ON TOTALLY REAL ISOTOPY CLASSES

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. Given two totally real embeddings $j_0, j_1 : M^n \to \mathbb{C}^n$ is it possible to find an isotopy joining j_0 and j_1 and such that j_t is totally real for every t ? A positive answer obviously implies that j_0 and j_1 belong to the same arcwise connected component of the space $I_{TR} = I_{TR}(M^n, \mathbb{C}^n)$ of totally real immersions and of the space E of ordinary embeddings. Let $E_{TR} = I_{TR} \cap E$ be the space of totally real embeddings and $\epsilon \in \pi_0(E)$, $\iota \in \pi_0(I_{TR})$. In what follows we give an upper bound to the cardinality of $\pi_0(E_{TR} \cap \epsilon \cap \iota)$. In some cases - for instance if the dimension n is 3 modulo 4 - this upper bound is one and so we deduce the computation of $\pi_0(E_{TR})$.

1. Introduction and main results.

Let M^n be a n-dimensional manifold without boundary and (W^{2n}, J) be a $2n$-dimensional almost complex manifold. An immersion $j : M^n \longrightarrow W^{2n}$ is said to be *totally real* if for every point p of M^n the complexified tangential map $dj_p^{\mathbb{C}} : T_pM^n \otimes \mathbb{C} \longrightarrow T_{j(p)}W^{2n}$ is of maximal rank. We denote by $I_{TR}(M^n, W^{2n})$ - or simply I_{TR} - the space of totally real immersions of M^n into W^{2n}, by E_{TR} the space of totally real embeddings, by I and E the spaces of ordinary immersions and ordinary embeddings (all these spaces are provided with the compact-open topology). Since the works of M. Gromov [9] and J. A. Lees [19], the space I_{TR} is well understood, so the next step is the study of the space E_{TR}. In [3], M. Audin gives a topological characterization of the manifolds M^n which admit a totally real embeddings into $\mathbb{C}^n$ (at least for $n \not\equiv 3$ [4]). Here we consider the following question.

Q : Given two totally real embeddings j_0 and j_1, is it possible to find an isotopy j_t joining j_0 and j_1 and such that j_t is totally real for every t ?

We call such an isotopy, a totally real isotopy. One trivial necessary condition for a positive answer is that the two embeddings must belong at once to the same component of E and of I_{TR}. Recall that $\pi_0(E)$ has been computed (when $n \geq 4$) by A. Haefliger and M. Hirsch [11],[14], and $\pi_0(I_{TR})$ by M. Gromov and J. A. Lees. So one can ask a more precise question :

Q': Given two totally real embeddings j_0 and j_1 that belong to the same component of E and of I_{TR}, is it possible to find a totally real isotopy joining j_0 and j_1 ?

In this article, we partially answer to this question. We mainly use a parametric version of a h-principle due to M. Gromov combined with the study of the pairs (I, I_{TR}) and (I, E).

Let $\epsilon \in \pi_0(E)$ and $\iota \in \pi_0(I_{TR})$, we put $E_{TR}(\epsilon,\iota) = E_{TR} \cap \epsilon \cap \iota$. Of course $E_{TR} = \cup_{\epsilon,\iota} E_{TR}(\epsilon,\iota)$ and some of the $E_{TR}(\epsilon,\iota)$ may be empty. We denote by $N(\epsilon,\iota)$ the cardinality of $\pi_0(E_{TR}(\epsilon,\iota))$. If $N(\epsilon,\iota)=1$ then the answer to the question Q' is positive for every couple of totally real embeddings such that $j_0, j_1 \in \epsilon \cap \iota$. Let $I^0_{TR} = \{j \in I_{TR}$: there exists a path inside I joining j to an ordinary embedding $\}$. If $\iota \notin \pi_0(I^0_{TR})$ then $E_{TR}(\epsilon,\iota) = \emptyset$ but the converse is still true. Let $\iota \in \pi_0(I^0_{TR})$, then for every ϵ in $\pi_0(E)$, one has $E_{TR}(\epsilon,\iota) \neq \emptyset$ i. e. $N(\epsilon,\iota) \geq 1$. This is a consequence of a theorem of M. Gromov ([9], p. 192). Using the h-principle for totally real embeddings and a combinatoric argument we obtain an upper bound for $N(\epsilon,\iota)$.

Theorem 1. – *Let M^n be a closed, connected, n-dimensional manifold admitting a totally real embedding in j into W^{2n}. Let $\epsilon = [j] \in \pi_0(E)$ and $\iota = [j] \in \pi_0(I^0_{TR})$. If $|\pi_1(I,j)|$ is finite, then : $N(\epsilon,\iota) \leq |\pi_1(I,j)|$.*

When $W^{2n} = \mathbb{C}^n$, we can improve theorem 1 and obtain the answer to the question Q' at least for some dimensions.

