ECG

IST-2000-26473

Effective Computational Geometry for Curves and Surfaces

ECG Technical Report No. : *ECG-TR-124102-02*

On the angular defect of triangulations and the pointwise approximation of curvatures

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Deliverable:	12 41 02
Site:	INRIA Sophia Antipolis
Month:	12

Project funded by the European Community
under the "Information Society Technologies"
Programme (1998–2002)

Abstract

Let S be a smooth surface of E^3, p a point on S, k_m, k_M, k_G and k_H the maximum, minimum, Gauss and mean curvatures of S at p. Consider a set $\{p_i p p_{i+1}\}_{i=1,\dots,n}$ of n Euclidean triangles forming a piece-wise linear approximation of S around p —with $p_{n+1} = p_1$. For each triangle, let γ_i be the angle $\angle p_i p p_{i+1}$, and let the angular defect at p be $2\pi - \sum_i \gamma_i$.

This paper establishes, asymptotically when the distances $\| pp_i \|$ go to zero, several approximation results relating the curvatures of S at p and normalized angular defects of meshes at p.

More precisely, if it is assumed that the p_is lie in normal sections two consecutive of which form an angle of $2\pi/n$, we provide the closed form expression of the angular defect as a function of the principal curvatures and $2\pi/n$. In particular, we show that $n = 4$ is the only value of n such that $2\pi - \sum_i \gamma_i$ depends upon the principal directions, and that $n = 6$ is the only value such that $2\pi - \sum_i \gamma_i$ provides an exact estimate for k_G. A corollary of these results is that the principal curvatures —whence k_G and k_H— can be computed from the angular defects of any two triangulations whose valences are not four.
By relaxing the hypothesis on the normal sections, we also show that the angular defect weighted by the so-called module of the mesh estimates k_G within an error bound depending upon k_m and k_M.

Since meshes are so ubiquitous in Computer Graphics and Computer Aided Design, and Discrete Differential operators are so necessary to their processing, we believe these contributions are one step forward the intelligence of the geometry of meshes, whence one step forward more robust algorithms.

Keywords. Smooth Surfaces, Meshes, Curvatures, Approximations, Differential Geometry.

1 Introduction

1.1 Smooth and triangulated surfaces

Meshes are ubiquitous in modern computer-related geometry. Meshes are easily obtained from physical objects through scanning and reconstruction. Meshes are commonly displayed by graphical hardware. Meshes are intuitive to deal with. They provide hierarchical representations that can be used for approximate representations. Meshes can be refined to smooth surfaces through subdivision.

In this context, and especially since meshes can be made dense enough so as to "look like" smooth surfaces, it is tempting to define a differential geometry of meshes which mimics that of smooth surfaces.

Example quantities well defined smooth surfaces that also look appealing for meshes are the surface area, the normal vector field, the curvatures, geodesics, the focal sets, the ridges, the medial axis, etc.

Interestingly, recent research in applied domains provides, for each of the notions just enumerated, several estimates adapted from classical differential geometry to the setting of piece-wise linear surfaces. Several definitions of normals, principal directions and curvatures over a mesh can be found in [Tau95], [DMSB]. Ridges of polyhedral surfaces as well as cuspidal edges of the focal sets are computed in [WB01]. Geodesics and discrete versions of the Gauss-Bonnet theorem are considered in [PS98]. But none of these contributions address the question of the accuracy of these estimates or that of their convergence when the mesh is refined.

As opposed to these approaches, the literature provides a few examples of *a priori* analysis. Given a discrete set —a point cloud or a mesh— which is assumed to sample a surface in a certain way, differential operators are derived together with theoretical guarantees about the discrepancy between the discrete estimates and the true value on the underlying smooth surface.

In [AB99] it is shown that the normal to a smooth surface sampled according to a criterion involving the skeleton can be estimated accurately from the Voronoi diagram of the sample points. The surface area of a mesh and its normal vector field versus those of a smooth surface are considered in [MT01]. More closely related to the question we address is [MW00], which provides some error bounds for estimates of the normal and the Gauss curvature of a sampled surface.

1.2 Smooth surfaces, polyhedra, Gauss curvature and angular defect

In this section, we hi-light striking parallels between the Gauss curvature of polyhedra and smooth surfaces of E^3. More precisely, we recall:

1. the definition of the Gauss curvature for smooth surfaces and polyhedra, as well as the Gauss-Bonnet theorem in both cases.
2. how the Gauss curvature of a smooth surface can be recovered from the angular defect of geodesic triangles.

Gauss curvature and the Gauss-Bonnet theorem for surfaces and polyhedra.
The Gauss curvature of an (abstract) oriented smooth Riemannian surface M is a smooth function k_G on M defined by using the metric tensor. It is well known that the Gauss curvature of a domain is identically equal to zero if and only if it is locally isometric to a portion of plane. One can associate to k_G a curvature measure K_G on M by integration over any domain U of M:

$$K_G(U) = \int_U k_G da,$$

where da denotes the area form of M. Suppose now that M is isometrically embedded in E^3. One way to evaluate its Gauss curvature at a point p is to make the product of the two principal curvatures of M at p, (this is nothing but *theorema egregium* of Gauss). An equivalent way it to use the Gauss map of the embedding and to calculate the limit A'/A, where A is the surface area of a region around p, A' the signed area of the image of A on S^2 by the Gauss map, the limit being taken (*roughly speaking*) as the region A around p becomes smaller and smaller [spi99, Chapter III].

Consider now an abstract Riemannian polyhedron P. Any point of P which is not a vertex has a neighborhood isometric to a plane, and one can define its Gauss curvature to be 0, by analogy with the smooth case. Moreover, if p is a vertex of P, one can assign to p the angular defect $\alpha_p = 2\pi - \sum_i \gamma_i$ at p, where the γ_is stand for the angles at p of the facets incident to p. We call this angular defect *the Gauss curvature $k_G(p)$ of the vertex p*, [Res93, Section 5]. Remark that it is now possible to define a curvature measure on P, as a measure concentrated on the vertices of P, by setting over any domain U of P

$$K_G(U) = \sum_{p \text{ vertex in } U} \alpha(p).$$

If P is isometrically embedded in E^3, then this angular defect is exactly the signed area of the image on S^2 by the Gauss map of any arbitrarily small neighborhood of p, which is a property analogous to the smooth case.

