The infimum of the Energy of unit vector fields on odd-dimensional spheres

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Abstract

We construct a one-parameter family of unit smooth vector fields globally defined on the sphere $\mathbb{S}^{2k+1}$ for $k \geq 2$, with energy converging to the energy of the unit radial vector field, which is defined on the complementary of two antipodal points. So we prove that the infimum of the energy of globally defined unit smooth vector fields is :

$$\left(\frac{2k+1}{2} + \frac{k}{2k-1}\right) \text{ vol } (\mathbb{S}^{2k+1}).$$

1 Introduction

Let M be a closed, n-dimensional Riemannian manifold and T^1M be the unit tangent bundle of M considered as a closed Riemannian manifold with the Sasaki metric. The energy of a unit smooth vector field $v \in \chi^1(M)$ is :

$$E(v) = \frac{1}{2}\int_M (n + \|\nabla v\|^2)\ dM,$$

where dM is the density defined by the metric and ∇ is its Levi-Civita connection [9].

It is clear that a lower bound for the functional E is given by $\frac{n}{2}$ vol (M). Moreover, it is realized if and only if v is parallel on M. The problem of minimizing the energy E among unit vector fields becomes interesting when M does not admit a global parallel unit vector field, as it has been pointed out in [9]. This is the case for an odd-dimensional sphere.

In [2], the second author established that Hopf vector fields are the unique absolute minima in the case of $\mathbb{S}^3$. The energy of Hopf vector fields on $\mathbb{S}^{2k+1}$ is

$$\left(\frac{2k+1}{2} + k\right) \text{ vol } (\mathbb{S}^{2k+1}).$$

Let us consider a unit vector field T, defined on the complementary of two antipodal points $\{p, -p\}$, such that it is tangent to radial geodesics issuing from p.

It is easy to see that $E(T) = (\frac{2k+1}{2} + \frac{k}{2k-1})$ vol $(\mathbb{S}^{2k+1}) := c_k$ and then for $k \geq 2$ they have less energy that Hopf vector fields. In fact, in [3] *it has been shown that for every unit smooth vector field v (possibly with isolated singularities) $E(v) \geq c_k$.* Moreover the equality holds if and only if v is geodesic and $v^\perp$ is integrable and with umbilical leaves. Since the only vector fields with these properties are radial vector fields, *this lower bound is not attained on $\chi^1(\mathbb{S}^{2k+1})$.*

In this paper, we prove the following theorem.

Theorem. – The infimum of E among all globally defined unit smooth vector fields of the sphere $\mathbb{S}^{2k+1}$ $(k \geq 2)$ is

$$(\frac{2k+1}{2} + \frac{k}{2k-1}) \text{ vol } (\mathbb{S}^{2k+1}).$$

To obtain the result, we construct a family T_R of unit vector fields such that their energy goes to c_k when R goes to zero. The vector field T_1 $(R = 1)$ is taken equal to T outside a neighbourghood G_1 of one of the geodesic segments joining the two antipodal singularities. The desingularization of T is done inside G_1 by a Riemannian analogue of the "insertion of a dipole", a method used by several authors to study the density problem in functional spaces [8]. The whole family $(T_R)_{R\leq 1}$ is obtained by performing a contraction of G_1 in the directions orthogonal to the geodesic segment. Inside the resulting neighbourghood G_R, the unit vector field T_R is the composition of this contraction with T_1 and outside, T_R is T. At this step, the problem is to carry out this construction while controling the energy. Since $Vol(G_R)$ decreases like R^{2k} when R goes to zero, every odd dimension greater or equal to 5, $||\nabla T_R||^2$ should increase only like $R^{-\alpha}, \alpha < 4$. This is part of the difficulty and explains the technicalities of the construction.

