Linking class of Lagrangian or totally real embeddings

Vincent Borrelli

1 Introduction and main results.

Given any totally real embedding j of the torus $\mathbb{T}^2$ into $\mathbb{C}^2$ and any non-vanishing vector field v in $\mathbb{T}^2$, one can construct a 1-class $\sigma_1(j,v) \in H^1(\mathbb{T}^2,\mathbb{Z})$ called *the linking class* of j in the direction v : if $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{T}^2$ is a loop which represents a generator of $H_1(\mathbb{T}^2,\mathbb{Z})$, one defines a differential map h_ϵ by :

$$\begin{array}{rccc} h_\epsilon : & \mathbb{S}^1 & \longrightarrow & \mathbb{C}^2 \\ & t & \longmapsto & j \circ \gamma(t) + J\epsilon v_{\gamma(t)} \end{array}$$

where J is the complex structure of $\mathbb{C}^2$. For sufficiently small ϵ, this map is injective and $h_\epsilon(\mathbb{S}^1) \cap j(\mathbb{T}^2) = \emptyset$. Thus, one can speak about the linking of h_ϵ and j. One puts $\sigma_1(j,v)([\gamma]) = lk(j,h_\epsilon)$ (cf. [2] for instance). This class is an effective tool to detect totally real isotopy classes of $\mathbb{T}^2$ which are not detected by the Maslov class. Nevertheless, if j is a *Lagrangian* embedding and τ is a translation field on $\mathbb{T}^2$, one has, according to Y. Eliashberg and L. Polterovich, $\sigma_1(j,\tau) = 0$ ([2]). This result does not generalize to totally real embeddings ([3],[9]).

In the first part of this article, we generalize the construction of the linking class of a torus to any (compact, oriented, without boundary) manifold M^n admitting a totally real embedding j in $\mathbb{C}^n$. We associate to every totally real embedding $j : M^n \longrightarrow \mathbb{C}^n$ and to every non vanishing vector field v on M^n an element $\sigma_{n-1}(j,v)$ of $H^{n-1}(M^n,\mathbb{Z})$: its linking class. As in the case of the torus $\mathbb{T}^2$, the linking class σ_{n-1} allows us to detect arcwise connected components of the space of totally real embeddings $EmbTR(M^n,\mathbb{C}^n)$.

In the second part of this article, we show the non triviality of this linking class by building families of totally real embeddings into $\mathbb{C}^n$ of (topological) product manifolds of the form $M^{n-1} \times \mathbb{S}^1$. We get the following

Theorem 1-2. – *Let M^{n-1} be a compact, oriented manifold without boundary. Let v be a non-vanishing vector field of $M^{n-1} \times \mathbb{S}^1$ tangent to the factor*

$\mathbb{S}^1$ *of* $M^{n-1} \times \mathbb{S}^1$.

1. *If* n *is even and if* M^{n-1} *admits a totally real immersion into* $\mathbb{C}^{n-1}$ *with an algebraic self-intersection number equals to 0 (resp. equals to 1), then, for all* $p \in \mathbb{Z}$, *there exists a totally real embedding* $j : M^{n-1} \times \mathbb{S}^1 \longrightarrow \mathbb{C}^n$ *such that*

$$< \mu(j), [\mathbb{S}^1] >= 2 \text{ and } < \sigma_{n-1}(j,v), [M^{n-1}] >= 2p \quad (\text{resp. } 2p+1),$$

($\mu(j)$ *denotes the Maslov class of* j*).*

2. *If* n *is odd and if* M^{n-1} *is an hypersurface of* $\mathbb{R}^n$ *such that* $\frac{\chi(M^{n-1})}{2} \equiv 1 \bmod 2$, *then, there exists two totally real embeddings* $j_0, j_1 : M^{n-1} \times \mathbb{S}^1 \longrightarrow \mathbb{C}^n$ *such that :*

$$< \sigma_{n-1}(j_0,v), [M^{n-1}] >= 0,$$

$$< \sigma_{n-1}(j_1,v), [M^{n-1}] > \neq 0.$$

As an illustration of this theorem, we obtain

Corollary 1.– *Let* M^n *be one of the following manifolds :*
1) $\mathbb{S}^1 \times \mathbb{S}^{2q-1}$,
2) the unitary group $U(2q)$,
3) the oriented Lagrangian Grassmannian $\Lambda(k)$ *with* $k = 4q$ *or* $k = 4q+3$,
Then, the space $EmbTR(M^n, \mathbb{C}^n)$ *has a countable family of arcwise connected components which are not detected by the Maslov class.*

These results broaden those of L. Polterovich in [9]. In the third part of this article, we study the relationship between isotopy classes of *ordinary* embeddings and the linking class. In the case where M^n is parallelizable, we obtain the following

Theorem 3.– *Let* M^n $(n \geq 4)$ *be a compact, parallelizable manifold without boundary. Let* j_0, j_1 *be two totally real (or Lagrangian) embeddings which belong to the same regular homotopy class of totally real (or Lagrangian) immersions. Then, the following propositions are equivalent :*

1. j_0 *and* j_1 *belong to the same isotopy class of ordinary embeddings,*

2. *there exists a non-vanishing vector field* v *on* M^n *such that :*
$$\begin{cases} \sigma_{n-1}(j_0,v) & = & \sigma_{n-1}(j_1,v) & \text{if } n \text{ is odd,} \\ \mu^*(\sigma_{n-1}(j_0,v)) & = & \mu^*(\sigma_{n-1}(j_1,v)) & \text{if } n \text{ is even,} \end{cases}$$

($\mu^* : H^{n-1}(M^n, \mathbb{Z}) \longrightarrow H^{n-1}(M^n, \mathbb{Z}_2)$ denotes the coefficients homomorphism).