An important relationship relating the Gauss curvature to topological properties is the Gauss-Bonnet theorem. Consider a closed orientable surface S and a closed polyhedron P, the latter with vertices $p_1, \ldots, p_n$. Let $\chi(S)$ and $\chi(P)$ stand for their Euler characteristic —that is $V - E + F$ in the usual jargon. The global Gauss-Bonnet theorem respectively states for S and P that

$$2\pi\chi(S) = \int\int_S k_G d\sigma, \tag{1}$$

$$2\pi\chi(P) = \sum_{i=1,\ldots,n} k_G(p_i). \tag{2}$$

These remarkable results actually state that if the topology is fixed —i.e. the genus of the surface or the polyhedron is given, the curvature distributes itself on S or P so as to comply with that topology. Notice that local versions of both theorems exist. (For the geodesic curvature involved in the local Gauss-Bonnet theorem, see [Res93, PS98].)

Remark that in this context, the Gauss curvature of a point of a polyhedron is dimensionless, while that of a surface is homogeneous to the inverse of a surface area. The topological invariant $2\pi\chi$ is dimensionless, which is coherent with equations (1) and (2).

Gauss curvature and geodesic triangles.

An important property of geodesic triangles also which involves the Gauss curvature is the following —see [CMS84] and [Laf86].

Let τ_i be a geodesic triangle on S, p, p_i and p_{i+1} its vertices, l_p, l_i, l_{i+1} the lengths of the geodesic arcs opposite to the vertices, and $l = \sup\{l_i, l_{i+1}\}$. Let T_i be the Euclidean triangle whose edges have the same lengths as those of τ_i. At last, let β_i be the angle of τ_i at p, and α_i be the angle $\angle p_i p p_{i+1}$ of T_i.
The angles β_i and α_i differ by a term involving the Gauss curvature k_G at p. More precisely, we have

Proposition. 1 *The angles β_i and α_i associated to a geodesic triangle τ_i satisfy*

$$\beta_i = \alpha_i + \frac{1}{6} \sin\alpha_i \ k_G \ l_i \ l_{i+1} + o(l^2). \tag{3}$$

Consider now a geodesic triangulation around p, that is a set of n geodesic triangles having p as common vertex and forming a topological disk around p. By looking at the tangents to the geodesics from p to the p_is, we have $\sum_i \beta_i = 2\pi$. Summing Equation (3) for the n geodesic triangles immediately yields the following

Theorem. 1 *Let $\mathcal{T}$ be a geodesic triangulation of a smooth surface M of E^3. Let p be a vertex of $\mathcal{T}$, and let $A(p)$ be the sum of the areas of the triangles T_i associated to the geodesic triangles τ_is. Then*

$$2\pi - \sum_i \alpha_i = \frac{A(p)}{3} k_G + o(l^2).$$

This result is of little help from a practical standpoint since the knowledge of geodesics is required. The question addressed in this paper is actually to study the quantity estimated by the angular defect when one replaces the geodesics by the Euclidean line-segments pp_i.

1.3 Contributions

Let S be a surface of E^3, and let p be a point of S. Also suppose that we are given a set $\{p_i p p_{i+1}\}_{i=1,\ldots,n}$ of n Euclidean triangles forming a piece-wise linear approximation of S around p. We shall refer to these triangles as the mesh, and to the p_is as the one-ring neighbors of p. For each triangle, let γ_i be the angle $\angle p_i p p_{i+1}$, and let the angular defect at p be $2\pi - \sum_i \gamma_i$. Also, for a one-ring neighbor p_i, let η_i stand for the Euclidean distance between p and p_i.

The question we address in this paper is:

> How precisely can one estimate the curvatures at a point p of a smooth surface using the angular defect of the triangles surrounding p?

Before engaging into the technicalities, several comments are in order.

Dimensionality. In order for the previous question to make sense, we shall pay a special attention to the dimensionality of the quantities involved. As already pointed out, the Gauss curvature of a smooth surface is homogeneous to the inverse of a surface area while that of a polyhedron is dimensionless. To estimate the curvatures of a smooth surface from a polyhedron, we will therefore have to normalize by lengths (for the principal curvatures) or surface areas (Gauss curvature).

Smooth surfaces and asymptotic estimates. Estimating the curvatures of a smooth surface S from a single mesh is obviously an hopeless target. The folds of S may indeed occur at a resolution much lower than that of the triangulation, so that the mesh may fall short from providing accurate information on the point-wise curvatures. The problem becomes more tractable if one assumes S belongs to a restricted class of surfaces —e.g. with Lipchitz like conditions on the variation of the normal— or assumes that a sequence of meshes with edges' lengths going to zero is available. We shall in the paper follow this latter perspective and derive asymptotic results.

Soundness of the angular defect for the Gauss curvature of smooth surfaces. Since the angular defect over geodesic triangles provides an estimate of k_G, it is tempting to believe that when the edges lengths $\| pp_i \|$ go to zero, one can safely replace the geodesic arcs from p to the p_is on S by the Euclidean segments pp_is. (It should be noticed that estimating the curvatures is a local process. It is therefore implicitly assumed that the edges of the triangulation have lengths small enough to avoid topological problems.) From an application perspective, the answer to the above question is especially interesting since the angular defect is widely used to estimate the Gauss curvature of triangulated surfaces —see [Cal86], [DMSB].

1.4 Paper overview

The paper is organized as follows. Section 2 provides the notations used throughout the paper. In section 3 we present approximation results for the curvature of plane curves. These results are used in section 4 to derive a formula on Euclidean triangles providing a piece-wise linear approximation of a smooth surface. Using restricted hypothesis on the geometry of these triangles, we show in section 5 that the angular defect does not provide, in general, an estimate for k_G. In section 6, the hypothesis of section 5 are alleviated and a general result about the accuracy of the angular defect is proved.