For the construction of these vector fields it has been very convenient to use stereographic projection from a point situated on the equator determined by the singularities so that the choosen geodesic segment becames a segment in the coordinate axe. In this chart, T has rotational symmetry with respect to this axe and it is obtained by rotating a vector field defined on the plane (except on the two singular points). This symmetry will be preserved by T_R. But, since it is impossible to obtain a unit vector field without singularities on $\mathbb{S}^2$, the construction should be then more delicate and of course it can be only done when the dimension is odd.
This result has been announced in [1]. Variational aspects of E were focused in [7], [9], [4],[5] and [6].

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2 Proof of the Theorem

Last assertion in the statement has been shown in [3], therefore to prove the theorem we have to show :

There exists a family $\{T_R \mid R \in]0,1]\}$ *of smooth globally defined unit vector fields on* $\mathbb{S}^{2k+1}$ *such that :* $\lim_{R\to 0} E(T_R) = c_k$.

Let p be a point of $\mathbb{S}^{2k+1} \subset \mathbb{R}^{2k+2}$ and let T be the unit vector field tangent to radial geodesics from p. The field T is defined on $\mathbb{S}^{2k+1} - \{-p, p\}$. Our purpose is to construct a globally defined field T_R by modifying T in a neighbourhood of one geodesic joining p and $-p$.

CONSTRUCTION OF T_R. – Let us assume for simplicity that $p = (1, 0, ..., 0)$. We represent by $(x, y) \in \mathbb{R} \times \mathbb{R}^{2k}$ the coordinates corresponding to the stereographic projection from the north pole $(0, ..., 0, 1)$ to the plane tangent to the sphere at the south pole. In this chart p has coordinates $(2, 0)$ and the geodesic segment joining p and $-p$ is just $I = [-2, 2] \times \{0\}$. Let E_0, E_y be two orthogonal vector fields that are unit with respect to the metric of the sphere and such that E_0 is on the direction of the x-axis and E_y is a radial field on the plane $\{x = 0\}$. Of course, since T is invariant by rotation around the x-axis it can be written as :

$$T(x, y) = a(x, \rho)E_0(x, y) + b(x, \rho)E_y(x, y),$$

where $\rho(y) = \sum_{i=1}^{2k} y_i^2$. (We have plotted in fig. 1 some trajectories of T).

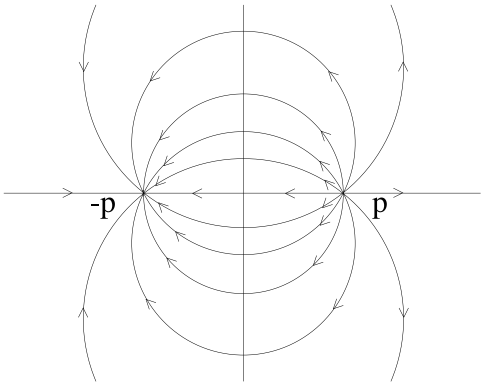

Fig.1

For each $R > 0$ we consider the open subset G_R bounded by the ellipsoide $\frac{x^2}{(2+R)^2} + \frac{\rho(y)}{R^2} = 1$. That is :

$$G_R = \{(x, y) \in \mathbb{R} \times \mathbb{R}^{2k} | d_R(x, y) < 1\}$$

where :

$$d_R(x, y) = \frac{\sqrt{R^2x^2 + \rho(y)(2 + R)^2}}{R(2 + R)}.$$

The field T_R will be defined such that :

- it is invariant by rotation,
- it is equal to T on the complementary of G_R.