Remark 1. – In fact, if $\sigma_{n-1}(j_0, v) = \sigma_{n-1}(j_1, v)$ for one non-vanishing vector field v, then $\sigma_{n-1}(j_0, w) = \sigma_{n-1}(j_1, w)$ for every non-vanishing vector field w.

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2 The linking class σ_{n-1}.

2.1 Notations and definitions.

In what follows, we denote by M^n a compact, oriented manifold without boundary and by T^1M^n its unitary bundle (for some metric g which plays no role). We also denote by Ω the symplectic form of $\mathbb{C}^n$ and by J the canonical complex structure. An immersion $j : M^n \longrightarrow \mathbb{C}^n$ is said to be *Lagrangian* if $j^*\Omega = 0$. It is said to be *totally real* if :

$$\forall p \in M^n, T_{j(p)}j(M^n) \oplus JT_{j(p)}j(M^n) = \mathbb{C}^n.$$

We denote by $Imm(M^n, \mathbb{R}^{2n})$ the space of immersions of M^n in $\mathbb{R}^{2n}$, by $Emb(M^n, \mathbb{R}^{2n})$ the space of embeddings, by $ImmTR(M^n, \mathbb{C}^n)$ the space of totally real immersions, by $ImmLag(M^n, \mathbb{C}^n)$ the space of Lagrangian immersions, etc.

2.2 Construction of σ_{n-1}.

Let $j : M^n \longrightarrow \mathbb{R}^{2n} \simeq \mathbb{C}^n$ be a totally real embedding and v a section of T^1M^n. (Such a section exists because the Euler-Poincaré characteristic $\chi(M^n)$ of a manifold admitting a totally real embeddings is 0).We build, from j, a family of differentiable maps j_ϵ :

$$\begin{array}{rccc} j_\epsilon : & M^n & \longrightarrow & \mathbb{C}^n \\ & b & \longmapsto & j(b) + \epsilon J dj_b(v). \end{array}$$

Since j is a totally real embedding, $j_\epsilon(M^n) \cap j(M^n) = \emptyset$ for small ϵ. The (oriented) manifold $j_\epsilon(M^n)$ represents a n-cycle of $\mathbb{C}^n \setminus j(M^n)$. The exact sequence of the pair $(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n))$ and the Alexander duality allow us to make the identifications :

$$H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H_{n+1}(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}).$$

Definition. – The *linking class* associated to the totally real embedding j and to the non-vanishing vector field v is the class $\sigma_{n-1}(j,v) \in H^{n-1}(M^n,\mathbb{Z})$ represented in $H_n(\mathbb{C}^n \setminus j(M^n),\mathbb{Z})$ by the n-cycle $[j_\epsilon(M^n)]$.

The class σ_{n-1} is obviously invariant for isotopy classes of totally real embeddings. Indeed, it is easy to check that if $j_1, j_2 : M^n \longrightarrow \mathbb{C}^n$ are two totally real embeddings and if v is an non-vanishing vector field on M^n such that $\sigma_{n-1}(j_1,v) \neq \sigma_{n-1}(j_2,v)$, then j_1 and j_2 are not isotopic through totally real embeddings.

Remark 2. – One can broaden this construction to obtain other invariants for isotopy classes of totally real embeddings. Let $0 < k < n-1$. Assume that T^1M^n admits a $(n-k-1)$-sphere subbundle whose total space $S^{n-k-1}M^n$ is oriented. The manifold $S^{n-k-1}M^n$ is connected, compact, without boundary and $(2n-k-1)$ dimensional. Denote by $[S^{n-k-1}M^n]$ the element of $H_{2n-k-1}(T^1M^n,\mathbb{Z})$ represented by $S^{n-k-1}M^n$. One can build a family of differential maps :

$$\begin{array}{rccc} j_\epsilon : & S^{n-k-1}M^n & \longrightarrow & \mathbb{C}^n \\ & (b,v) & \longmapsto & j(b)+\epsilon J dj_b(v). \end{array}$$

The (oriented) manifold $j_\epsilon(S^{n-k-1}M^n)$ represents a $(2n-k-1)$-cycle of $\mathbb{C}^n \setminus j(M^n)$ (for small ϵ).The exact sequence of the pair $(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n))$ and the Alexander duality allow us to make the identifications :

$$H_{2n-k-1}(\mathbb{C}^n \setminus j(M^n),\mathbb{Z}) \simeq H_{2n-k}(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n),\mathbb{Z}) \simeq H^k(M^n,\mathbb{Z}).$$

One can define the k-th *linking class* associated to the totally real embedding j and to the oriented $(n-k-1)$-sphere bundle $S^{n-k-1}M^n$ to be the class $\sigma_k(j, S^{n-k-1}M^n) \in H^k(M^n,\mathbb{Z})$ represented in $H_{2n-k-1}(\mathbb{C}^n \setminus j(M^n),\mathbb{Z})$ by the $(2n-k-1)$-cycle $[j_\epsilon(S^{n-k-1}M^n)]$.

