Minimal Legendrian submanifolds of S^{2n+1} and absolutely area-minimizing cones

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Abstract

We use minimal Legendrian submanifolds in spheres to construct examples of absolutely area-minimizing cones and we prove a result about Legendrian 2-tori in S^5.

1 Introduction

A Legendrian submanifold of a $(2n+1)$-dimensional contact manifold is an n-dimensional integral submanifold of the contact distribution. In this paper we consider the unit sphere S^{2n+1} equipped with the contact structure inherited from its standard embedding in $\mathbb{C}^{n+1}$. To an arbitrary immersion $f : M \to S^{2n+1}$ one can associate its cone map $\hat{f} : M \times (0, +\infty) \to \mathbb{R}^{2n+2}$ where $\hat{f}(x,t) = tf(x)$ and we identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. It turns out that f is a Legendrian immersion if and only if $\hat{f}$ is a Lagrangian immersion with respect to the standard symplectic structure of $\mathbb{R}^{2n+2}$. We also note that the normal vectors to f in S^{2n+1} at x are exactly the normal vectors to $\hat{f}$ at (x,t) for $t > 0$. Furthermore, if the second fundamental form of f in S^{2n+1} with respect to a normal vector ν in x has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the second fundamental form of $\hat{f}$ with respect to ν in (t,x) has eigenvalues $t\lambda_1, \ldots, t\lambda_n, 0$. It follows that f is a minimal immersion if and only if $\hat{f}$ is a minimal immersion.

The manifold of all Lagrangian planes in $\mathbb{R}^{2n+2}$ is called the *Lagrangian Grassmannian*, and it is easy to see that this manifold is diffeomorphic to the homogeneous space $\mathbf{U}(n+1)/\mathbf{O}(n+1)$. To an arbitrary Lagrangian immersion $g : P \to \mathbb{R}^{2n+2}$ one can now naturally associate a *Gauss map* $G(g) : P \to \mathbf{U}(n+1)/\mathbf{O}(n+1)$. Consider the composition

$$P \xrightarrow{G(g)} \mathbf{U}(n+1)/\mathbf{O}(n+1) \xrightarrow{\det^2_{\mathbb{C}}} S^1. \tag{1}$$

The pull-back of the volume form of S^1 under (1) is called the *Maslov form* $\mu(g)$ of g, and the class $[\mu(g)]$ it defines is called the *Maslov class* of g. A result of Morvan [15] states that the Maslov form of g is given by $\mu(g) = -\frac{n+1}{\pi}\langle JH, \cdot\rangle$, where H is the mean curvature vector

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of g, J the complex structure in $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ and $\langle \cdot, \cdot \rangle$ denotes the real inner product. As a corollary, a Lagrangian immersion is minimal if and only if it has a vanishing Maslov form.

Notice that a Lagrangian immersion has a vanishing Maslov form precisely when the tangent planes at any two distinct points differ by an element of $\mathbf{SU}(n+1)$. In [8], section III, Harvey and Lawson define a $(n+1)$-dimensional submanifold of $\mathbb{R}^{2n+2}$ to be *special Lagrangian* if the tangent planes at any two distinct points differ by an element of $\mathbf{SU}(n+1)$. As an application of the theory of calibrations introduced in the same paper, they show that special Lagrangian submanifolds have the property of being absolutely area-minimizing. In particular, we conclude from the above discussion that *the cone over a compact minimal Legendrian submanifold of S^{2n+1} is absolutely area-minimizing*. It is interesting to notice that the same conclusion can be drawn from Proposition 2.17 in [8], which in fact contains a restatement of Morvan's formula (namely, equation (2.19)).

In section 2 of this paper, we use this conclusion to construct two series of examples of absolutely area-minimizing cones. The first series generalizes Corollary 3.21 and Example 3.22 in [8]. The second series unifies the proofs of the absolute area-minimality of some examples in [4, 5].

In section 3, we prove a result about Legendrian 2-tori in S^5. Any minimal Legendrian 2-torus projects down via the Hopf fibration to a minimal Lagrangian 2-torus in the complex projective plane $\mathbb{C}P^2$, and there is now a growing literature on minimal Legendrian 2-tori in S^5 using this correspondence, see [2, 10, 12, 14]. The following theorem is a weak form of the converse to the fact that any minimal Legendrian 2-torus of S^5 has a vanishing Maslov class.

Theorem 1. – *Let $f : T^2 \to S^5$ be a Legendrian immersion. Then f is regularly homotopic through Legendrian immersions to a minimal Legendrian embedding if and only if $[\mu(\hat{f})] = 0$.*

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2 Examples of minimal Legendrian submanifolds of S^{2n+1}

In this section we present examples of compact homogeneous minimal Legendrian submanifolds of S^{2n+1}. The cones over these submanifolds are area-minimizing. Although some of the examples are already known, they are here presented from a new and unified perspective.