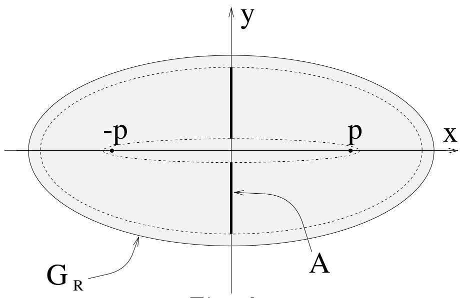

Fig. 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $0 \leq f \leq 1$, $f_{|]-\infty, \frac{1}{10}]} \equiv 0$ and $f_{|[\frac{9}{10}, \infty[} \equiv 1$. The function $h_R = f \circ d_R$ vanishes on $G_{\frac{R}{10}}$ and is equal to 1 on the complementary of $G_{\frac{9R}{10}}$. We are going to use this function to define a smooth vector field U on the complementary of $A = \{0\} \times (B_{\frac{9R}{10}} - B_{\frac{R}{10}})$ (see fig. 2). Here $B_r = \{(0, y)| \ \ |y| < r\}$ where $|y|$ is the Euclidean norm (we reserve the notation $||.||$ for the norm of the usual metric of the sphere). Let φ be the angle of T with the horizontal axis so that $a = \cos\varphi$ and $b = \sin\varphi$. Let $G_R^+ = \{(x, y) \in G_R | x > 0\}$. On $G_R^+ - \{p\}$ the angle φ is between 0 and π. We define $U_{|G_R^+}$ to be :

$$U = \cos(h_R \varphi) E_0 + \sin(h_R \varphi) E_y.$$

According to the definition of h_R, the angle of U with the horizontal axis is 0 near the axis and is φ near the boundary of G_R. On $G_R^- = \{(x, y) \in G_R | x < 0\}$ we define U with the same formula but now the angle φ is taken to be between $-\pi$ and 0. Notice that on the set $\{x = 0\}$ our choices do not match together except for $\{0\} \times (B_{\frac{R}{10}} \cup (B_R - B_{\frac{9R}{10}}))$. Outside G_R the vector field U is taken to be equal to T. So, U is a smooth unit vector field, invariant by rotation, defined on the connected set A^c. In order to overcome this difficulty, we introduce the disc D_R (resp. $\overline{D}_R$) on the plane $\{y_2 = ... = y_{2k} = 0\}$ with center $(0, R/2)$ and radius $R/2$ (resp. $\frac{2}{5}R$) (see fig. 3).

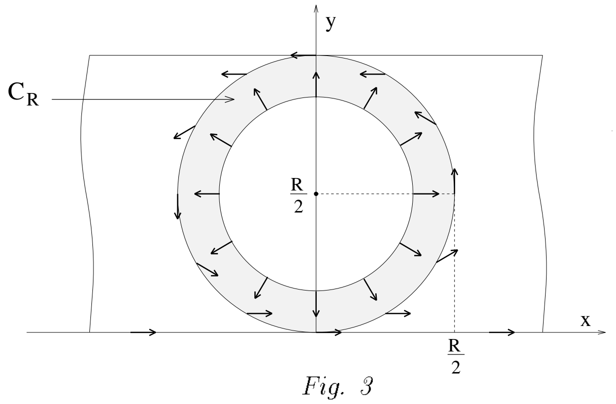

Fig. 3

We are going to modify U on $C_R = (D_R - \overline{D}_R) \times \mathbb{S}^{2k-1}$ such that the resulting vector field V has the following properties :

1) V restricted to $\partial D_R \times \mathbb{S}^{2k-1}$ is U,
2) V restricted to $\partial \overline{D}_R \times \mathbb{S}^{2k-1}$ is the outward normal field.

Since V is going to be also invariant by rotation, it is enough to define it on $D_R - \overline{D}_R$. To do this, let us use polar coordinates on D_R. On these coordinates, U has the form :

$$U(r, \theta) = \cos\psi(r, \theta)E_r + \sin\psi(r, \theta)E_\theta.$$

Note that $\psi(r, \theta) \in]-\pi, \pi[$. We take V to be :

$$V = \cos(g_R\psi)E_r + \sin(g_R\psi)E_\theta$$

where $g_R(r, \theta) = f(\frac{10r}{R} - 4)$. By construction, V satisfies conditions 1) and 2).

