t}(g) \qquad t \in [0, 1]$$

Slowing down

For $\varepsilon > 0$ find $f^\varepsilon, g^\varepsilon$ such that

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and then

$$\rho_t^\varepsilon := P_{\varepsilon t}(f^\varepsilon) \, P_{\varepsilon(1-t)}(g^\varepsilon) \qquad t \in [0,1]$$

Informal statement of the convergence as $\varepsilon \downarrow 0$

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The actual statement is:

- on abstract spaces
- a statement about Gamma-convergence
- the main assumption is that the heat kernel satisfies the natural large deviation principle

How to find the functions $f^\varepsilon, g^\varepsilon$ (1/2)

Let R_ε be the measure on M^2 given by

$$\mathrm{d}R_\varepsilon(x,y) := r_\varepsilon(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{for} \qquad r_\varepsilon(x,y) := \frac{\mathrm{d}P_\varepsilon(\delta_x)}{\mathrm{d}x}(y).$$

$f^\varepsilon, g^\varepsilon$ solve our problem if and only if

$$f^\varepsilon \otimes g^\varepsilon\, R_\varepsilon \quad \text{is a transport plan from } \rho_0 \text{ to } \rho_1$$

where $f^\varepsilon \otimes g^\varepsilon(x,y) := f^\varepsilon(x) g^\varepsilon(y)$

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Its Euler equation is

$$\int \log\left(\frac{\mathrm{d}\pi^\varepsilon}{\mathrm{d}R_\varepsilon}\right)\mathrm{d}\sigma \qquad \text{for every } \sigma \text{ such that } \pi^1_*\sigma = \pi^2_*\sigma = 0$$

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This forces

$$\log\left(\frac{\mathrm{d}\pi^\varepsilon}{\mathrm{d}R_\varepsilon}\right) = a^\varepsilon \oplus b^\varepsilon$$

for some $a^\varepsilon, b^\varepsilon$.

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Thus for $f^\varepsilon := \exp(a^\varepsilon)$, $g^\varepsilon := \exp(b^\varepsilon)$ we have

$$\pi^\varepsilon = f^\varepsilon \otimes g^\varepsilon \, R_\varepsilon$$

A first link with optimal transport

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we see that

$$\begin{aligned}
\varepsilon H(\pi | R_\varepsilon) &= \varepsilon \int \log\left(\frac{\mathrm{d}\pi}{\mathrm{d}R_\varepsilon}\right) \mathrm{d}\pi \\
&\sim -\varepsilon \int \log\left(\frac{\mathrm{d}R_\varepsilon}{\mathrm{d}x\,\mathrm{d}y}\right) \mathrm{d}\pi \\
&\sim \frac{1}{2} \int d^2(x,y)\, \mathrm{d}\pi
\end{aligned}$$

The precise formulation involves large deviations and Gamma-convergence.

The dual problem

Some manipulation show that the dual of the problem

Minimize $\varepsilon H(\pi|R_\varepsilon)$ *among all transport plan* π *from* ρ_0 *to* ρ_1

is

$$\textit{Maximize} \int \varphi \,\mathrm{d}\rho_0 + \int \psi \,\mathrm{d}\rho_1 - \varepsilon \log\Big(\int e^{\frac{\varphi \oplus \psi}{\varepsilon}} \,\mathrm{d}R_\varepsilon \Big)$$

among all $\varphi, \psi \in C(M)$

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$$\textit{among all } \varphi, \psi \in C(M)$$

Moreover, if π^ε is a minimizer and $\varphi^\varepsilon, \psi^\varepsilon$ maximizers we have

$$\pi^\varepsilon = e^{\frac{\varphi \oplus \psi}{\varepsilon}} R_\varepsilon$$

Second link with optimal transport

Using again $r_\varepsilon(x,y) \sim c_\varepsilon(x) e^{-\frac{d^2(x,y)}{2\varepsilon}}$ we get

$$\varepsilon \log\Big(\int e^{\frac{\varphi\oplus\psi}{\varepsilon}}\,\mathrm{d}R_\varepsilon\Big) \sim \varepsilon \log\Big(\int e^{\frac{\varphi\oplus\psi - d^2/2}{\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y\Big)$$

$$\sim \max_{x,y} \varphi(x) + \psi(y) - \frac{d^2(x,y)}{2}$$

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thus the 'limit problem' is to