3 Non triviality of σ_{n-1}.

3.1 Preliminaries.

Let $j : M^n \longrightarrow \mathbb{R}^{2n}$ be an immersion such that all self-intersection points are isolated, double and transversal. We denote by $I_1, ..., I_N$ each of the double points of $j(M^n)$ and by $p_1, q_1, ..., p_N, q_N$ the $2N$ points of M^n such that $j(p_k) = j(q_k) = I_k$. We also denote by $(e_1, ..., e_{2n}, e_{2n+1})$ the canonical basis of $\mathbb{R}^{2n} \oplus \mathbb{R}$. We want to alter the immersion j into an embedding of M^n in $\mathbb{R}^{2n+1}$ by "pushing out" $j(p_k)$ and $j(q_k)$ along the line spanned by e_{2n+1}, but in opposite directions. For each couple (p_k, q_k), there are two ways to do that. Either, one deforms a neighbourhood of p_k in the direction $+e_{2n+1}$ and a neighbourhood of q_k in the direction $-e_{2n+1}$, or one does the opposite. Actually there are 2^N different ways to alter j into an embedding by this process. Let θ be a sequence of N signs $+$ or $-$. We define a function $\alpha_\theta : M^n \longrightarrow \mathbb{R}$ such that if the k-th term of the sequence θ is a $+$, α_θ is a positive bump function on a small disk $D^n(p_k)$ of M^n centered in p_k and a negative bump function on a small disk $D^n(q_k)$ centered in q_k. If the k-th term is a $-$, it is the opposite. Out of these $2N$ disks, we put : $\alpha_\theta = 0$. We define an embedding j_θ of M^n in $\mathbb{R}^{2n+1}$ by

$$\begin{array}{rccc} j_\theta : & M^n & \longrightarrow & \mathbb{R}^{2n} \oplus \mathbb{R} \\ & x & \longmapsto & (j(x), \alpha_\theta(x)), \end{array}$$

and, for every $\epsilon > 0$, an immersion $j_{\theta,\epsilon}$ by

$$\begin{array}{rccl} j_{\theta,\epsilon} : & M^n & \longrightarrow & \mathbb{R}^{2n} \oplus \mathbb{R} \\ & x & \longmapsto & (j(x), \alpha_\theta + \epsilon). \end{array}$$

If $\|\alpha_\theta\|_1$ and ϵ are sufficiently small, the field e_{2n+1} is everywhere transversal to j_θ. Hence, $j_{\theta,\epsilon}$ is an embedding and moreover $j_\theta(M^n) \cap j_{\theta,\epsilon}(M^n) = \emptyset$.

Lemma 1. – *Let Lk be the following map :*

$$\begin{array}{rccc} Lk : & \{0, 1, ..., 2^N - 1\} & \longrightarrow & \mathbb{Z} \\ & \theta & \longmapsto & Linking(j_\theta, j_{\theta,\epsilon}). \end{array}$$

One has :

	N *even*	N *odd*
n *even*	$Im\ Lk = \{2k\}$ *with* $-N \leq 2k \leq N$	$Im\ Lk = \{2k+1\}$ *with* $-N \leq 2k+1 \leq N$
n *odd*	$Im\ Lk = \{-N, ..., -2, 0, 2, ..., N\}$	$Im\ Lk = \{-N, ..., -1, 1, ..., N\}$

In particular, $Linking(j_\theta, j_{\theta,\epsilon})$ has the same parity as N. If N is odd, it can not be zero.

Proof of lemma 1. – We give here the main arguments of the proof. One puts $M^n_\theta = j_\theta(M^n_\theta)$ and $M^n_{\theta,\epsilon} = j_{\theta,\epsilon}(M^n_{\theta,\epsilon})$. We first assume that there exists a submanifold W^{n+1} of $\mathbb{R}^{2n} \oplus \mathbb{R}$ such that $\partial W^{n+1} = M^n_\theta$. The following formula is well-known :

$$Lk(M^n_\theta, M^n_{\theta,\epsilon}) = (-1)^n W^{n+1}.M^n_{\theta,\epsilon}. \qquad (*)$$

(Here $W^{n+1}.M^n_{\theta,\epsilon}$ denotes the algebraic intersection number of W^{n+1} and $M^n_{\theta,\epsilon}$). We assume besides that there are exactly N intersection points and

that all of them are transversal. When deforming $M^n_{\theta,\epsilon}$ around an intersection point from the direction $+e_{2n+1}$ to the direction $-e_{2n+1}$, the sign of the index of the intersection point changes if n is odd but remains unchanged if n is even. This explains the values that appear in the array.
In general, there does not exist submanifold W^{n+1} of $\mathbb{R}^{2n} \oplus \mathbb{R}$ such that $\partial W^{n+1} = M^n_\theta$. Nevertheless, one can always build a submanifold W^{n+1} with the following properties :

1) $\partial W^{n+1} = M^n_\theta \cup V^n_\theta$ where V^n_θ is a submanifold of $\mathbb{R}^{2n} \oplus \mathbb{R}$, such that $Lk(V^n_\theta, M^n_{\theta,\epsilon}) = 0$.
2) there are exactly N intersection points of W^{n+1} with $M^n_{\theta,\epsilon}$ and all of them are transversal.

The condition on V^n_θ allows us to use equality $(*)$. Hence, the previous arguments apply. $\square$

3.2 Self-intersection of totally real immersions.

3.2.1 A Lagrangian model

Let

$$\begin{array}{rccc} \gamma: & I = [0,1] & \longrightarrow & \mathbb{R} \oplus i\mathbb{R} \\ & t & \longmapsto & (x(t), y(t)) \end{array}$$

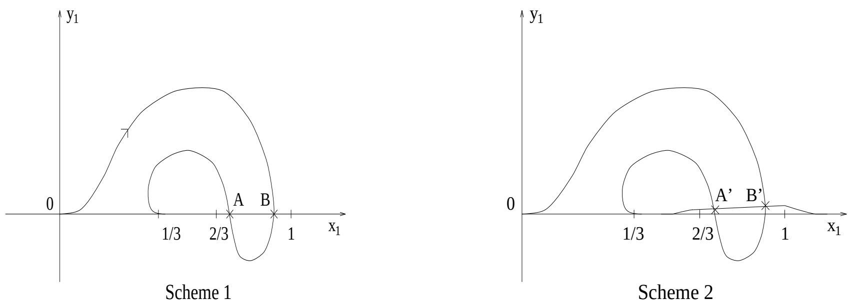
be any curve of $T^*\mathbb{R}$ such that :

$$\begin{array}{llll} 1) & \forall t \in [0,\epsilon] \ : \ \gamma(t) = (t,0), & 2) & \gamma([0,1]) \subset \{x \geq 0\}, \\ 3) & \gamma([1-\epsilon,1]) \subset \{x > 0, y = 0\}, & 4) & \|\frac{\partial \gamma}{\partial t}(t)\| > 0 \text{ for all } t \in [0,1]. \end{array}$$