2.1 Twisted normal bundles

In [8] Harvey and Lawson defined a submanifold M of S^n to be *austere* if the set of eigenvalues of the Weingarten operator of M with respect to any normal vector is invariant under multiplication by -1. Then they showed that for any compact austere submanifold M of S^n, the cone over

$$\Phi : N^1M \times S^1 \to S^{2n+1}, \qquad \Phi(\nu_x, e^{i\theta}) = (\cos\theta\, x, \sin\theta\, \nu_x),$$

where N^1M denotes the unit normal bundle of M in S^n, is special Lagrangian.

Note that the map Φ fails to be an immersion for $\sin\theta = 0$ if the codimension of M in S^{2n+1} is bigger than one, and it fails to be an immersion for $\cos\theta = 0$ if 0 is an eigenvalue of the Weingarten operator of M with respect to some unit normal. Therefore we will consider the following variation of this construction. Let M be a submanifold of the unit sphere S^n and NM be the normal bundle of M in S^n. We denote by $\widetilde{M_+}$, $\widetilde{M_-}$ two copies of $NM^{\leq\frac{\pi}{2}} = \{\nu \in NM : |\nu| \leq \pi/2\}$ and by $\widetilde{M}$ the double of $NM^{\leq\frac{\pi}{2}}$ i. e. $\widetilde{M} = \widetilde{M_+} \cup \widetilde{M_-}/\partial\widetilde{M_+} \sim \partial\widetilde{M_-}$. Note that if M is a closed hypersurface of S^n then $\widetilde{M}$ is diffeomorphic to the product $M \times S^1$. Define

$$\Psi : \widetilde{M} \to S^{2n+1}, \qquad \Psi(\nu_x) = \begin{cases} (\epsilon\cos|\nu_x|\, x, \sin|\nu_x|\, \frac{\nu_x}{|\nu_x|}) & \text{if } \nu_x \neq 0, \\ (\epsilon x, 0) & \text{if } \nu_x = 0, \end{cases}$$

where $\epsilon = \pm 1$ according to whether $x \in \widetilde{M_\pm}$. It is easy to see that Ψ has the same image as Φ.

Proposition 1. – *If 0 is never an eigenvalue of the Weingarten operator of M with respect to any unit normal, then $\Psi : \widetilde{M} \to S^{2n+1}$ is a Legendrian immersion. If in addition M is austere, then Ψ is minimal.*

Proof. – We will first compute the differential of Ψ at a normal vector ν_{x_0}. Near x_0 we choose an orthonormal frame field $e_1, \ldots, e_p$ and a normal frame field $\nu_1, \ldots, \nu_q$ where $p + q = n$ and $e_1, \ldots, \nu_q$ is positively oriented. Without loss of generality, we may assume that $(\nabla\nu_k)^N_{x_0} = 0$ where $()^N$ denotes the orthogonal projection onto the normal space $N_{x_0}M$. We can also take $e_1, \ldots, e_p$ to be the eigenvectors of the Weingarten operator A^{ν_1}, namely $A^{\nu_1}(e_k) = \lambda_k e_k$. Upon a linear change of coordinates we may further assume that at x_0 the vectors $x_0, e_1, \ldots, \nu_q$ give the standard basis of $\mathbb{R}^{n+1} \supset S^n \supset M$.

If $\nu_{x_0} \neq 0$ we may assume $\nu_1 = \frac{\nu_{x_0}}{|\nu_{x_0}|}$. The differential of Ψ at ν_{x_0} is then given by

$$\Psi_*(e_j) = (\epsilon\cos|\nu_{x_0}|\, e_j, -\lambda_j \sin|\nu_{x_0}|\, e_j) \quad \text{for } 1 \leq j \leq p \text{ and}$$

$$\Psi_*(\nu_j) = \begin{cases} (-\epsilon\sin|\nu_{x_0}|\, x_0, \cos|\nu_{x_0}|\, \frac{\nu_{x_0}}{|\nu_{x_0}|}) & \text{if } j = 1, \\ (0, \sin|\nu_{x_0}|\, \frac{\nu_j}{|\nu_{x_0}|}) & \text{if } 2 \leq j \leq q. \end{cases}$$

If $\nu_{x_0} = 0$ we have

$$\Psi_*(e_j) = (\epsilon e_j, 0) \quad \text{for } 1 \leq j \leq p \text{ and}$$

$$\Psi_*(\nu_j) = (0, \nu_j) \quad \text{for } 1 \leq j \leq q.$$

In these formulae the fields e_j, ν_j are evaluated at x_0.