2 Notations

Consider a point p of a smooth surface together with n Euclidean triangles $\{p_i p p_{i+1}\}_{i=1,\dots,n}$ —with $p_{n+1} = p_1$— forming a piece-wise linear approximation of S around p. The following notations are used throughout the paper —see Figure 1:

Normal sections; Π_i, φ_i. Consider the vertical plane Π_i containing the normal $\mathbf{n}$ and p_i. We assume Π_i is defined by its angle φ_i with respect to some coordinate system in the tangent plane —for example the one associated with the principal directions.

Angle between normal sections; β_i. The angle between two consecutive normal sections Π_i and Π_{i+1} is denoted β_i, that is $\beta_i = \varphi_{i+1} - \varphi_i$. Put differently, β_i measures, in the tangent plane, the angle between the tangents to the plane curves $S \cap \Pi_i$ and $S \cap \Pi_{i+1}$.

Directional curvatures; λ_i. We let λ_i stand for the curvature of the plane curve $S \cap \Pi_i$. Notice that if Π_i contains a principal direction, λ_i reduces to the corresponding principal curvature k_M or k_m.

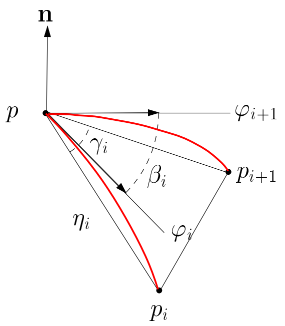

Figure 1: Notations

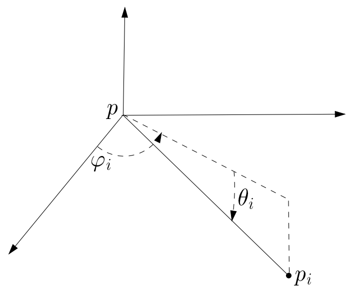

Figure 2: p_i in polar coordinates

3 Plane curves

In this section, we provide an estimate for the curvature of a plane curve from the angular defect of an inscribed polygon.

3.1 A lemma on plane curves

Let C be a C^∞ smooth regular curve. Let p_0 be a point of C. It is well known that C can locally be represented by the graph $(x, f(x))$ of a smooth function f, such that $p_0 = (0,0)$, (*i.e.* $f(0) = 0$), and such that the tangent to C at p_0 is aligned with the x-axis, (*i.e.* $f'(0) = 0$). Let k be the curvature of C at the origin (*i.e.* $k = f''(0)$), and let $\nu = f'''(0)$. Near the origin we have

$$f(x) = \frac{kx^2}{2} + \frac{\nu x^3}{6} + o(x^3). \tag{4}$$

Let us now use polar coordinates: $x = \eta\cos\theta$ and $y = \eta\sin\theta$. In order to approximate the curvature of C at p_0, we shall need an expression of θ as a function of η. Obtaining such an expression involves the implicit function theorem, and the reader is referred to section 8 for the proof of the following

Lemma. 1 *Let $f(x)$ be a C^∞ smooth regular function with $k = f^{''}(0)$ and $\nu = f^{'''}(0)$. For a point $p = (\eta \cos\theta, \eta \sin\theta)$ on the graph of f near the origin, one has:*

$$\theta = \frac{k\eta}{2} + \frac{\nu\eta^2}{6} + o(\eta^2) \text{ if } x \geq 0, \tag{5}$$

$$\theta = \frac{k\eta}{2} - \frac{\nu\eta^2}{6} + o(\eta^2) \text{ if } x \leq 0. \tag{6}$$

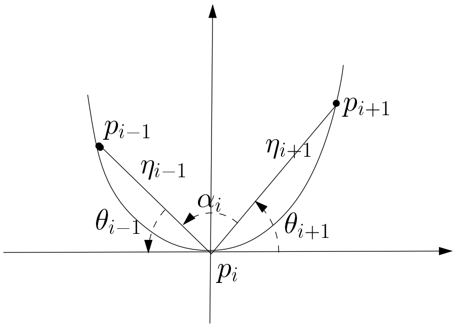

Figure 3: Smooth curve and inscribed polygon

3.2 Approximating the curvature of a plane curve

The previous lemma can be used to estimate the curvature of a curve from the angular defect α_i of an inscribed polygon. See Figure 3 for the notations.

Theorem. 2 *Let p_{i-1}, p_i and p_{i+1} be three points as indicated on Figure 3, with η_{i-1} (η_{i+1}) the distance from p_i to p_{i-1} (p_{i+1}). Also let $\overline{\eta_i} = (\eta_{i-1} + \eta_{i+1})/2$. The angular defect α_i at p_i and the curvature k satisfy:*

- *if $\eta_{i-1} = \eta_{i+1} = \eta$:*

$$\frac{\pi - \alpha_i}{\eta} = k + o(\eta), \tag{7}$$

- *if $\eta_{i-1} \neq \eta_{i+1}$:*

$$\frac{\pi - \alpha_i}{\overline{\eta_i}} = k + o(1). \tag{8}$$

Proof. Since the proofs of the two statements are similar, we focus just on the first one.

Equations (5) and (6) applied to p_{i+1} and p_{i-1} yield $\theta_{i-1} + \theta_{i+1} = k\eta + o(\eta^2)$. But we also have $\theta_{i-1} + \theta_{i+1} = \pi - \alpha_i$, whence the result. $\square$

Interestingly, the speed of convergence if faster when the two neighbors are located at the same distance from p_i.

Remark. The previous Theorem can be extended to space curves.

4 A lemma on normal sections and Euclidean triangles

This section is devoted to a general result involving Euclidean triangles and smooth surfaces. This result is the cornerstone of the next two sections.

Using the notations of section 2, we aim at finding a dependence relationship between β_i, γ_i, η_i and η_{i+1}. More precisely, we shall consider that the normal sections Π_i and Π_{i+1} are fixed, and study the dependence between γ_i, η_i and η_{i+1}. Obviously, two of these variables are independent. For example, setting η_i and η_{i+1} automatically impose the value of γ_i.