It remains to define the vector field on $\overline{D}_R \times \mathbb{S}^{2k-1}$. Let $X \in \chi^1(\mathbb{S}^{2k-1}(1))$ and $\alpha(r) = \frac{\pi}{2} f(\frac{10r}{9R})$. On $\{(r, \theta, s) \in \mathbb{R}^{2k+1} | r < \frac{9R}{10}$ and $|s| = \frac{R}{2}\}$ we put :

$$W_R(r, \theta, s) = \cos\alpha(r)X_R(s) + \sin\alpha(r)E_r(r, \theta, s)$$

with $X_R(s) = \frac{1}{\lambda}X(\frac{2s}{R})$ where λ is the conformal factor of the metric on the sphere, that is $\lambda(x, y) = 4/(4 + x^2 + \rho)$. In particular $X_R(s) = \frac{16+R^2}{16}X(\frac{2s}{R})$.

Now T_R is defined to be W_R on $\overline{D}_R \times \mathbb{S}^{2k-1}$, V on C_R, U on $G_R - (D_R \times \mathbb{S}^{2k-1})$ and T otherwise. The figure 4 shows some of the trajectories of the field T_R outside $D_R \times \mathbb{S}^{2k-1}$.

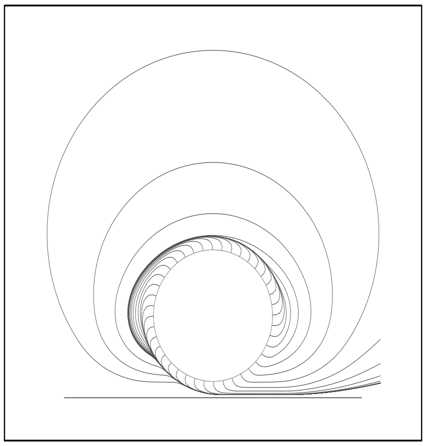

Fig. 4

Remark. – It is easy to see that the distribution orthogonal to T_R is integrable if $X \in \chi^1(\mathbb{S}^{2k-1})$ is such that $X^\perp$ is integrable. The figure below shows the leaves of the orthogonal distribution out of $D_R \times \mathbb{S}^{2k-1}$.

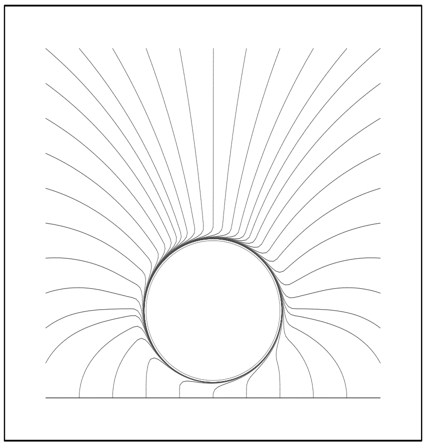

Fig. 5

Energy of T_R. – Let $T = aE_0 + bE_y$ with $a = \cos\varphi$ and $b = \sin\varphi$, then (see Appendix) :

$$\begin{aligned} \|\nabla T\|^2 &= \left(\frac{1}{\lambda}\frac{\partial\varphi}{\partial x} + \frac{\sqrt{\rho}}{2}\right)^2 + \left(\frac{2\sqrt{\rho}}{\lambda}\frac{\partial\varphi}{\partial\rho} - \frac{x}{2}\right)^2 \\ &\quad + (n-2)\left((4 + x^2 - \rho)\frac{\sin\varphi}{4\sqrt{\rho}} - \frac{x}{2}\sqrt{\rho}\cos\varphi\right)^2 . \end{aligned}$$

We use this formula to prove the following lemma :

Lemma 1. *–One has :*

$$\forall R \leq 1, \ \forall m \in G_1, \quad \|\nabla T_R(m)\|^2 \leq F(R) + 4h_R^2(m)\|\nabla T(m)\|^2$$

where T is the unit radial vector field and $F(R)$ is of the form $\dfrac{Const_3}{R^3} + \dfrac{Const_2}{R^2} + \dfrac{Const_1}{R} + Const_0$.