The above calculation immediately shows that Ψ is an immersion (if $\lambda_j \neq 0$ for $1 \leq j \leq p$) and Legendrian (since $\langle\Psi_*(v), J\Psi(\nu_{x_0})\rangle = \langle\Psi_*(v), J\Psi_*(v')\rangle = 0$ where each of v, v' is one of $e_1, \ldots, \nu_q$ and $J : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ is given by $J(x, y) = (-y, x)$). Next we compute the

Maslov form of the cone over Ψ. For this purpose, we view $\mathbb{R}^{2n+2}$ as $\mathbb{C}^{n+1}$ and J as the multiplication by i, and we represent $G(\hat{\Psi})(\nu_{x_0})$ by a unitary matrix which we specify by its column vectors and then compute the squared determinant. If $\nu_{x_0} \neq 0$ the matrix has as columns the unit vectors

$$\begin{array}{lll} \epsilon\cos|\nu_{x_0}|\, x_0 + i\sin|\nu_{x_0}|\nu_1, & \frac{\epsilon\cos|\nu_{x_0}| - i\lambda_1\sin|\nu_{x_0}|}{\cos^2|\nu_{x_0}| + \lambda_1^2\sin^2|\nu_{x_0}|}e_1, \ldots, & \frac{\epsilon\cos|\nu_{x_0}| - i\lambda_p\sin|\nu_{x_0}|}{\cos^2|\nu_{x_0}| + \lambda_p^2\sin^2|\nu_{x_0}|}e_p, \\ -\epsilon\sin|\nu_{x_0}|\, x_0 + i\cos|\nu_{x_0}|\,\nu_1, & i\nu_2, \ldots, i\nu_q. & \end{array}$$

Therefore

$$(\det)^2 G(\hat{\Psi})(\nu_{x_0}) = \epsilon^2 i^{2q}\left[\prod_{j=1}^{p}\frac{(\epsilon\cos|\nu_{x_0}| - i\lambda_j\sin|\nu_{x_0}|)^2}{(\cos^2|\nu_{x_0}| + \lambda_j^2\sin^2|\nu_{x_0}|)^2}\right](\det)^2(x_0, e_1, \ldots, e_p, \nu_1, \nu_2, \ldots \nu_q).$$

If M is austere this expression equals $(-1)^q$ (because

$$(\epsilon\cos|\nu_{x_0}| - i\lambda\sin|\nu_{x_0}|)(\epsilon\cos|\nu_{x_0}| + i\lambda\sin|\nu_{x_0}|) = \cos^2|\nu_{x_0}| + \lambda^2\sin^2|\nu_{x_0}|$$

for $\lambda \in \mathbb{R}$). If $\nu_{x_0} = 0$ we get:

$$(\det)^2(\epsilon x_0, \epsilon e_1, \ldots, \epsilon e_p, i\nu_1, \ldots, i\nu_q) = \epsilon^{2p} i^{2q}(\det)^2(x_0, e_1, \ldots, e_p, \nu_1, \ldots, \nu_q) = (-1)^q.$$

Thus $(\det)^2 G(\hat{\Psi})$ is a constant map. This implies that the Maslov form of $\hat{\Psi}$ vanishes, hence Ψ is minimal. □

2.2 Twisted normal bundles of isoparametric submanifolds

We shall now apply the construction discussed in section 2.1 to certain leaves of certain isoparametric foliations of spheres. An *isoparametric hypersurface* M^{n-1} of S^n is a hypersurface with constant principal curvatures. A good introduction to the theory of isoparametric hypersurfaces in spheres can be found in chapter 3 of [3], and a comprehensive survey, including generalizations, can be found in [21]. It follows from the formulae for Weingarten operators of tubes that if M is isoparametric, then so are all of its parallel hypersurfaces. A parallel submanifold of an arbitrary submanifold M is a submanifold of the same dimension obtained by exponentiating a parallel normal vector field in M, and a focal point of M is a critical value of the exponential map restricted to the normal bundle of M. The focal set of M, which is the set of all the focal points of M, need not have a manifold structure. However, in the case of an isoparametric hypersurface, the exponential map has constant rank along a parallel normal vector field, and this can be used to show that the focal set of a complete isoparametric hypersurface decomposes into a union of submanifolds (see [17]).

Let M be an isoparametric hypersurface of S^n. Münzner [16], generalizing previous work by Cartan, showed that the distinct principal curvatures $\lambda_1 > \cdots > \lambda_g$ of M satisfy $\lambda_k = \cot(\theta + (k-1)\pi/g)$ for $k = 1, \ldots, g$, where $0 < \theta < \pi/g$, and the multiplicities $m_1, \ldots, m_g$ satisfy $m_1 = \ldots = m_g$ if g is odd, and $m_1 = m_3 = \ldots = m_{g-1}$, $m_2 = m_4 = \ldots = m_g$ if g is even. Münzner's computation also shows that the focal manifolds must be minimal. Beginning with these facts, he proved that the isoparametric parallel hypersurfaces of M can be represented as open subsets of level hypersurfaces in S^n of a homogeneous polynomial F of degree g on $\mathbb{R}^{n+1}$ which satisfies the Cartan-Münzner differential equations:

$$|\mathrm{grad}F(x)|^2 = g^2|x|^{2g-2}, \quad \Delta F = \frac{g^2(m_2 - m_1)}{2}|x|^{g-2},$$

where g is the number of distinct principal curvatures and m_1, m_2 are the two (possibly equal) multiplicities of the principal curvatures. This result implies that any piece of isoparametric hypersurface can be extended to a compact isoparametric hypersurface. His construction also shows that there are only two focal manifolds, even though there may be more than two principal curvatures; these correspond precisely to the only two singular levels of the polynomial F; and they sit apart at a distance of π/g. Further, each element of the family of isoparametric parallel hypersurfaces divides the ambient sphere into two ball bundles over the respective focal submanifolds. From this topological information, he was able to determine the cohomology rings of the hypersurfaces and the focal submanifolds, and then by using delicate cohomological and algebraic arguments, he obtained the splendid result that the number g of distinct principal curvatures must satisfy $g = 1$, 2, 3, 4 or 6.

A large class of examples of isoparametric families of hypersurfaces in spheres is provided by the orbital foliations induced by orthogonal representations of compact Lie groups that have cohomogeneity two in Euclidean space. These examples are of course homogeneous, and, in fact, a classification of homogeneous isoparametric families follows from the classification of orthogonal representations of cohomogeneity two that is given in [9]. Examples of inhomogeneous isoparametric hypersurfaces are known only in the case $g = 4$, see [18, 19, 7]. So far no complete classification of isoparametric hypersurfaces is known, although Cartan classified the case $g \leq 3$, and in the case $g = 6$ the unsolved problem is whether there exists an inhomogeneous isoparametric hypersurface with $m_1 = m_2 = 2$. See section 1 in [21] for more details.

Let $\{M_t\}$ for $0 < t < \pi/g$ be an isoparametric family of parallel hypersurfaces of S^n with g distinct principal curvatures, and let M_0 and $M_{\pi/g}$ denote the focal manifolds; here t is the distance from M_0 to M_t. Then the principal curvatures of M_t are $\lambda_k(t) = \cot(t+(k-1)\pi/g)$ for $k = 1, \ldots, g$, and the principal curvatures of the focal manifold M_0 (resp. $M_{\pi/g}$) with respect to any unit normal are $\lambda_k(0)$ for $k = 2, \ldots, g$ (resp. $\lambda_k(\pi/g)$ for $k = 1, \ldots, g-1$), with the multiplicities satisfying the same identities as above. As we said before, our aim is to apply the construction of section 2.1 to some element of $\{M_t\}$. A moment's thought shows that we can find $t \in [0, \pi/g]$ such that M_t is austere with no zero principal curvature with respect to any unit normal if and only if $m_1 = m_2$. So we make this assumption from now on, and we proceed as follows.

1. If g is odd then the focal manifolds are congruent under the antipodal map of S^n. Since $m_1 = m_2$, they are compact austere submanifolds of S^n, and since they have no principal curvature equal to zero, the construction of section 2.1 yields an example of a minimal Legendrian embedding of $\widetilde{M}$ into S^{2n+1}. The global structure theorem of Münzner says that each compact isoparametric hypersurface divides S^n into two manifolds with boundary, each of which is a normal ball bundle over one of the two focal manifolds. It follows that $\widetilde{M}$ is diffeomorphic to the sphere S^n.

2. If g is even, it follows from $m_1 = m_2$ that the isoparametric hypersurface $M_{\pi/2g}$ is a compact austere submanifold of S^n all of whose principal curvatures are nonzero. Therefore the construction of section 2.1 yields an example of a minimal Legendrian immersion of $\widetilde{M}$ into S^{2n+1}. It is easy to use the fact that $M_{\pi/2g}$ is invariant under the antipodal map of S^n to show that this immersion induces a minimal Legendrian embedding of $\widetilde{M}/\mathbb{Z}_2 \simeq M_{\pi/2g} \times_{\mathbb{Z}_2} S^1$ into S^{2n+1} (the nontrivial element of $\mathbb{Z}_2$ acts on each factor of the product $M_{\pi/2g} \times S^1$ by the antipodal map).