One denotes by D^{n-1} the $(n-1)$-dimensional closed disk of $\mathbb{R}^{n-1} \oplus i\{0\}$ with radius 1 and centered at 0 and by $(t,u) \in \mathbb{R}^*_+ \times \mathbb{S}^{n-2}$ the spherical coordinates in $\mathbb{R}^{n-1} \setminus \{0\}$. We define a Lagrangian immersion f_γ of D^{n-1} into $\mathbb{R}^{n-1} \oplus i\mathbb{R}^{n-1}$ (with boundary lying inside $\mathbb{R}^{n-1} \oplus i\{0\}$) by putting :

$$\begin{cases} f_\gamma(t,u) = (x(t)u, y(t)u) & \text{if } (t,u) \in D^{n-1} \setminus \{0\}, \\ f_\gamma(0) = 0. \end{cases}$$

Let Π be the Lagrangian n-plane $\{y_1 = 0, ..., y_n = 0\}$ and γ be one suitable parametrization of the following curve (scheme 1):



The intersection set between f_γ and Π contains two $(n-2)$-spheres : $\mathcal{C}_A = \{uA/u \in \mathbb{S}^{n-2}\}$ and $\mathcal{C}_B = \{uB/u \in \mathbb{S}^{n-2}\}$. One can perturb a little this situation to obtain four transversal double points instead of these two $(n-2)$-spheres. (For example, change the n-plane Π to $\Pi_\epsilon = \{y_1 = \epsilon x_1, y_2 = \epsilon, ..., y_n = \epsilon\}$ in the region $\{2/3 \leq \|x\| \leq 1\}$ and smooth the corners, *cf.* scheme 2).

3.2.2 Adding self-intersection points

Let $j : M^{n-1} \longrightarrow \mathbb{C}^{n-1}$ be a Lagrangian immersion having $N(j)$ double points. From j, it is not difficult to build a totally real immersion h having more self-intersection points:

1) We apply the theorem of the symplectic tubular neighbourhood of Weinstein ([11]) to glu a copy of the Lagrangian model f_γ in the neighborhood of a regular point of j. We call j_1 the resulting immersion.

2) We pertub j_1 a little to obtain a new (totally real) immersion h which has only transversal self-intersection points. According to §3.2.1 there exists (at least) one perturbation which add four intersection points.

3.3 The class σ_{n-1} of some embeddings of $M^{n-1} \times \mathbb{S}^1$ with n even.

Let $f : M^n \longrightarrow \mathbb{C}^n$ be an immersion. We denote by $I(f)$ the *algebraic* number of double points of any approximation of f with transversal double points. If n is even (and M^n orientable), $I(f) \in \mathbb{Z}$, otherwise, $I(f) \in \mathbb{Z}_2$.

Theorem 1. – *Let M^{n-1} be a compact, oriented odd dimensional manifold without boundary. If M^{n-1} admits a totally real immersion j into $\mathbb{C}^{n-1}$ such that $I(j) = 0$ (resp. such that $I(j) = 1$), then, for all $p \in \mathbb{Z}$, there exists a totally real embedding $h : M^{n-1} \times \mathbb{S}^1 \longrightarrow \mathbb{C}^n$ such that*

$$< \mu(h), [\mathbb{S}^1] >= 2 \textit{ and } < \sigma_{n-1}(h, v), [M^{n-1}] >= 2p \quad (\textit{resp. } 2p + 1),$$

($\mu(h)$ denotes the Maslov class of h and v a non-vanishing vector field of $M^{n-1} \times \mathbb{S}^1$ tangent to the factor $\mathbb{S}^1$).

Proof of theorem 1. – Let $j : M^{n-1} \longrightarrow \mathbb{C}^{n-1}$ be a totally real immersion and let N be the number of self-intersection points of j. As in §3.1, we associate to j the embedding :

$$\begin{array}{rccl} j_\theta : & M^{n-1} & \longrightarrow & \mathbb{R}^{2n-2} \oplus \mathbb{R} \subset \mathbb{C}^{n-1} \times \mathbb{C} \\ & x & \longmapsto & (j(x), 1 + \alpha_\theta(x)). \end{array}$$

Without loss of generality, we can always assume that $j_\theta(M^{n-1})$ lies inside $\mathbb{R}^{2n-2}_+ \oplus \mathbb{R} = \{x_1 > 0, ..., x_{2n-2} > 0\}$. We put

$$\begin{array}{rccc} h_\theta : & M^{n-1}\times \mathbb{S}^1 & \longrightarrow & \mathbb{C}^{n-1}\times\mathbb{C} \\ & (x,\varphi) & \longmapsto & (j(x),(1+\alpha_\theta(x))e^{i\varphi}). \end{array}$$