Note that the theorem is a consequence of this lemma. Indeed :

$$\lim_{R\to 0} \int_{G_R} h_R^2 \|\nabla T\|^2 \, dM = 0$$

since $\int \|\nabla T\|^2 < \infty$, and :

$$\int_{G_R} F(R) \, dM \sim \frac{1}{R^3} vol(G_R) \sim R^{2k-3}.$$

The hypothesis $2k+1 \geq 5$ implies that :

$$\lim_{R\to 0} \int_{G_R} F(R) \, dM = 0$$

and thus : $\lim_{R\to 0} E(T_R) = E(T) = c_k$.

Proof of lemma 1.– Let us compute first $\|\nabla T_R\|^2$ on the domain $G_R - (D_R \times \mathbb{S}^{2k-1})$. Here $T_R = \cos(\varphi_R)E_0 + \sin(\varphi_R)E_y$ with $\varphi_R = h_R\varphi$ and $h_R = f \circ d_R$. It is easy to see that on G_1,

$$|\frac{\partial d_R}{\partial x}| \leq \frac{1}{2+R} \leq \frac{1}{2} \text{ and } |\frac{\partial d_R}{\partial \rho}| \leq \frac{1}{2R\sqrt{\rho}}.$$

Thus :

$$\begin{aligned}
\left(\tfrac{1}{\lambda}\tfrac{\partial \varphi_R}{\partial x} + \tfrac{\sqrt{\rho}}{2}\right)^2 = & \left(\tfrac{1}{\lambda}\tfrac{\partial d_R}{\partial x} f'\varphi + \tfrac{1}{\lambda} h_R \tfrac{\partial \varphi}{\partial x} + \tfrac{\sqrt{\rho}}{2}\right)^2 \\
\leq & \ 2\left(|\tfrac{1}{\lambda}\tfrac{\partial d_R}{\partial x} f'\varphi + \tfrac{(1-h_R)\sqrt{\rho}}{2}|^2 + h_R^2 |\tfrac{1}{\lambda}\tfrac{\partial \varphi}{\partial x} + \tfrac{\sqrt{\rho}}{2}|^2\right) \\
\leq & \ 2\left((\tfrac{\pi}{\lambda}\tfrac{1}{2}\sup|f'| + \tfrac{\sqrt{\rho}}{2})^2 + h_R^2\|\nabla T\|^2\right).
\end{aligned}$$

On the other hand :

$$\begin{aligned}
\left(\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial \varphi_R}{\partial \rho} - \tfrac{x}{2}\right)^2 = & \left(\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial d_R}{\partial \rho} f'\varphi + \tfrac{2\sqrt{\rho}}{\lambda} h_R \tfrac{\partial \varphi}{\partial \rho} - \tfrac{x}{2}\right)^2 \\
\leq & \ 2\left(|\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial d_R}{\partial \rho} f'\varphi + \tfrac{(h_R-1)x}{2}|^2 + h_R^2 |\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial \varphi}{\partial \rho} - \tfrac{x}{2}|^2\right) \\
\leq & \ 2\left((\tfrac{2\pi}{\lambda R}\sup|f'| + \tfrac{3}{2})^2 + h_R^2\|\nabla T\|^2\right).
\end{aligned}$$

It remains to bound : $(n-2)\left((4+x^2-\rho)\frac{\sin h_R\varphi}{4\sqrt{\rho}} - \frac{x}{2}\sqrt{\rho}\cos\varphi\right)^2$. The only non-trivial term is the quotient $\frac{\sin h_R\varphi}{\sqrt{\rho}}$. Let $B_R = \{\rho \leq (\frac{R}{15})^3\} \cap G_1$. Of course on $B_R^c \cap G_1$, $\frac{|\sin h_R\varphi|^2}{\rho} \leq (\frac{15}{R})^3$ and on $B_R \cap G_{\frac{R}{10}}$, $\frac{\sin h_R\varphi}{\sqrt{\rho}} =$

0. Thus we have to find a bound of $\frac{\sin h_R \varphi}{\sqrt{\rho}}$ on $D = G^c_{\frac{R}{10}} \cap B_R \cap G_1 = D_- \cup D_+$ (see the figure below).