Next, we consider all the known examples of isoparametric foliations of S^n that satisfy $m_1 = m_2 = m$. It turns out that these are homogeneous, namely the orbital foliation induced on S^n by an orthogonal representation ρ of a compact connected Lie group G on $\mathbb{R}^{n+1}$ of cohomogeneity two. Some of these examples are discussed in detail in section 7 of chapter 3 of [3]; see also [22]. According to whether g is odd or even, we have possibility 1 or 2, as described above.

g=1: Here $m = n - 1$. The foliation is by umbilic hypersurfaces, namely the hyperspheres parallel to the equator. The focal manifolds are points and $\widetilde{M}$ is a totally geodesic totally real $S^n \subset S^{2n+1}$.

g=2: Here n is odd and $m = \frac{n-1}{2}$. The foliation is by hypersurfaces parallel to the Clifford torus $M_{\pi/4} = S^{\frac{n-1}{2}}(1/\sqrt{2}) \times S^{\frac{n-1}{2}}(1/\sqrt{2}) \subset S^n$, and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $(S^{\frac{n-1}{2}} \times S^{\frac{n-1}{2}}) \times_{\mathbb{Z}_2} S^1$. In particular we get a minimal Legendrian immersion

$$S^1 \times S^{\frac{n-1}{2}} \times S^{\frac{n-1}{2}} \to S^{2n+1}.$$

This example first appears in [8], Example 3.22.

g=3: Here $m = 1, 2, 4, 8$. Let G be the group of 3×3 $\mathbb{F}$-unitary matrices, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions), $\mathbb{O}$ (octonions), according to whether $m = 1, 2, 4, 8$. Note that $G = \mathbf{O}(3)$, $\mathbf{U}(3)$, $\mathbf{Sp}(3)$, $\mathbf{F}_4$, respectively. Let V be the real vector space of 3×3 traceless $\mathbb{F}$-Hermitean matrices. Note that $\dim V = 3m + 2$. Now $\rho : G \to \mathbf{O}(\mathbb{R}^{3m+2})$ is conjugation. The focal manifolds are the so called standard embeddings of the projective planes, namely $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{O}P^2$. The construction gives minimal Legendrian embeddings

$$S^4 \to S^9, \quad S^7 \to S^{15}, \quad S^{13} \to S^{27}, \quad S^{25} \to S^{51}.$$

Since these embeddings are not totally geodesic (in fact, a simple computation shows that Ψ can be totally geodesic in S^{2n+1} only if M is totally geodesic in S^n), they are not congruent to the standard embedding $S^n \subset \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$ of the unit sphere. The case $S^4 \to S^9$ first appears in [8], Example 3.21.

g=4: Here $m = 1, 2$. If $m = 1$ then the representation $\rho : \mathbf{U}(2) \to \mathbf{O}(\mathbb{R}^6)$ is on the space of 2×2 complex symmetric matrices and is given by $\rho(A)X = AXA^t$. The isoparametric hypersurfaces are diffeomorphic to $\mathbf{U}(2)/\mathbb{Z}_2^2$ and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $(\mathbf{U}(2)/\mathbb{Z}_2^2) \times_{\mathbb{Z}_2} S^1$. Since $\mathbf{U}(2)$ is diffeomorphic to $S^1 \times S^3$, in particular we get a minimal Legendrian immersion

$$S^1 \times S^1 \times S^3 \to S^{11}.$$

If $m = 2$ then $\rho : \mathbf{Sp}(2) \to \mathbf{O}(\mathbb{R}^{10})$ is the adjoint representation. The isoparametric hypersurfaces are diffeomorphic to $\mathbf{Sp}(2)/T^2$ where $T^2 \subset \mathbf{Sp}(2)$ is a maximal torus and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $\mathbf{Sp}(2)/T^2 \times_{\mathbb{Z}_2} S^1$.

g=6: Here $m = 1, 2$. If $m = 1$ then ρ is the only almost faithful real irreducible 8-dimensional representation of $\mathbf{SO}(4)$. The isoparametric hypersurfaces are diffeomorphic to $\mathbf{SO}(4)/\mathbb{Z}_2^2$ and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $(\mathbf{SO}(4)/\mathbb{Z}_2^2) \times_{\mathbb{Z}_2} S^1$. Since $\mathbf{SO}(4)$ is finitely covered by $S^3 \times S^3$, in particular we get a minimal Legendrian immersion[1]

$$S^1 \times S^3 \times S^3 \to S^{15}.$$

If $m = 2$ then $\rho : \mathbf{G}_2 \to \mathbf{O}(\mathbb{R}^{14})$ is the adjoint representation. The isoparametric hypersurface are diffeomorphic to $\mathbf{G}_2/T^2$ where $T^2 \subset \mathbf{G}_2$ is a maximal torus and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $\mathbf{G}_2/T^2 \times_{\mathbb{Z}_2} S^1$.