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which is the same as

$$\textit{maximize} \qquad \int \varphi\,\mathrm{d}\rho_0 + \int \psi\,\mathrm{d}\rho_1 \qquad \textit{among } \varphi\oplus\psi \le \frac{d^2}{2}$$

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Geodesics in $(\mathscr{P}_2(M), W_2)$

A W_2-geodesic (μ_t) on $\mathscr{P}_2(M)$ solves

$$\partial_t \mu_t + \operatorname{div}(\nabla \phi_t \, \mu_t) = 0$$

for functions (ϕ_t) such that

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Problem: no matter how nice μ_0, μ_1 are, in general the ϕ_t's are only semiconcave.

Can we approximate geodesics with smooth curves?

The problem informally stated

Given a geodeisc (μ_t), can we find curves (μ_t^ε) which are smooth and produce a second order approximation of (μ_t)?

First and second order differentiation formula

Given (μ_t) smooth define $(\phi_t), (a_t)$ by

$$\partial_t \mu_t + \mathrm{div}(\nabla \phi_t \, \mu_t) = 0$$
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Then for every f smooth we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}\mu_t = \int \langle \nabla f, \nabla\phi_t \rangle \, \mathrm{d}\mu_t$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int f \, \mathrm{d}\mu_t = \int \mathrm{Hess}(f)(\nabla\phi_t, \nabla\phi_t) + \langle \nabla f, \nabla a_t \rangle \, \mathrm{d}\mu_t$$

Rigorous statement of the problem

Given M smooth and compact and μ_0, μ_1 with bounded density, find (μ_t^ε) so that
0th order: (μ_t^ε) uniformly W_2-converges to the only W_2-geodesic (μ_t) from μ_0 to μ_1 with densities uniformly bounded

1st order: Up to subsequences $\phi_t^{\varepsilon_n} \to \phi_t$ in $W^{1,2}$, with (ϕ_t) a choice of Kantorovich potentials associated to (μ_t).

2nd order: For every $f \in W^{2,2}(M)$ and $\delta \in (0, 1/2)$ it holds

$$\int_\delta^{1-\delta} \int \langle \nabla f, \nabla a_t^\varepsilon \rangle \, \rho_t^\varepsilon \, \mathrm{dvol} \, \mathrm{d}t \qquad \to \qquad 0$$

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The estimates should depend only on ,

- the L^∞-norms of the densities of μ_0, μ_1
- a lower bound on the Ricci curvature of M
- an upper bound on the dimension of M
- an upper bound on the diameter of M

A natural attempt: viscous approximation of HJ

Let the geodesic (μ_t) be given as well as a Kantorovich potential φ_0.

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Let the geodesic (μ_t) be given as well as a Kantorovich potential φ_0.
Fix $\varepsilon > 0$ and solve

$$\partial_t \phi_t^\varepsilon + \frac{|\nabla \phi_t^\varepsilon|^2}{2} = \varepsilon \Delta \phi^\varepsilon \qquad \phi_0^\varepsilon = \varphi_0$$

and then the initial value problem

$$\partial_t \mu_t^\varepsilon + \mathrm{div}(\nabla \phi_t^\varepsilon \mu_t^\varepsilon) = 0 \qquad \mu_0^\varepsilon = \mu_0$$

Useful inequalities concerning the Hamilton-Jacobi-Bellman equation

Let (u_t) be a positive solution of the heat equation on the compact Riemannian manifold M.
Then:
$\sim$Hamilton's gradient estimate

$$|\nabla \log(u_t)| \leq \frac{C_1}{t} \qquad \forall t \in (0, 1]$$

Li-Yau Laplacian estimate

$$\Delta \log(u_t) \geq -\frac{C_2}{t} \qquad \forall t \in (0, 1]$$

The constants C_1, C_2 depend only on a lower bound on the Ricci curvature and an upper bound on dimension and diameter of M.

Where this leads

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However, we *cannot* obtain from PDE estimates convergence to 0 of the acceleration in any reasonable sense.

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Setting

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ρ_0, ρ_1 are bounded probability densities on M

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The dependence on ε, t of the various functions will sometimes be omitted.