Lemma. 2 *With the above notations, let $\eta = \max(\eta_i, \eta_{i+1})$. Define the following sum and product functions*

$$s(\eta_i, \eta_{i+1}) = \frac{\lambda_i^2 \eta_i^2 + \lambda_{i+1}^2 \eta_{i+1}^2}{8}, \; p(\eta_i, \eta_{i+1}) = \frac{\lambda_i \lambda_{i+1} \eta_i \eta_{i+1}}{4}.$$

The β_i, γ_i, η_i and η_{i+1} quantities satisfy

$$\beta_i = \gamma_i + \frac{p(\eta_i, \eta_{i+1})}{\sin \gamma_i} - s(\eta_i, \eta_{i+1}) \cot \gamma_i + o(\eta^2) \tag{9}$$

and

$$\beta_i = \gamma_i + \frac{p(\eta_i, \eta_{i+1})}{\sin \beta_i} - s(\eta_i, \eta_{i+1}) \cot \beta_i + o(\eta^2). \tag{10}$$

Proof. The proofs of the two claims following the same guideline, we just prove the second one. Assume point p is at the origin. Let us write the coordinates of p_i in polar form with the conventions of Figure 2 —that is φ measures an angle in the tangent plane of S at p. If X^t stands for the transpose of X, the coordinates of p_i are $p_i = (\eta_i \cos\theta_i \cos\varphi_i, \eta_i \cos\theta_i \sin\varphi_i, \eta_i \sin\theta_i)^t$, and similarly for p_{i+1}.

Since $\beta_i = \varphi_{i+1} - \varphi_i$, the two expressions of the dot product $\mathbf{pp_i} \cdot \mathbf{pp_{i+1}}$ yield

$$\cos\gamma_i = \cos\beta_i \;\; \cos\theta_i \cos\theta_{i+1} + \sin\theta_i \sin\theta_{i+1}. \tag{11}$$

Since the curve $\Pi_i \cap S$ is a plane curve, using Equation (5) which expresses θ as a function of η, we have

$$\cos\theta_i = 1 - \frac{\lambda_i^2 \eta_i^2}{8} + o(\eta_i^2), \quad \sin\theta_i = \frac{\lambda_i \eta_i}{2} + o(\eta_i).$$

Plugging these values into Equation (11) yields

$$\cos\gamma_i = (1 - s(\eta_i, \eta_{i+1})) \cos\beta_i + p(\eta_i, \eta_{i+1}) + o(\eta^2).$$

To turn the previous expression into a relationship between γ_i and β_i, we use the Taylor formula of order one to $f(x) = \arccos(x)$ together with $a = \cos\beta_i$ and $b = \cos\gamma_i$:

$$\gamma_i = \beta_i + \frac{-1}{\sin\beta_i}\big(-s(\eta_i, \eta_{i+1}) \cos\beta_i + p(\eta_i, \eta_{i+1}) + o(\eta^2)\big) + o(\eta^2).$$

Re-arranging the terms completes the proof. $\square$

5 Surfaces and regular polygons

5.1 Result and applications

Using the lemma proved in the previous section, we are now ready to analyze the angular defect under a restrictive hypothesis:

Definition. 1 *Let p be a point of a smooth surface S and let $p_i, i = 1, \ldots, n$ be its one ring neighbors. Point p is called a* regular vertex *if (i)the p_is lie in normal sections two consecutive of which form an angle of $\theta(n) = 2\pi/n$ (ii)the η_is all take the same value η.*

Let a stand for the angle between the direction associated with the maximum curvature and the normal section containing p_1 —see Figure 4. For a regular vertex, the β_is are constant and equal to $\theta(n)$, so that $\varphi_i = a + (i-1)\theta(n)$. Under these hypothesis, we prove that the angular defect does not provide, in general, an estimate for the Gauss curvature.

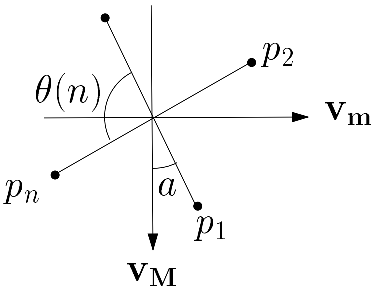

Figure 4: Regular triangulation around p: tangent plane seen from above

Theorem. 3 *Consider a regular vertex of valence n. The following holds:*

1. *There exists two functions $A(a,n)$ and $B(a,n)$ such that*

$$2\pi - \sum_{i=1}^{n} \gamma_i = \left[A(a,n)\ k_G + B(a,n)\ (k_M^2 + k_m^2)\right]\eta^2 + o(\eta^2). \tag{12}$$

2. *The only value of n such that the functions $A(a,n)$ and $B(a,n)$ depend upon a is $n = 4$, and then*

$$2\pi - \sum_{i=1}^{n} \gamma_i = \left[(1 - 2\cos^2 a \sin^2 a)\ k_G + \cos^2 a \sin^2 a\ (k_M^2 + k_m^2)\right]\eta^2 + o(\eta^2). \tag{13}$$

3. *If $n \neq 4$:*

$$2\pi - \sum_{i=1}^{n} \gamma_i = \frac{n}{16 \sin 2\pi/n}\left[\left(2 - \cos\frac{4\pi}{n} - \cos\frac{2\pi}{n}\right)\ k_G + \left(1 + \frac{1}{2}\cos\frac{4\pi}{n} - \frac{3}{2}\cos\frac{2\pi}{n}\right)\ (k_M^2 + k_m^2)\right]\eta^2 + o(\eta^2). \tag{14}$$

In particular, the only value of n such that $B(a,n) = 0$ is $n = 6$, and then $A(a,6) = \sqrt{3}/2$, that is

$$2\pi - \sum_i \gamma_i = \frac{\sqrt{3}}{2} k_G\ \eta^2 + o(\eta^2). \tag{15}$$

The graphs of the functions $A(a,4)$ and $B(a,4)$, as well as $A(a,n)$ and $B(a,n)$ for $n \neq 4$ are presented on Figures 5 and 6. Notice again that the angular defects provides an estimate for the Gauss curvature only for $n=6$ whatever the angle a is, and for $n=4$ when the one-ring neighbors are aligned with the principal directions.