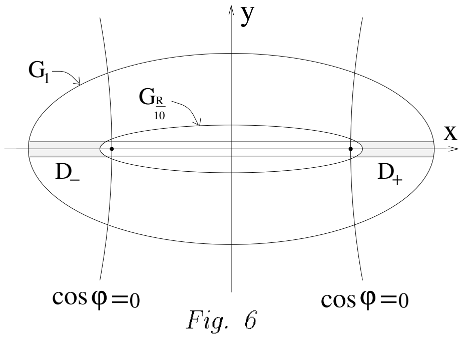

Fig. 6

Note that the hyperboloid of two sheets $H = \{\cos\varphi = 0\} = \{x^2 = \rho + 4\}$ (see Appendix) intersects the boundary of $G_{\frac{R}{10}}$ at

$$\rho = \left(\frac{R}{10}\right)^3 \frac{4 + \frac{R}{10}}{\left(\frac{R}{10}\right)^2 + (2 + \frac{R}{10})^2} \geq \frac{4\frac{R^3}{1000}}{\left(\frac{1}{10}\right)^2 + \left(2 + \frac{1}{10}\right)^2} > \left(\frac{R}{15}\right)^3.$$

Thus on D_+, $0 \leq \varphi < \frac{\pi}{2}$ and $\frac{|\sin h_R\varphi|}{\sqrt{\rho}} \leq \frac{|\sin\varphi|}{\sqrt{\rho}}$. By the lemma 3 of the appendix :

$$\frac{|\sin\varphi|^2}{\rho} = \frac{4x^2}{(4-x^2)^2 + \rho^2 + 2x^2\rho + 8\rho}.$$

It is easily checked that :

$$\min_{\partial G_R} (4-x^2)^2 + \rho^2 + 2x^2\rho + 8\rho = \frac{4R^3(2+R)^2}{1+R},$$

and therefore :

$$\min_{G_1 - G_{\frac{R}{10}}} (4-x^2)^2 + \rho^2 + 2x^2\rho + 8\rho = 4\frac{R^3}{10^3}\frac{(2+\frac{R}{10})^2}{1+\frac{R}{10}} \geq \frac{4R^3}{275}.$$

Thus on D_+ and by symmetry on D_-,

$$\frac{|\sin\varphi|^2}{\rho} \leq 4(2+1)^2 \frac{275}{4R^3}.$$

Finally on $G_R - (D_R \times \mathbb{S}^{2k-1})$, we have obtained :

$$\|\nabla T_R(m)\|^2 \leq \frac{Const_3}{R^3} + \frac{Const_2}{R^2} + \frac{Const_1}{R} + Const_0 + 4h_R^2\|\nabla T(m)\|^2$$

where the $Const_i$ depend on $\sup|f'|$.

The same method obviously works for the remaining part of G_R. Note that there is a subtility concerning W_R because, in this case, the unit vector field X_R depends on R. So, instead of the function $\cos\alpha$, one must consider the function $\frac{16+R^2}{16}\cos\alpha$. Anyway, the behaviour of this function near 0 is similar to the other cases. □

3 Appendix

Lemma 2. – *Let $T(x,y) = a(x,\rho)E_0(x,y) + b(x,\rho)E_y(x,y)$ as in §2, then :*

$$\begin{aligned}\|\nabla T\|^2 &= \left(\tfrac{1}{\lambda}\tfrac{\partial a}{\partial x} - \tfrac{\sqrt{\rho}}{2}b\right)^2 + \left(\tfrac{1}{\lambda}\tfrac{\partial b}{\partial x} + \tfrac{\sqrt{\rho}}{2}a\right)^2 + \left(\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial a}{\partial \rho} + \tfrac{x}{2}b\right)^2 \\ &+ \left(\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial b}{\partial \rho} - \tfrac{x}{2}a\right)^2 + (n-2)\left(\tfrac{4+x^2-\rho}{4\sqrt{\rho}}b - \tfrac{xa}{2}\right)^2.\end{aligned}$$