If $\|\alpha_\theta\|_1$ is sufficiently small, h_θ is a totally real embedding. We want to compute $<\sigma_{n-1}(h_\theta,v),[M^{n-1}]>$, where v is the non-vanishing vector field given by $v(x,\varphi)=-ie^{i\varphi}$. We choose the element $M_\theta^{n-1}=j_\theta(M^{n-1})$ to represent $[M^{n-1}]$. The computation of $<\sigma_{n-1}(h_\theta),[M^{n-1}]>$ reduces the problem to find the linking number between h_θ and $j_{\theta,\epsilon}$ where

$$\begin{array}{rccc} j_{\theta,\epsilon} : & M^{n-1} & \longrightarrow & \mathbb{C}^{n-1}\times\mathbb{C} \\ & x & \longmapsto & h_\theta(x,0)+J\epsilon v(x,0)=(j(x),1+\alpha_\theta(x)+\epsilon). \end{array}$$

A computation shows that : $<\sigma_{n-1}(h_\theta,v),[M^{n-1}]>=(-1)^n Lk(j_\theta,j_{\theta,\epsilon})$. Since the dimension of M^{n-1} is odd, lemma 1 implies :

1) if N is even, $Lk(j_\theta,j_{\theta,\epsilon})$ takes all the even values between $-N$ and N when θ varies,

2) if N is odd, $Lk(j_\theta,j_{\theta,\epsilon})$ takes all the odd values between $-N$ and N.

The hypothesis $I(j)=0$ (resp. $I(j)=1$) implies N even (resp. N odd). Therefore : $<\sigma_{n-1}(h_\theta,v),[M^{n-1}]>=2p$ (resp. $2p+1$). According to §3.1, we can take N as large as we want. Indeed, due to the theorem of Gromov-Lees, if M^n admits a totally real immersion j into $\mathbb{C}^n$ with $I(j)=0$ (resp. $I(j)=1$), then it admits a Lagrangian immersion $\tilde{\jmath}$ with $I(\tilde{\jmath})=0$ (resp. $I(\tilde{\jmath})=1$). One can begin with $\tilde{\jmath}$ and use §3.1 to add self-intersection points.

Regarding the Maslov class of h_θ on $[\mathbb{S}^1]$, an easy computation shows that $<\mu(h_\theta),[\mathbb{S}^1]>=2$. $\square$

Remark 3. – When the dimension of M^{n-1} is even, the same approach shows that the integer $<\sigma_{n-1}(h_\theta,v),[M^{n-1}]>$ can not be zero if N is odd.

Corollary 1. – *Let M^n be one of the following manifolds :*

1) $\mathbb{S}^1\times\mathbb{S}^{2q-1}$,

2) the unitary group $U(2q)$,

3) the oriented Lagrangian Grassmannian $\Lambda(k)$ with $k=4q$ or $k=4q+3$,

Then, the space $EmbTR(M^n,\mathbb{C}^n)$ has a countable family of arcwise connected components which are not detected by the Maslov class.

Proof of corollary 1. – Since topologically, $U(2q)\simeq\mathbb{S}^1\times SU(2q)$ and $\Lambda(k)=\mathbb{S}^1\times SU(k)/SO(k)$, it is enough to show that $\mathbb{S}^{2q-1}$, $SU(2q)$ and $SU(k)/SO(k)$ admit a totally real immersion. This is well known for $\mathbb{S}^{2q-1}$ and $SU(2q)$. For $SU(k)/SO(k)$, one has to notice that the oriented Lagrangian Grassmannian $\Lambda(k)$ admits a Lagrangian embedding in $\mathbb{C}^{\frac{k(k+1)}{2}}$ (*cf.*

[8], for instance). Thus $T\Lambda(k)\otimes\mathbb{C}$ is trivial and so is $T(SU(k)/SO(k))\otimes\mathbb{C}$. Using the Gromov-Lees theorem, we deduce that $SU(k)/SO(k)$ admits a Lagrangian immersion in $\mathbb{C}^{\frac{k(k+1)}{2}-1}$. The values $k=4q$ and $k=4q+3$ are those for which $SU(k)/SO(k)$ is odd dimensional. $\square$

3.4 The class σ_{n-1} of some embeddings of $M^{n-1}\times\mathbb{S}^1$ with n odd.

Theorem 2. – *Let M^{n-1} be a compact, oriented, even dimensional hypersurface without boundary embedded in $\mathbb{R}^n$ such that $\frac{\chi(M^{n-1})}{2}\equiv 1 \mod 2$. Then, there exists two totally real embeddings $j_0, j_1 : M^{n-1}\times\mathbb{S}^1 \longrightarrow \mathbb{C}^n$ such that :*

$$< \sigma_{n-1}(j_0,v),[M^{n-1}] >= 0,$$
$$< \sigma_{n-1}(j_1,v),[M^{n-1}] >\neq 0,$$

(v denotes a non-vanishing vector field tangent to the factor $\mathbb{S}^1$ of $M^{n-1}\times\mathbb{S}^1$).

Proof of theorem 2. – An oriented hypersurface is stably parallelizable, thus it admits a Lagrangian immersion h. The algebraic number $I(h)$ of self-intersection points is given by the formula :

$$I(h) = (-1)^{\frac{n+1}{2}}\frac{\chi(M^{n-1})}{2}.$$

Therefore the number $N(h)$ of self-intersection points is odd. We deduce from remark 3, the existence of a totally real embedding $j_1 : M^{n-1}\times\mathbb{S}^1 \longrightarrow \mathbb{C}^n$ such that $< \sigma_{n-1}(j_1),[M^{n-1}] >\neq 0$.