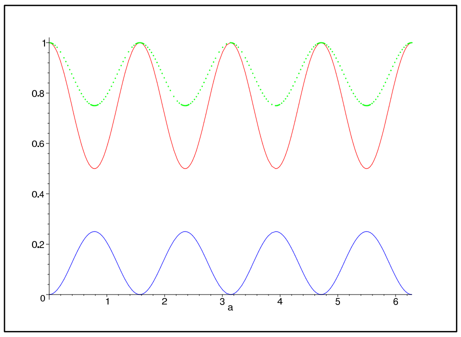

Figure 5: The coefficients $A(a,4)$, $B(a,4)$ and $A(a,4)+B(a,4)$ — respectively (solid curve, top), (solid curve, bottom), (dotted curve)

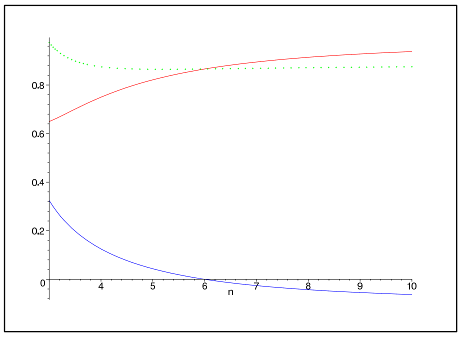

Figure 6: The coefficients $A(a,n)$, $B(a,n)$ and $A(a,n)+B(a,n)$ — respectively (solid curve, top), (solid curve, bottom), (dotted curve)

Using the previous theorem, we have the following

Corollary. 1 *The principal curvatures of a smooth surface can be computed from the angular defects of any two meshes whose valence are not four.*

Proof. Assume we are given two meshes with valences n_1 and n_2 different from four. Let $e(n_i)$ be the limit value of $(2\pi - \sum_i \gamma_i)/\eta^2$, and $a(n_i), b(n_i)$ be the coefficients of k_G and $k_m^2 + k_M^2$ in Equation (12) have:

$$\begin{cases} e(n_1) = a(n_1)k_G + (k_m^2 + k_M^2)b(n_1) \\ e(n_2) = a(n_2)k_G + (k_m^2 + k_M^2)b(n_2) \end{cases}$$

Since $n_1 \neq n_2 \Rightarrow a(n_1)b(n_2) - a(n_2)b(n_1) \neq 0$, we have

$$k_G = \frac{e(n_1)b(n_2) - e(n_2)b(n_1)}{a(n_1)b(n_2) - a(n_2)b(n_1)}. \tag{16}$$

If e refers to one of the $e(n_i)$ and similarly for a and b, once the Gauss curvature is known, we are left with the system

$$\begin{cases} k_m\ k_M = k_G \\ (k_m^2 + k_M^2)b = e - ak_G. \end{cases}$$

Letting $d = e - ak_G$ and observing that $b \neq 0$ since $n \neq 4$ yields

$$b\ k_m^4 - d\ k_m^2 + b\ k_G^2 = 0, \tag{17}$$

which solves to

$$k_m^2 = \frac{e - ak_G - \sqrt{(e - ak_G)^2 - 4b^2k_G^2}}{2b}. \tag{18}$$

Notice that the sign of the square root chosen in the previous expression does not matter since the two solutions corresponding to $\pm\sqrt{}$ are actually conjugated with respect to $k_m\ k_M = k_G$, that is

$$k_M^2 = \frac{k_G^2}{k_m^2} = \frac{e - ak_G + \sqrt{(e-ak_G)^2 - 4b^2k_G^2}}{2b}. \tag{19}$$

Once k_m^2 and k_M^2 have been computed, k_m and k_M are determined observing that the product of their signs is the sign of k_G, and that $k_m \leq k_M$. Once k_m and k_M are known, the mean curvature k_H is $k_H = (k_m + k_M)/2$.

Remark. Umbilics can be detected from the two angular defects without computing the principal curvatures since in this case $k_m = k_M$. When $k_m = k_M$ we indeed have

$$(e - ak_G)^2 - 4b^2k_G^2 = 0$$

and $sign(k_G) \geq 0$. $\Box$

Remark. It is tempting to infer the position of the principal directions using the valence four mesh and equation (13). Notice however that this yields the value of $\cos^2 a$. The index of the one-ring neighbor involved in the definition of the offset angle a is still missing.

Remark. It is important to notice that corollary 1 compares the Gauss curvature and the squares of the principal curvatures of S against the angular defect normalized by η^2 —see section 1.2 for a discussion of the dimensionality issues.

Remark. In section 1.2 we recalled the Gauss-Bonnet theorem for a polyhedra P and a compact orientable surface S. Assume P and S are homeomorphic. An interesting question in the global convergence of the sum of the angular defects over P when P is refined so as to converge to S —in the Hausdorff sense. This question is especially interesting since the angular defect usually involves the squares of the principal curvatures. See [Fu93] for similar concerns.

5.2 Proofs

The proof of Theorem 3 consists of working out the expressions of $A(a,n)$, $B(a,n)$ and $C(a,n)$ from Equation (10). We begin with a lemma about trigonometric sums.

Lemma. 3 *Define the three trigonometric sums*

$$t_1(a,n) = \sum_{i=1}^{n} cos(2\varphi_i), \quad t_2(a,n) = \frac{1}{2}\sum_{i=1}^{n} cos(2\varphi_i + 2\varphi_{i-1}), \quad t_3(a,n) = \sum_{i=1}^{n} cos(4\varphi_i). \tag{20}$$

For a regular uniform mesh of valence n, we have

$$2\pi - \sum_{i=1}^{n} \gamma_i = \left[A(a,n)\ k_G + B(a,n)\ k_M^2 + C(a,n)\ k_m^2)\right]\ \eta^2 + o(\eta^2) \tag{21}$$

with

$$A(a,n) = \frac{1}{16\sin\theta(n)}\big(2n - n\cos 2\theta(n) - n\cos\theta(n) - t_2(a,n) + t_3(a,n)\cos\theta(n)\big), \tag{22}$$

$$B(a,n) = \frac{1}{16\sin\theta(n)}\big(n + \frac{n}{2}\cos 2\theta(n) - \frac{3n}{2}\cos\theta(n) + 2(1-\cos\theta(n))t_1(a,n) + t_2(a,n) - \frac{\cos\theta(n)}{2}t_3(a,n)\big), \tag{23}$$

$$C(a,n) = \frac{1}{16\sin\theta(n)}\big(n+\frac{n}{2}\cos 2\theta(n) - \frac{3n}{2}\cos\theta(n) - 2(1-\cos\theta(n))t_1(a,n) + t_2(a,n) - \frac{\cos\theta(n)}{2}t_3(a,n)\big). \tag{24}$$