Proof. – We have that $E_0 = \frac{1}{\lambda}\frac{\partial}{\partial x}$, $E_i = \frac{1}{\lambda}\frac{\partial}{\partial y^i}$, for $i = 1, \ldots, 2k$ and then

$$E_y = \frac{1}{\sqrt{\rho}}\sum_{i=1}^{n} y^i E_i.$$

Since the Christoffel symbols in this chart are :

$$\Gamma_{ij}^l = -\frac{\lambda}{2}\left(\delta_{il}y^j + \delta_{jl}y^i - \delta_{ij}y^l\right),$$

then :

$$g(\nabla_{E_i}E_j, E_l) = \frac{1}{2}\left(\delta_{ij}y^l - \delta_{il}y^j\right).$$

Here we take $i = 0, \ldots\ 2k$ and $y^0 = x$. Now, a straightforward computation shows :

$$\begin{aligned}
g(\nabla_{E_0}T, E_0) &= E_0(a) - \tfrac{\sqrt{\rho}}{2}b = \tfrac{1}{\lambda}\tfrac{\partial a}{\partial x} - \tfrac{\sqrt{\rho}}{2}b \\
g(\nabla_{E_0}T, E_l) &= y^l\left(\tfrac{E_0(b)}{\sqrt{\rho}} + \tfrac{a}{2}\right) \\
&= y^l\left(\tfrac{1}{\lambda\sqrt{\rho}}\tfrac{\partial b}{\partial x} + \tfrac{a}{2}\right) \\
g(\nabla_{E_l}T, E_0) &= E_l(a) + y^l\tfrac{bx}{2\sqrt{\rho}} \\
&= y^l\left(\tfrac{2}{\lambda}\tfrac{\partial a}{\partial \rho} + \tfrac{bx}{2\sqrt{\rho}}\right) \\
g(\nabla_{E_l}T, E_j) &= \delta_{lj}\left(-\tfrac{ax}{2} - \tfrac{b\sqrt{\rho}}{2}\right) + E_l(b)\tfrac{y^j}{\sqrt{\rho}} \\
&\quad + bE_l\left(\tfrac{y^j}{\sqrt{\rho}}\right) + \tfrac{y^l y^j b}{2\sqrt{\rho}} \\
&= \delta_{lj}\left(\tfrac{b}{\lambda\sqrt{\rho}} - \tfrac{ax}{2} - \tfrac{b\sqrt{\rho}}{2}\right) \\
&\quad + \tfrac{y^l y^j}{\sqrt{\rho}}\left(\tfrac{2}{\lambda}\tfrac{\partial b}{\partial \rho} - b\tfrac{4+x^2-\rho}{4\rho}\right).
\end{aligned}$$

By taking the squares of these expressions, adding up and simplifying we get the result. □

Lemma 3. – *The expression of the radial vector field is $T(x,y) = a(x,\rho)E_0(x,y) + b(x,\rho)E_y(x,y)$ where :*

$$\begin{aligned}a(x,\rho) &= \tfrac{\lambda(x^2-\rho-4)}{4\sqrt{1-\lambda^2x^2}} \\ b(x,\rho) &= \tfrac{\lambda\sqrt{\rho}x}{2\sqrt{1-\lambda^2x^2}}.\end{aligned}$$

Proof.- At a point of coordinates (x,y), the differential of the stereographic projection maps T into the unit vector that is tangent to the circle passing through $(2,0)$, $(-2,0)$ and (x,y), oriented from $(2,0)$ to $(-2,0)$ and contained in the plane determined by the x-axis and the vector $(0,y)$, if $y \neq 0$. If $y = 0$, this circle degenerates in a line. The proof follows now from elementary arguments. $\square$

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