Let $f : M^{n-1} \longrightarrow \mathbb{R}^n$ denote the inclusion $M^{n-1}\subset\mathbb{R}^n$ and n_x the normal vector of $f(M^{n-1})$ at $f(x)$. In [7], T. Kawashima shows that, for small η, the map

$$\begin{array}{rccc} j_0 : & M^{n-1}\times\mathbb{S}^1 & \longrightarrow & \mathbb{C}^n \\ & (x,\varphi) & \longmapsto & f(x)+\eta e^{i\varphi}n_x \end{array}$$

is a Lagrangian embedding of $M^{n-1}\times\mathbb{S}^1$ in $\mathbb{C}^n$. As in lemma 1, the computation of $< \sigma_{n-1}(j_0,v),[M^{n-1}] >$ reduces the problem to a computation of intersection points. It is easy so see that this number is zero. We conclude :

$$< \sigma_{n-1}(j_0,v),[M^{n-1}] >= 0. \quad \square$$

4 The linking class σ_{n-1} and ordinary embeddings

The first two sections of this part are devoted to the results of A. Haefliger and M. Hirsch about the classification of immersions and embeddings. These results are needed for the two last sections in which we prove theorem 3.

4.1 Classification of immersions of M^n into $\mathbb{R}^{2n}$ ([6],[4]).

Let M^n be a compact, oriented manifold without boundary. One denotes by FM^n the frame bundle of M^n and $V_{2n,n}$ the Stiefel-Whitney manifold of n-frames in $\mathbb{R}^{2n}$. The group $Gl(n,\mathbb{R})$ acts on FM^n and on $V_{2n,n}$ by right multiplication. One denotes by $T_{2n,n}(M^n)$ the associated bundle to FM^n with fibre $V_{2n,n}$.

If $f : M^n \longrightarrow \mathbb{R}^{2n}$ is an immersion, its derivative induces a map :

$$\begin{array}{rccc} df: & FM^n & \longrightarrow & V_{2n,n} \\ & (X_1,...,X_n) & \longmapsto & (df(X_1),...,df(X_n)). \end{array}$$

The map df is equivariant for the action of $Gl(n,\mathbb{R})$, i.e. $df(XA) = df(X)A$. Therefore, it induces a section $s(f)$ of $T_{2n,n}(M^n)$. Let $\Gamma(T_{2n,n}(M^n))$ be the space of all sections of $T_{2n,n}(M^n)$ and $\pi_0(Imm(M^n,\mathbb{R}^{2n}))$ be the set of regular homotopy classes of immersions of M^n in $\mathbb{R}^{2n}$. The central result is the

Immersion classification theorem (M. Hirsch).
The following map is a bijection :

$$\begin{array}{ccc} \pi_0(Imm(M^n,\mathbb{R}^{2n})) & \longrightarrow & \pi_0(\Gamma(T_{2n,n}(M^n))) \\ [f] & \longmapsto & [s(f)]. \end{array}$$

4.2 Classification of embeddings of M^n into $\mathbb{R}^{2n}$ ([5]).

Let M^n be a compact, oriented manifold without boundary. Let M_0^n denotes M^n minus a point p. If $f : M^n \longrightarrow \mathbb{R}^{2n}$ is an embedding, we denote by $NM_0^n(f)$ the normal bundle of $f(M_0^n)$ in $\mathbb{R}^{2n}$. It is well known that two embeddings of M_0^n into $\mathbb{R}^{2n}$ are isotopic. Thus, one can take off f in the notation of the normal bundle NM_0^n. Let ν be a section of the unit normal bundle $N^1M_0^n$. From f and ν, one can define a new embedding $f_\nu : M^n \longrightarrow \mathbb{R}^{2n}$ by putting :

$$f_\nu(x) = \begin{cases} f(x) + \lambda(x)\nu(x) & \text{si } x \in M^n \setminus D_1, \\ f(x) & \text{si } x \in D_1, \end{cases}$$

where D_1 is a disk of (small) radius r of M^n centered at p and $\lambda : M^n \longrightarrow \mathbb{R}$ is a C^∞ function equal to 0 on D_1 and to ϵ (small) on $M^n \setminus D_2$, D_2 beeing a concentric disk of radius $2r$. One has

Proposition (A. Haefliger, M. Hirsch). – *Let $f : M^n \longrightarrow \mathbb{R}^{2n}$ be an embedding. There exists a section ν of $N^1M_0^n$ such that $f_\nu(M^n)$ is homologous to zero in $\mathbb{R}^{2n} \setminus f(M^n \setminus D_2)$. Moreover, this section is unique, up to homotopy (between sections of $N^1M_0^n$).*

Definition. – Such a section of $N^1M_0^n$ is said to be *associated* to the embedding f. One denotes by $\nu(f)$ any associated section to f.

Let $V_{2n,n+1}$ be the Stiefel-Whitney manifold of $(n+1)$-frames in $\mathbb{R}^{2n}$ and let $T_{2n,n+1}(M^n)$ be the associated bundle to FM^n with fibre $V_{2n,n+1}$ (the group $Gl(n,\mathbb{R})$ acting on the first n vectors of a $(n+1)$-frame in $\mathbb{R}^{2n}$). Every embedding $f : M^n \longrightarrow \mathbb{R}^{2n}$, induces a section $\tilde{s}(f) = (s(f), \nu(f))$ of $T_{2n,n+1}(M_0^n)$ ($s(f)$ is the section built in 4.1). One denotes by $\Gamma(T_{2n,n+1}(M_0^n))$ the space of sections of $T_{2n,n+1}(M_0^n)$. The central result is the

Embedding classification theorem (A. Haefliger, M. Hirsch). *If $n \geq 4$, then the following map is a bijection :*

$$\begin{array}{ccc} \pi_0(Emb(M^n, \mathbb{R}^{2n})) & \longrightarrow & \pi_0(\Gamma(T_{2n,n+1}(M_0^n))) \\ [f] & \longmapsto & [\tilde{s}(f)]. \end{array}$$

4.3 Some properties of the linking class.

4.3.1 Existence of a field w such that $\sigma_{n-1}(j,w) = 0$.

Let G be a n-plane of the Euclidian space $\mathbb{E}^{2n} \simeq \mathbb{C}^n$. We denote by $\pi_G : \mathbb{E}^{2n} \to G$ the orthogonal projection and we define a linear map $F_G : G^\perp \to G$ by putting $F_G = \pi_G \circ J$. Note that F_G is invertible if and only if G is a totally real n-plane. If j is a totally real embedding of M^n into the Euclidian space $\mathbb{E}^{2n} \simeq \mathbb{C}^n$ we denote by $F_{j(m)}$ (or simply F) the map $F_{T_{j(m)}j(M^n)}$.