Proof. Since the angles in the tangent plane are known, we work out the angular defect using Equation (10) rather than Equation (9). Assuming that $\eta_i = \eta$ and $\beta_i = \theta(n)$, we rewrite Equation (10) as follows:

$$\frac{4\sin\theta(n)}{\eta^2}(\beta_i - \gamma_i) = \mu_i + o(1) \tag{25}$$

with

$$\mu_i = \lambda_i\lambda_{i+1} - \frac{1}{2}(\lambda_i^2 + \lambda_{i+1}^2)\cos\theta(n). \tag{26}$$

Expressing the directional curvatures λ_i with Euler's relation, that is

$$\lambda_i = \cos^2\varphi_i\ k_M + \sin^2\varphi_i\ k_m, \tag{27}$$

and substituting into $\sum_{i=i}^{n}\mu_i$ yields a homogeneous polynomial of degree four in sines and cosines. We linearize this polynomial using the standard formulae $\cos^2 a = (1+\cos 2a)/2, \sin^2 a = (1-\cos 2a)/2, \cos a\cos b = (\cos(a+b)/2+\cos(a-b))/2$, as well as $\cos^2 a\sin^2 a = (1-\cos 4a)/8,\ \cos^4 a = (\cos 4a + 4\cos 2a + 3)/8,\ \sin^4 a = (\cos 4a - 4\cos 2a + 3)/8$. Theses calculations simplify to Equations (22), (23) and (24).

Notice that the expressions of $B(a,n)$ and $C(a,n)$ just differ by the sign of the coefficient of $t_1(a,n)$. $\square$

Lemma. 4 *Let α an β be positive integers, and consider the sum*

$$S(\alpha,\beta,n) = \sum_{k=1}^{n}\cos(\alpha + k\frac{\beta\pi}{n}). \tag{28}$$

If $\beta\pi/n = 0(2\pi)$, then $S(\alpha,\beta,n) = n\cos\alpha$. If $\beta\pi/n \neq 0(2\pi)$ and $\beta\pi = 0(2\pi)$, then $S(\alpha,\beta,n) = 0$.

Proof. If $\beta\pi/n = 0(2\pi)$, the result is trivial. Otherwise, consider the two terms:

$$S(\alpha,\beta,n) = \sum_{k=1}^{n}\cos(\alpha + k\frac{\beta\pi}{n}),\ T(\alpha,\beta,n) = \sum_{k=1}^{n}\sin(\alpha + k\frac{\beta\pi}{n}). \tag{29}$$

Let $\beta(n) = \beta\pi/n$. Using complex numbers we have the geometric sum

$$S(\alpha,\beta,n) + iT(\alpha,\beta,n) = \sum_{k=1}^{n} e^{i\alpha}(e^{i\beta(n)})^k = e^{i(\alpha+\beta(n))}\frac{1-e^{i\ n\beta(n)}}{1-e^{i\beta(n)}} \tag{30}$$

whose numerator is null if $n\beta(n) = \beta\pi = 0(2\pi)$. $\square$

We are now ready to prove Theorem 3:

Proof. To prove **1.**, we need to show that $B(a,n) = C(a,n)$ in lemma 3. Using the trigonometric sum of lemma 4, we have $t_1(a,n) = S(2a,4,n),\ t_2(a,n) = S(4a-4\pi/n,8,n),\ t_3(a,n) = S(4a,8,n)$.

But $t_1(a,n) = 0$ for all values of n since, by the same lemma, we never have $4\pi/n = 0(2\pi)$ for $n \geq 3$. Since B and C just differ by the sign of the coefficient of t_1, $B = C$ for all ns.

For **2.**, $A(a,n)$ and $B(a,n)$ depend upon a if the trigonometric sums t_2 or t_3 are non vanishing. According to the above lemma the condition is $8\pi/n = 0(2\pi)$, which occurs for $n = 4$ only.

For **3.**, it is easily checked that the only value of n such that $B(a,n)$ vanishes is $n = 6$. $\square$

6 Surfaces: the general case

This section generalizes the analysis carried out in the previous section. We relax the hypothesis on the edges' lengths η_is as well as on the angles φ_is, and provide an expression of the angular defect as a Taylor expansion in the edges' lengths.

6.1 Angular defect and k_G

We shall need the following definition:

Definition. 2 *Consider the one-ring neighbors of p. For the sake of conciseness, let $c_i = \cos\varphi_i$, $s_i = \sin\varphi_i$, and define the following quantities:*

$$a_i = \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1}(c_i^2 s_{i+1}^2 + s_i^2 c_{i+1}^2) - \frac{\cos\gamma_i}{2}(\eta_i^2(2c_i^2 s_i^2) + \eta_{i+1}^2(2c_{i+1}^2 s_{i+1}^2))\right], \tag{31}$$

$$b_i = \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1}(c_i^2 c_{i+1}^2) - \frac{\cos\gamma_i}{2}(\eta_i^2 c_i^4 + \eta_{i+1}^2 c_{i+1}^4)\right], \tag{32}$$

$$c_i = \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1}(s_i^2 s_{i+1}^2) - \frac{\cos\gamma_i}{2}(\eta_i^2 s_i^4 + \eta_{i+1}^2 s_{i+1}^4)\right], \tag{33}$$

$$S_p = A + B + C \text{ with } A = \sum_i a_i,\ B = \sum_i b_i,\ C = \sum_i c_i. \tag{34}$$

The quantity S_p is called the module *of the mesh at p.*

We have:

Lemma. 5 *The module of the mesh at p is independent from the angles $\varphi_1, \dots, \varphi_n$.*

Proof. Simply observe that

$$a_i + b_i + c_i = \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1} - \frac{\cos\gamma_i}{2}(\eta_i^2 + \eta_{i+1}^2)\right].$$