2.3 Orbits of isotropy representations of Hermitian symmetric spaces

We shall now show that certain orbits of the isotropy representations of certain Hermitian symmetric spaces give rise to examples of Legendrian submanifolds in spheres. The results quoted without proof can all be found in [11] and [13]. Let X be an Hermitian symmetric space of compact type, G its connected group of isometries and K the isotropy subgroup. So $X = G/K$ is a product of compact irreducible Hermitian symmetric spaces, and G is the product of the corresponding simple groups of isometries. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the decomposition of the Lie algebra $\mathfrak{g}$ of G into the ± 1-eigenspaces of the involution, which will be denoted by σ. Here $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is naturally identified with the tangent space of X at the base point. Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$. We choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{k}$ (this is possible, since $\mathfrak{k}$ is of maximal rank in $\mathfrak{g}$). Then $\mathfrak{h}^c$ is a Cartan subalgebra in $\mathfrak{g}^c$. The roots of $\mathfrak{g}^c$ which are also roots of $\mathfrak{k}^c$ are called compact roots. Introduce an ordering in the system of roots. To each root α we define the coroot $H_\alpha \in \mathfrak{h}^c$ by the rule $\alpha(H) = (H_\alpha, H)$ for all $H \in \mathfrak{h}^c$ (here $(\cdot, \cdot)$ denotes the Cartan-Killing form of $\mathfrak{g}^c$) and we associate a root vector X_α so that

$$(X_\alpha - X_{-\alpha}), \quad i(X_\alpha + X_{-\alpha}) \in \mathfrak{g} \quad \text{and} \quad [X_\alpha, X_{-\alpha}] = \frac{2H_\alpha}{|\alpha|^2}.$$

There exists an element Z in the center of $\mathfrak{k}$ such that, for every noncompact positive root α, $[Z, X_\alpha] = -iX_\alpha$ and $[Z, X_{-\alpha}] = iX_{-\alpha}$. The restriction of $\mathrm{ad}(Z)$ to $\mathfrak{p}$ is a complex structure, which will be denoted by J. By means of this J, we view $\mathfrak{p}$ as $\mathbb{C}^{\dim X}$. The roots α, β are

[1] This immersion is of course different from the immersion of $S^1 \times S^3 \times S^3$ in S^{15} given above in the case $g = 2$, $n = 7$, because their images are not diffeomorphic.

called strongly orthogonal if $\alpha \pm \beta$ are not roots. There exists a set Δ of strongly orthogonal noncompact positive roots such that the real subspace $\mathfrak{a}$ spanned by the $i(X_\alpha + X_{-\alpha})$, $\alpha \in \Delta$, is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$.

Define the Cayley transform $c = \exp \frac{\pi}{4} \sum_{\alpha\in\Delta} i(X_\alpha + X_{-\alpha}) \in G$ and let $\tau = \mathrm{ad}(c)^2$, viewed as an automorphism of $\mathfrak{g}$. Then $\tau^4 = 1$ and the following are equivalent (see [13], Remark 1, p. 286 and Proposition 4.4, p. 273):

- the noncompact dual X^* of X is of tube type;
- $\tau(Z) = -Z$;
- $\tau(\mathfrak{k}) = \mathfrak{k}$;
- $\tau^2 = 1$.

Hereafter we shall assume that the Hermitian symmetric space X^* is of tube type, that is, holomorphically equivalent to the tube domain over a self-dual cone. Since τ is an involution of $\mathfrak{g}$ which preserves $\mathfrak{k}$, and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, we have that τ also preserves $\mathfrak{p}$, and this yields the splittings $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$, $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$ into the ± 1-eigenspaces of τ. Now $\mathrm{ad}(c)$ preserves $\mathfrak{k}^+$ and $\mathfrak{p}^+$, and interchanges $\mathfrak{k}^-$ with $\mathfrak{p}^-$, and J interchanges $\mathfrak{p}^+$ with $\mathfrak{p}^-$. Note that $Z \in \mathfrak{k}^-$ and define $p = \mathrm{ad}(c)Z \in \mathfrak{p}^-$. Let $\rho : K \to \mathbf{U}(\mathfrak{p})$ be the linear isotropy representation of X and consider the K-orbit through p. The tangent space $T_pK(p)$ is

$$\begin{aligned}
[\mathfrak{k}, p] &= \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}, Z] \\
&= \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}^+, Z] + \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}^-, Z] \\
&= \mathrm{ad}(c)[\mathfrak{k}^+, Z] + \mathrm{ad}(c)\, J\, \mathrm{ad}(c)^{-1}\mathfrak{k}^- \\
&= 0 + \mathrm{ad}(c)\, J\, \mathfrak{p}^- \\
&= \mathrm{ad}(c)\, \mathfrak{p}^+ \\
&= \mathfrak{p}^+.
\end{aligned}$$