Let $\nu(j)$ be a section of $N^1M_0^n$ associated to the embedding j. This section can be extended to a section $\tilde{\nu}(j)$ of N^1M^n, (because $\chi(NM^n) = \chi(TM^n) = 0$). According to the definition of both $\nu(j)$ and σ_{n-1}, it is easy to check that $\sigma_{n-1}(j, F\nu(j)) = 0$. We have obtained

Proposition 1. – *Let $j : M^n \longrightarrow \mathbb{C}^n$ be a totally real embedding, $\nu(j)$ a section of $N^1M_0^n$ associated to the embedding j and $\tilde{\nu}(j)$ an extension of $\nu(j)$ on M^n (such an extension always exists). Then, one has :*

$$\sigma_{n-1}(j, F\nu(j)) = 0.$$

4.3.2 A formula connecting $\sigma_{n-1}(j,v)$ and $\sigma_{n-1}(j,v')$

Proposition 2. – *Let v, v' be two sections of the unitary tangent bundle T^1M^n. Then, one has :*

$$\sigma_{n-1}(j,v) - \sigma_{n-1}(j,v') = d(v,v'),$$

where the difference class $d(v,v') \in H^{n-1}(M^n, \mathbb{Z})$ is the first obstruction to build a homotopy from the section v to the section v'.

The proof of this proposition is left to the reader (it is only a reformulation of an argument appearing in the proof 4.3 of [5]). When M^n is parallelizable, we have an explicit expression for $d(v, v')$. Let $R = (e_1, ..., e_n)$ denote a direct frame field on M^n and v a non vanishing section of TM^n. We define a function $f_{R,v} : M^n \longrightarrow \mathbb{S}^{n-1} \subset \mathbb{E}^n$ by associating to each point p of M^n, the (normalized) coordinates of v_p in R_p. Proposition 2 gives :

$$\sigma_{n-1}(j, v) = \sigma_{n-1}(j, e_1) + (-1)^{n(n+1)/2} f^*_{R,v} u.$$

where u is the generator of $H^{n-1}(\mathbb{S}^{n-1}, \mathbb{Z})$ such that $< u, [\mathbb{S}^{n-1}] >= 1$.

4.4 Relationship between σ_{n-1} and ordinary embeddings.

Theorem 3. – *Let M^n $(n \geq 4)$ be a compact, parallelizable manifold without boundary. Let j_0, j_1 be two totally real (or Lagrangian) embeddings which belong to the same regular homotopy class of totally real (or Lagrangian) immersions. Then, the following propositions are equivalent :*

1. *j_0 and j_1 belong to the same isotopy class of ordinary embeddings,*

2. *there exists a non-vanishing vector field v on M^n such that :*
$$\begin{cases} \sigma_{n-1}(j_0, v) = \sigma_{n-1}(j_1, v) & \text{if } n \text{ is odd,} \\ \mu^*(\sigma_{n-1}(j_0, v)) = \mu^*(\sigma_{n-1}(j_1, v)) & \text{if } n \text{ is even,} \end{cases}$$

($\mu^* : H^{n-1}(M^n, \mathbb{Z}) \longrightarrow H^{n-1}(M^n, \mathbb{Z}_2)$ denotes the coefficients homomorphism).

Proof of theorem 3.– Since M^n is parallelizable, the fibre bundles $T_{2n,n}$ (M^n_0) and $T_{2n,n+1}(M^n_0)$ are trivial and $\pi_0(Emb(M^n, \mathbb{C}^n)) \simeq [M^n_0, V_{2n,n+1}]$. Let $V_{TR}(n)$ (resp. $V_{Lag}(n)$) be the submanifold of $V_{2n,n}$ which is defined as the set of n-frames such that the n-planes they span are totally real (resp. Lagrangian). Since j_0 and j_1 belong to the same regular homotopy class of totally real (resp. Lagrangian) immersions, there exists a path s_t among the sections of $V_{TR}(n)$ (resp. $V_{Lag}(n)$) such that :

$$s_0 = s(j_0), \qquad s_1 = s(j_1).$$

The manifold $V_{2n,n+1}$ is a bundle over $V_{2n,n}$. The section s_t lifts to a section $\tilde{s}_t = (s_t, \nu_t)$ such that $\tilde{s}_0 = (s_0, \nu_0) = (s(j_0), \nu(j_0))$.

$$\begin{tikzcd}
 & V_{2n,n+1} \arrow[d] \\
M^n_0 \arrow[ur, "\tilde{s}_t"] \arrow[r, "s_t"] & V_{2n,n}
\end{tikzcd}$$

Therefore, we get two sections of $T_{2n,n+1}(M_0^n)$ over $s(j_1)$:

$$\tilde{s}_1 = (s(j_1), \nu_1), \quad \tilde{s}(j_1) = (s(j_1), \nu(j_1)).$$

The manifold $V_{2n,n}$ is $(n-1)$-connected, thus : $[M_0^n, V_{2n,n}] \simeq \{0\}$. There exists a homotopy σ_t among the sections of $V_{2n,n}$ such that :

$$\sigma_0 = s(j_1), \qquad \sigma_1 = \text{ constant section.}$$

One can lift this homotopy to a section $M_0^n \longrightarrow V_{2n,n+1}$ in to ways : either to a homotopy $\tilde{\sigma}_t^1 = (\sigma_t, n_t^1)$ where $n_0^1 = \nu_1$, or to an other homotopy $\tilde{\sigma}_t^2 = (\sigma_t, n_t^2)$ where $n_0^2 = \nu(j_1)$. Similarly, the manifold $V_{2n,2n} \simeq Gl(2n, \mathbb{R})$ beeing a fibration over $V_{2n,n}$, the homotopy σ_t also lifts to a homotopy $\Sigma_t = (\sigma_t, \mathcal{R}_t)$ of $V_{2n,2n}$. One can always take $\mathcal{R}_0 = F_{j_1}^{-1}\sigma_0$.