□

Lemma. 6 *Let $\eta = \sup_i \eta_i$. The angular defect and the principal curvatures satisfy*

$$2\pi - \sum_{i=1}^{n}\gamma_i = (A\ k_G + B\ k_M^2 + C\ k_m^2) + o(\eta^2) \tag{35}$$

Proof. We express the directional curvature λ_i using Euler's relation. Plugging the values of λ_i and λ_{i+1} into Equation (9), summing over the n one-ring neighbors and grouping terms in k_G, k_M^2 and k_m^2 yields Equation (35). □

Theorem. 4 *Let T_m be a sequence of meshes on a surface having p as a commun vertex. Consider the one-ring around p. Let $\overline{\eta}_m = \sup_i \eta_{m_i}, \underline{\eta}_m = \inf_i \eta_{m_i}$. Suppose that*

1. *there exist two positive constants $\gamma_{min}, \gamma_{max}$ such that $\forall i, \forall m, 0 < \gamma_{min} \leq \gamma_{m_i} \leq \gamma_{max}$.*

2. there exist two positive constants η_1, η_2 such that $\forall m, \eta_1 \leq \overline{\eta}_m / \underline{\eta}_m \leq \eta_2$.

Then, there exists a positive constant $\mathcal{C}$ such that

$$\limsup_m \mid \frac{2\pi - \sum_i \gamma_{i_m}}{S_{p_m}} - k_G \mid \leq \frac{n\mathcal{C}}{2\sin\gamma_{min}} \left[(k_M - k_m)^2 + \mid k_M^2 - k_m^2 \mid \right] \tag{36}$$

Proof. Let us consider a particular mesh in the sequence, and for the sake of clarity, let us omit its index m. We have $b_i \leq \eta^2/(2\sin\gamma_i)$ whence $B \leq n\eta^2/(2\sin\gamma_i)$. The same inequalities hold for c_i and C. From Equation (35), we get

$$\begin{aligned} \mid 2\pi - \sum_i \gamma_i - (A+B+C)k_G \mid &= \mid B(k_M^2 - k_G) + C(k_m^2 - k_G) \mid + o(\eta^2) \\ &= \mid \frac{B+C}{2}(k_M - k_m)^2 + \frac{B-C}{2}(k_M^2 - k_m^2) \mid + o(\eta^2) \\ &\leq \frac{B+C}{2}(k_M - k_m)^2 + \mid \frac{B-C}{2} \mid \mid k_M^2 - k_m^2 \mid + o(\eta^2) \end{aligned} \tag{37}$$

Using the upper bounds on B and C and since S_p does not depend on the angles $\varphi_1, \ldots, \varphi_n$, we get:

$$\mid \frac{2\pi - \sum_i \gamma_i}{S_p} - k_G \mid \leq \frac{\eta^2}{\mid S_p \mid} \frac{n}{2\sin\gamma_{min}} \left((k_M - k_m)^2 + \mid k_M^2 - k_m^2 \mid \right) + o(1). \tag{38}$$

Let us now get back to the sequence of meshes i.e. consider that the previous equation is indexed by m —that is $\eta = \overline{\eta}_m$, $\gamma_i = \gamma_{m_i}$, $S_p = S_{p,m}$. The asumptions on the γ_{m_i} angles and those on $\overline{\eta}_m / \underline{\eta}_m$ imply that there exists a constant $\mathcal{C}$ such that $\eta^2 / \mid S_p \mid \leq \mathcal{C}$, whence the result. $\square$

Theorem 4 shows that the error uncured when approximating the Gauss curvature by $(2\pi - \sum_i \gamma_i)/S_p$ depends upon k_M and k_m. In particular, this error in minimum if $k_M = k_m$ if $k_G > 0$ —or $k_M = -k_m$ if $k_G < 0$. Although the Gauss curvature in intrinsic i.e. invariant upon isometric transformations of the surface, the accuracy of the estimate depends upon the particular embedding considered.

Theorem 4 also deserves the following comment. In using geodesic triangulations to estimate k_G, the natural quantity to divide the angular defect by is the area of the triangles surrounding p. As the previous analysis shows, using Euclidean triangles induces the module of the mesh rather than its area. (Notice however that in both cases, the denominator is homogeneous to a surface area.)

At last, it should be observed that the error term may vanish under very special circumstances. We have encountered two of them in section 5, namely for a valence six triangulation, or a valence four triangulation with $a = 0(\pi/2)$. Other cancellations can certainly be obtained exploiting the independence of the η_is and the φ_is.

Remark. Theorem 4 refines the $O(1)$ upper bound provided in [MW00].

6.2 Angular defect: illustrations

Consider the monkey saddle of Figure 7, point p being at the origin. Using a triangulation whose one-ring neighbors are located above or below the $z = 0$ plane results in a positive angular defect —as if we were processing an elliptic vertex. Using a triangulation whose points are distributed on the two sides of the tangent plane results in a negative angular defect —as if an hyperbolic vertex was processed.

But for such a surface, the second fundamental form II_p is null so that the point is a planar point and any directional curvature around p is also null. No matter what triangulation is used, the angular defect will converge to zero which corresponds to a null Gauss curvature.

Consider now the two surfaces of Figure 8 and assume they differ by a small "fold" located in-between two consecutive one-ring neighbors of the mesh. If II_p does not change —the fold affects third or higher order terms of the Monge form of S at p, neither the directional curvatures not the angular defect are affected. If II_p does change, the angular defect may not change due to cancellations in the error terms.

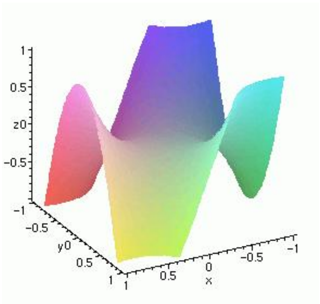

Figure 7: Monkey saddle

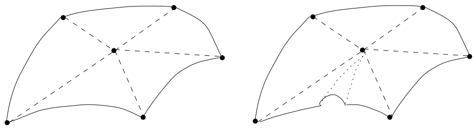

Figure 8: Local fold

7 Conclusion

Let S be a smooth surface, p a point of S, and consider a mesh providing a piece-wise linear approximation of S around p. This paper establishes, asymptotically, several approximation results relating the curvatures of S at p and normalized angular defects of meshes at p. In particular, we show that the angular defect does not provide in general, an accurate point-wise estimate of the Gauss curvature.