Now $[\mathfrak{k}, p]$ is a Lagrangian subspace of $\mathfrak{p}$. Let $\mathfrak{k}'$ be the Cartan-Killing orthogonal complement of Z in $\mathfrak{k}$ and let K' be the analytic subgroup of K corresponding to $\mathfrak{k}'$. Since $[\mathfrak{k}, p] = [\mathfrak{k}', p] \oplus \mathbb{R}[Z, p] = [\mathfrak{k}', p] \oplus \mathbb{R}Jp$, we have that $[\mathfrak{k}', p] \oplus \mathbb{R}p$ is also a Lagrangian subspace of $\mathfrak{p}$. So the tangent space $T_pK'(p)$ is a Legendrian subspace of $\mathfrak{p}$. At this point, we assume that X is irreducible. Then $\rho(K') \subset \mathbf{SU}(\mathfrak{p})$, so that $K'(p)$ is a Legendrian submanifold and its Maslov form vanishes. Hence $K'(p)$ is a compact minimal Legendrian submanifold of the unit sphere in $\mathfrak{p} \cong \mathbb{C}^{\dim X}$. The remark after Proposition 4.4 in [13] can be interpreted as saying that X^* being of tube type is equivalent to the system of restricted roots of X^* being reduced (compare p. 456 in [11]). According to the classification of Hermitian symmetric spaces (see, for example, section 8.11 in [23]), one sees that there are five types of irreducible compact Hermitian symmetric spaces X whose noncompact duals are of tube type. These are listed in the table below, which also describes in each case the diffeomorphism type M of $K'(p)$.

X	$M \subset S^{\dim X - 1}$
$\mathbf{SO}(n+2)/\mathbf{SO}(2) \times \mathbf{SO}(n)$ $(n \geq 3)$	$S^{n-1} \subset S^{2n-1}$
$\mathbf{SU}(2n)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$ $(n \geq 2)$	$\mathbf{SU}(n) \subset S^{2n^2-1}$
$\mathbf{SO}(4n)/\mathbf{U}(2n)$ $(n \geq 2)$	$\mathbf{SU}(2n)/\mathbf{Sp}(n) \subset S^{4n^2-2n-1}$
$\mathbf{Sp}(n)/\mathbf{U}(n)$ $(n \geq 3)$	$\mathbf{SU}(n)/\mathbf{SO}(n) \subset S^{n^2+n-1}$
$\mathbf{E}_7/\mathbf{SO}(2)\mathbf{E}_6$	$\mathbf{E}_6/\mathbf{F}_4 \subset S^{53}$

The first four minimal Legendrian embeddings in the table are given as follows: $S^{n-1} \subset S^{2n-1}$ is totally real; $\mathbf{SU}(n) \subset S^{2n^2-1}$ comes from the standard inclusion of $\mathbf{SU}(n)$ into the $n \times n$ complex matrices; $\mathbf{SU}(n)/\mathbf{SO}(n) \subset S^{n^2+n-1}$ is the $\mathbf{SU}(n)$-orbit $\{AA^t : A \in \mathbf{SU}(n)\}$ in the space of $n \times n$ complex symmetric matrices; $\mathbf{SU}(2n)/\mathbf{Sp}(n) \subset S^{4n^2-2n-1}$ is the $\mathbf{SU}(2n)$-orbit $\{A \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} A^t : A \in \mathbf{SU}(2n)\}$ in the space of $2n \times 2n$ complex skew-symmetric matrices. In [4, 5], these examples are discussed, with the curious exception of $\mathbf{SU}(2n)/\mathbf{Sp}(n)$ for $n \neq 3$, on a case by case basis (note however the misprint in the title of [5]).

2.4 Yet another example

Consider the 4-dimensional complex irreducible representation of $\mathbf{SU}(2)$, namely $\rho : \mathbf{SU}(2) \to \mathbf{SU}(4)$ (ρ can be realized as the natural action of $\mathbf{SU}(2)$ on the space of homogeneous complex polynomials of degree 3 in two variables z_1, z_2). Upon suitable choices of basis elements, we can write

$$\rho \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} = \begin{pmatrix} \alpha^3 & \sqrt{3}\alpha\beta^2 & \beta^3 & \sqrt{3}\alpha^2\beta \\ \sqrt{3}\alpha\bar\beta^2 & \alpha\bar\alpha^2 - 2\bar\alpha\beta\bar\beta & \sqrt{3}\bar\alpha^2\beta & \beta\bar\beta^2 - 2\alpha\bar\alpha\bar\beta \\ -\bar\beta^3 & -\sqrt{3}\bar\alpha^2\bar\beta & \bar\alpha^3 & \sqrt{3}\bar\alpha\bar\beta^2 \\ -\sqrt{3}\alpha^2\bar\beta & 2\alpha\bar\alpha\beta - \beta^2\bar\beta & \sqrt{3}\bar\alpha\beta^2 & \alpha^2\bar\alpha - 2\alpha\beta\bar\beta \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. Therefore the induced representation on the Lie algebra level maps the basis elements of the Lie algebra $\mathfrak{su}(2)$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

respectively into

$$\begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -2 \\ 0 & -\sqrt{3} & 0 & 0 \\ -\sqrt{3} & 2 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} 0 & 0 & 0 & i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 2i \\ 0 & i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 2i & 0 & 0 \end{pmatrix}.$$

Let M be the orbit through the point $p = (1, 0, 1, 0) \in \mathbb{C}^4$, which is the quotient of S^3 by a cyclic subgroup of order 3. The easily seen identities

$$\langle Xp, iYp \rangle = \langle Xp, ip \rangle = 0$$

for $X, Y \in \mathfrak{su}(2)$ show that the tangent space of M at p is a Legendrian subspace. Since $\rho[\mathbf{SU}(2)] \subset \mathbf{SU}(4)$, it follows that M is a Legendrian submanifold of S^7 with vanishing Maslov form. Therefore it is minimal and the cone over M is absolutely area-minimizing. Observe that this example is not the twisted normal bundle over a compact minimal surface in S^3 because of its topology.