In fact, M can be taken to be a bounded $\mathrm{RCD}^*(K, N)$ space.

Objects and PDEs involved

$$\rho = fg$$

$$\partial_t f = \frac{\varepsilon}{2}\Delta f \qquad\qquad -\partial_t g = \frac{\varepsilon}{2}\Delta g$$

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$$\begin{cases} \varphi := \varepsilon \log f \\ \partial_t \varphi = \frac{1}{2}|\nabla\varphi|^2 + \frac{\varepsilon}{2}\Delta\varphi \end{cases} \qquad\qquad \begin{cases} \psi := \varepsilon \log g \\ -\partial_t \psi = \frac{1}{2}|\nabla\psi|^2 + \frac{\varepsilon}{2}\Delta\psi \end{cases}$$

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$$\begin{aligned} \partial_t \rho + \mathrm{div}(\nabla\phi\rho) &= 0 \\ \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 &= \underbrace{-\frac{1}{8}\varepsilon^2(2\Delta\log(\rho) + |\nabla\log(\rho)|^2)}_{=:a} \end{aligned}$$

Statement of the convergence results

Theorem (G. Tamanini '16)
With the assumptions and notation just introduced we have:

0th order: (ρ_t^ε) uniformly W_2-converges to the only W_2-geodesic $(\bar{\rho}_t)$ from ρ_0 to ρ_1

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The estimates depend only on $\|\rho_0\|_{L^\infty}, \|\rho_1\|_{L^\infty}$, a lower bound on the Ricci curvature of M and an upper bound on the dimension and diameter of M.

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The estimates depend only on $\|\rho_0\|_{L^\infty}, \|\rho_1\|_{L^\infty}$, a lower bound on the Ricci curvature of M and an upper bound on the dimension and diameter of M.
Actually, the statement holds on bounded $\mathrm{RCD}^*(K, N)$ spaces.

Ingredients of the proof

0-th and 1-st order convergence are obtained as for the viscous approximation.

For the second order convergence we start from:
Theorem (Leonard)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(\rho_t^\varepsilon|\mathrm{vol}) = \frac{1}{2}\int \Big(\Gamma_2(\varphi_t^\varepsilon) + \Gamma_2(\psi_t^\varepsilon)\Big)\rho_t^\varepsilon \,\mathrm{dvol}$$
$$= \int \Big(\Gamma_2\big(\phi_t^\varepsilon\big) + \tfrac{\varepsilon}{2}\Gamma_2\big(\log(\rho_t^\varepsilon)\big)\Big)\rho_t^\varepsilon \,\mathrm{dvol}$$

where

$$\Gamma_2(h) := \Delta\frac{|\nabla h|^2}{2} - \langle \nabla h, \nabla\Delta h\rangle$$

A new controlled quantity

Say that $\mathrm{Ric}(M) \geq 0$ so that

$$\Gamma_2(h) \geq |\mathrm{Hess}(h)|^2$$

Then from Leonard's formula we deduce that

$$\sup_{\varepsilon \in (0,1)} \iint_{\delta}^{1-\delta} |\mathrm{Hess}(\phi_t^{\varepsilon})|^2 + \varepsilon^2 |\mathrm{Hess}(\log(\rho_t^{\varepsilon})|^2 \, \mathrm{d}t \, \mathrm{dvol} < \infty$$

for every $\delta \in (0, 1/2)$.

Second order differentiation formula on $\mathrm{RCD}^*(K, N)$ spaces

Theorem (G., Tamanini '16)

Let

- M be a compact $\mathrm{RCD}^*(K, N)$ space
- (μ_t) a W_2-geodesic with $\mu_0, \mu_1 \leq C\mathrm{vol}$
- $f \in H^{2,2}(M)$

Then
$$t \mapsto \int f \, \mathrm{d}\mu_t \qquad \text{is } C^2$$

and
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int f \, \mathrm{d}\mu_t = \int \mathrm{Hess}(f)(\nabla\phi_t, \nabla\phi_t) \, \mathrm{d}\mu_t$$

where $(\varphi_t) \subset W^{1,2}(M)$ is any continuous choice of functions such that

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In particular, the choice of evolved Kantorovich potential does the job.

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Thank you