The first non vanishing homotopy group of $V_{2n,n+1}$ is $\pi_{n-1}(V_{2n,n+1})$. It is $\mathbb{Z}$ if n is odd and $\mathbb{Z}_2$ if not. Thus :

$$[M_0^n, V_{2n,n+1}] \simeq H^{n-1}(M^n, \mathbb{Z} \text{ or } \mathbb{Z}_2).$$

A generator of $\pi_{n-1}(V_{2n,n+1})$ is given by the map :

$$\begin{array}{rccc} \alpha : & \mathbb{S}^{n-1} \subset \mathbb{R}^n & \longrightarrow & V_{2n,n+1} \\ & x & \longmapsto & (e_1, ..., e_n, x) \end{array}$$

where $(e_1, ..., e_n)$ is a fixed n-frame in $\mathbb{R}^{2n}$ and x is a vector of the unitary sphere of the orthogonal n-plane spanned by $(e_1, ..., e_n)$ (*cf.* [10], p. 132). Therefore the sections $\tilde{\sigma}_1^1$ and $\tilde{\sigma}_1^2$ of $V_{2n,n+1}$ are homotopic if and only if :

$$\left\{ \begin{array}{lcll} f^*_{\mathcal{R}_1, n_1^1} u & = & f^*_{\mathcal{R}_1, n_1^2} u & \text{if } n \text{ is odd,} \\ \mu^*(f^*_{\mathcal{R}_1, n_1^1} u) & = & \mu^*(f^*_{\mathcal{R}_1, n_1^2} u) & \text{if } n \text{ is even.} \end{array} \right.$$

(The notations $f^*_{\mathcal{R}_t, n_t^i} u$ are obvious generalizations of those introduced in §4.3.2). Thus, $\tilde{\sigma}_0^1 = \tilde{s}_1$ and $\tilde{\sigma}_0^2 = \tilde{s}(j_1)$ are homotopic if and only if :

$$\left\{ \begin{array}{lcll} f^*_{\mathcal{R}_0, \nu_1} u & = & f^*_{\mathcal{R}_0, \nu(j_1)} u & (n \text{ odd }), \\ \mu^*(f^*_{\mathcal{R}_0, \nu_1} u) & = & \mu^*(f^*_{\mathcal{R}_0, \nu(j_1)} u) & (n \text{ even }). \end{array} \right.$$

Since $\tilde{s}(j_0)$ and $\tilde{s}_1$ are homotopic, this equality tells us when $[\tilde{s}(j_0)] = [\tilde{s}(j_1)]$. In an other hand, according to propositions 1 and 2 and using the identification $H^{n-1}(M_0^n) \simeq H^{n-1}(M^n)$, we get :

$$\begin{array}{ll} \sigma_{n-1}(j_0, e_1) + (-1)^{n(n+1)/2} f^*_{s_0, F_{j_0}\nu(j_0)} u = 0, & (1) \\ \sigma_{n-1}(j_1, e_1) + (-1)^{n(n+1)/2} f^*_{s_1, F_{j_1}\nu(j_1)} u = 0. & (2) \end{array}$$

As s_t lies inside $V_{TR}(n)$ (resp. $V_{Lag}(n)$), one can write the equality (1) for each $t \in [0,1]$, i.e.

$$\sigma_{n-1}(j_0, e_1) + (-1)^{n(n+1)/2} f^*_{s_t, F_{Span(s_t)}\nu_t} u = 0.$$

In particular :

$$\sigma_{n-1}(j_0, e_1) + (-1)^{n(n+1)/2} f^*_{s_1, F_{j_1}\nu_1} u = 0.$$

As $s_1 = F_{j_1}\mathcal{R}_0$, one finally obtains :

$$[\tilde{s}(j_0)]{=}[\tilde{s}(j_1)] \Longleftrightarrow \begin{cases} \sigma_{n-1}(j_0, e_1) = \sigma_{n-1}(j_1, e_1) & (n \text{ odd }) \\ \mu^*(\sigma_{n-1}(j_0, e_1)) = \mu^*(\sigma_{n-1}(j_1, e_1)) & (n \text{ even }). \end{cases}$$

This finishes the proof of theorem 3. $\square$

Remark 4. – Theorem 3 can be seen as a first step to understand the map $EmbTR(M^n, \mathbb{C}^n) \longrightarrow \pi_0(Emb(M^n, \mathbb{C}^n))$. In the immersions case, the map $ImmLag(M^n, \mathbb{C}^n)$ (or $ImmTR(M^n, \mathbb{C}^n)$) $\longrightarrow \pi_0(Imm(M^n, \mathbb{C}^n))$ has been intensively studied by M. Audin. The interested reader can refer to [1].

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Institut Girard Desargues – UPRES-A 5028 du CNRS
Université Claude Bernard, Lyon 1
43, boulevard du 11 Novembre 1918
69622 Villeurbanne Cedex, France
email : borrelli@desargues.univ-lyon1.fr