3 Legendrian 2-tori in S^5

In this section we prove Theorem 1 which was stated in the introduction.

Let λ be the Liouville form. We denote by $I_{Leg}(T^2, S^5)$ the space of Legendrian immersions of T^2 into S^5 and by $\mathrm{Mono}_\lambda(TT^2, TS^5)$ the space of bundle monomorphisms $\sigma : TT^2 \to TS^5$ such that $\lambda_{|\sigma(F)} = 0$ and $d\lambda_{|\sigma(F)} = 0$ for each fiber F of TT^2. (These two spaces are provided with the compact-open topology). The h-principle for Legendrian immersions implies that the map:

$$\begin{array}{ccc} I_{Leg}(T^2, S^5) & \longrightarrow & \mathrm{Mono}_\lambda(TT^2, TS^5) \\ g & \longmapsto & dg, \end{array}$$

is a weak homotopy equivalence (see [6], p. 144).

Lemma. – *$\pi_0(I_{Leg}(T^2, S^5)) \simeq H^1(T^2, \mathbb{Z})$ and the identification is given by half the Maslov class.*

Proof of the lemma. – A 2-frame of T_xT^2 is said to be Legendrian if the plane it spans is Legendrian. The space of all Legendrian 2-frames can be identified with $\mathbf{GL}(3, \mathbb{C})$. Let $R = (e_1, e_2)$ be a global section of the frame bundle $\pi : FT^2 \to T^2$. There is a bijection:

$$\Psi : \ \mathrm{Mono}_\lambda(TT^2, TS^5) \longrightarrow C^0(T^2, \mathbf{GL}(3, \mathbb{C})), \quad \sigma \longmapsto \Psi(\sigma) = f_\sigma,$$

where f_σ is the map $x \mapsto (\sigma(\pi(R_x)), \sigma(e_1(x)), \sigma(e_2(x)))$. Thus

$$\pi_0(I_{Leg}(T^2, S^5)) \simeq [T^2, \mathbf{GL}(3, \mathbb{C})] \simeq [T^2, \mathbf{U}(3)],$$

since $\mathbf{U}(3)$ is a deformation retract of $\mathbf{GL}(3, \mathbb{C})$. Consider the projection $\det : \mathbf{U}(3) \to \mathbf{U}(1)$ and let u be the pull back of the (normalized) volume form w of $\mathbf{U}(1)$. The class u is a generator of $H^1(\mathbf{U}(3), \mathbb{Z})$ and, since $\pi_1(\mathbf{U}(3)) = \mathbb{Z}$, $\pi_2(\mathbf{U}(3)) = 0$, the degree map:

$$\begin{array}{rccc} \deg : & [T^2, \mathbf{U}(3)] & \longrightarrow & H^1(T^2, \mathbb{Z}) \\ & f & \longmapsto & f^*u \end{array}$$

is a bijection (see [20], p. 451 or [1], p. 510 for example). Let $g \in I_{Leg}(T^2, S^5)$ and let $\hat{g}$ be its associated cone map $\hat{g} : T^2 \times (0, \infty) \to \mathbb{C}^3$ given by $\hat{g}(t, x) = tg(x)$. Let $p : \mathbf{U}(3) \to \mathbf{U}(3)/\mathbf{O}(3)$ be the projection. The Gauss map of $\hat{g}$ at $(x, 0)$ is given by

$$G(\hat{g})(x, 0) = p \circ r \circ (g(x), dg(e_1(x)), dg(e_2)(x)) = p \circ r \circ f_{dg}(x, 0),$$

where $r : \mathbf{GL}(3,\mathbb{C}) \to \mathbf{U}(3)$ is a retraction. Hence

$$[\mu(\hat{g})] = (\det^2 \circ G(\hat{g}))^* w = f_{dg}^* \circ r^* \circ p^* \circ (\det^2)^* w.$$

It is readily checked that $p^* \circ (\det^2)^* w = 2u$ and, of course, $r^* = id$ so

$$[\mu(\hat{g})] = 2 f_{dg}^* u.$$

Thus $\pi_0(I_{Leg}(T^2, S^5))$ is classified by half the Maslov class. $\square$

The proof of the Theorem 1 is a consequence of this lemma combined with the fact that there actually exists a minimal Legendrian embedding of the 2-torus into S^5.

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