From a practical standpoint, these results are especially interesting since the angular defect is a widely used estimate for the Gauss curvature of triangulated surfaces. From a theoretical perspective, we believe these contributions might find applications for the many operations involving differential operators on meshes, that is fairing, smoothing, as well as subdivision. A clear understanding of the geometry of meshes in certainly one step forward more robust algorithms.

On a broader perspective, these contributions illustrate the difficulties one has to face in order to perform differential geometry on non smooth objects. It would therefore be very interesting to generalize the analysis presented in this paper to all the methods —least-square quadrics, gradient-based operators, etc— which are used to estimate the normal, mean curvature, principal directions, ridges, etc of triangulated surfaces.

Acknowledgements

Partially supported by the IST Programme of the EU as a Shared-cost RTD (FET Open) Project under Contract No IST-2000-26473 (ECG - Effective Computational Geometry for Curves and Surfaces)

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8 Appendix: proof of lemma 1

Using polar coordinates ($x = \eta\cos\theta, y = \eta\sin\theta$) to express the position of a point $p \in C$, we get that C is implicitly represented by $F(\eta,\theta) = 0$ with

$$F(\eta,\theta) = y - f(x) = \eta\sin\theta - f(\eta\cos\theta). \tag{39}$$

Obtaining an expression of θ as a function of η involves the implicit function theorem applied to to F at $(\eta = 0, \theta = 0)$. Unfortunately, $\partial F/\partial\theta(0,0) = 0$. We get around the difficulty using an auxiliary function Φ defined as follows:

Lemma. 7 *Let $\Phi(\eta,\theta)$ be defined by*

$$\eta \neq 0: \ \Phi(\eta,\theta) = \frac{F(\eta,\theta)}{\eta} = \sin\theta - \frac{f(\eta\cos\theta)}{\eta}, \tag{40}$$

$$\eta = 0: \ \Phi(0,\theta) = \sin\theta. \tag{41}$$

The function Φ is C^2, and the point $(\eta = 0, \theta = 0)$ is a regular point of Φ. Moreover, near the origin

$$\theta = A\eta + B\eta^2 + o(\eta^2).$$

*Proof.*The proof consists of three parts.

Part I. $\Phi(\eta,\theta)$ versus $F(\eta,\theta)$.
We first observe that working with $F(\eta,\theta)$ or $\Phi(\eta,\theta)$ is equivalent. When $\eta \neq 0$, the solutions of $F = 0$ and $\Phi = 0$ are the same. If $\eta = 0$, the only solution of $\Phi = 0$ is $(0, \theta = 0)$ which is also a solution of $F = 0$. To be more precise, any pair $(0,\theta)$ is a solution of $F = 0$, and working with Φ instead of F retains only one of these solutions, namely $(0,0)$.

Part II. $\Phi(\eta,\theta)$ is C^2.
To begin with, observe that

$$f(x) = kx^2/2 + \nu x^3/6 + o(x^3) \text{ and } f^{'}(x) = kx + \nu x^2/2 + o(x^2).$$

Using these two expressions in the following calculations are straightforward.
<u>Φ is C^0.</u>

$$\lim_{\eta\to 0} \frac{F(\eta,\theta)}{\eta} = \lim_{\eta\to 0} \sin\theta - f(\eta\cos\theta) = \sin\theta$$

<u>Φ is C^1.</u> We consider $\partial\Phi/\partial\eta(0,\theta)$ and $\partial\Phi/\partial\theta(0,\theta)$ in the two settings —$\eta \neq 0$ and $\eta = 0$.

- $\eta \neq 0$, $\partial\Phi/\partial\eta(0,\theta)$

$$\frac{\partial\Phi}{\partial\eta}(0,\theta) = \lim_{\eta\to 0} \frac{\partial\Phi}{\partial\eta}(\eta,\theta) = -\frac{k\cos^2\theta}{2}$$

- $\eta = 0$, $\partial\Phi/\partial\eta(0,\theta)$

$$\frac{\partial\Phi}{\partial\eta}(0,\theta) = \lim_{\eta\to 0} \frac{\Phi(\eta,\theta) - \Phi(0,\theta)}{\eta} = \lim_{\eta\to 0}(\sin\theta - \frac{f(\eta\cos\theta)}{\eta} - \sin\theta) = -\frac{k\cos^2\theta}{2}$$

- $\eta \neq 0$, $\partial\Phi/\partial\theta(0,\theta)$

$$\frac{\partial\Phi}{\partial\eta}(0,\theta) = \lim_{\eta\to 0} \frac{\partial\Phi}{\partial\theta}(\eta,\theta) = \cos\theta$$

• $\eta \neq 0$, $\partial\Phi/\partial\theta(0,\theta)$

$$\frac{\partial \Phi}{\partial \eta}(0,\theta) = \frac{\partial \sin\theta}{\partial \theta}(0,\theta) = \cos\theta$$

<u>Φ is C^2.</u> The equality of the four second order derivatives $\partial^2\Phi/(\partial u^1 \partial u^2)$ with $u^1 = \{\eta,\theta\}$ and $u^2 = \{\eta,\theta\}$ in the two settings are checked similarly.

Part III. Expression of θ as a function of (η).
To see that $(0,0)$ is a regular point, observe that $\partial\Phi/\partial\theta(0,0) = \cos 0 = 1$. Since Φ is C^2, applying the implicit function theorem yields

$$\theta = A\eta + B\eta^2 + o(\eta^2).$$

□

The proof of lemma is now straightforward:
Proof. From Equation (8), one easily derives the equivalents of $\cos\theta$ and $\sin\theta$ as a function of η. Plugging them into Equation (4) yields Equation (5). Equation (6) is proved in the same way observing that $p_{i-1} = (-\eta_{i-1}\cos\theta_{i-1}, \eta_{i-1}\sin\theta_{i-1})$. □