Theorem 2. – *Let M^n $(n \geq 3)$ be a closed, connected, n-dimensional manifold admitting a totally real embedding j into $\mathbb{C}^n$. Let $\epsilon = [j] \in \pi_0(E)$ and $\iota = [j] \in \pi_0(I^0_{TR})$, then*

n	M^n *orientable*	M^n *non orientable*
$n \equiv 0\ [2]$	$N(\epsilon,\iota)$ *is finite under condition* C_1	$N(\epsilon,\iota) \leq 2$
$n \equiv 1\ [4]$	$N(\epsilon,\iota) \leq 2$ $N(\epsilon,\iota) = 1$ *under condition* C_2	$N(\epsilon,\iota) \leq 4$
$n \equiv 3\ [4]$	$N(\epsilon,\iota) = 1$	$N(\epsilon,\iota) \leq 2$

where the C_i are the following conditions :

(C_1) $H_1(M^n,\mathbb{Z})$ is finite,

(C_2) Coker $Sq^2 \circ \rho_2 = \{0\}$ where $\rho_2 : H^{n-2}(M^n,\mathbb{Z}) \to H^{n-2}(M^n,\mathbb{Z}_2)$ is the reduction modulo 2 and $Sq^2 : H^{n-2}(M^n,\mathbb{Z}_2) \longrightarrow H^{2n}(M^n,\mathbb{Z}_2)$ the Steenrod squaring operation.

One deduces :

Corollary 2.1 – *Let M^n $(n \geq 3)$ be a closed, connected, orientable manifold admitting a totally real embedding j into $\mathbb{C}^n$.*

1) If $n \equiv 3\ [4]$,

2) or if $n \equiv 1\ [4]$ and M^n satisfies the condition C_2,

then :

$$\pi_0(E_{TR}) \simeq \pi_0(E) \times \pi_0(I^0_{TR})$$

and the bijection is given by the induced map of the diagonal map $E_{TR} \longrightarrow E \times I^0_{TR}$.

When $W^{2n} = \mathbb{C}^n$ the space I^0_{TR} has a very simple interpretation in terms of the algebraic self-intersection number $\mathcal{I}_a$.

1) If n is odd, there is a natural splitting of I_{TR} into $I^0_{TR} \cup I^1_{TR}$ given by $\mathcal{I}_a \in \mathbb{Z}_2$. Namely, a totally real immersion is in I^0_{TR} (resp. in I^1_{TR}) if $\mathcal{I}_a = 0$ (resp. $\mathcal{I}_a = 1$). M. Audin proves that $I_{TR} = I^0_{TR}$ if dim $M^n \neq 2^q - 1$ ([3], th. 0.1) or if M^n is

parallelizable and $n \neq 1, 3$ ([3], prop. 0.3). In an other hand, she shows that, for every q, there exists a manifold M^{2^q-1} (admitting a totally real embedding into $\mathbb{C}^n$) such that $I^1_{TR} \neq \emptyset$.

2) If n is even then $I^0_{TR} = I_{TR}$ since the algebraic self-intersection number must be equal (up to the sign) to half the Euler-Poincaré caracteristic $\chi(M^n)$ and this latter is zero because the tangent space and the normal space are isomorphic via J.

Thus in many case $I^0_{TR} = I_{TR}$ and according to [9],[19] (see also [7]) :

$$\pi_0(I_{TR}) = [M^n, U(n)] = \tilde{K}^{-1}(M^n),$$

where $\tilde{K}^{-1}(M^n)$ denote the reduced K-theory group of the suspension of M^n. Let us do, as examples, some explicite computations of $\pi_0(E_{TR})$.

The sphere $\mathbb{S}^3$. – In [10] M. Gromov shows that $\mathbb{S}^n$ admits a totally real embedding into $\mathbb{C}^n$ if and only if $n = 1$ or 3. An explicit example of a totally real embedding of $\mathbb{S}^3$ into $\mathbb{C}^3$ is built in [1]. As we shall see, there is in fact many "different" totally real embeddings of $\mathbb{S}^3$ into $\mathbb{C}^3$. Let $V_{6,3}$ be the Stiefel manifold of 3-frames of $\mathbb{R}^6$. According to [16] and [10], one has :

$$\begin{array}{lllllll} \pi_0(I_{TR}) & \simeq & [\mathbb{S}^3, U(3)] & \simeq & \pi_3(U(3)) & \simeq & \mathbb{Z} \\ \pi_0(I) & \simeq & [\mathbb{S}^3, V_{6,3}] & \simeq & \pi_3(V_{6,3}) & \simeq & \mathbb{Z}_2. \end{array}$$

With consistent identifications, the map $\pi_0(I_{TR}) \longrightarrow \pi_0(I)$ induced by the inclusion $I_{TR} \subset I$ is the same as the map $l_\sharp : \mathbb{Z} \simeq \pi_3(U(3)) \longrightarrow \pi_3(V_{6,3}) \simeq \mathbb{Z}_2$ induced by the inclusion $l : U(3) \to V_{6,3}$. This last map is the reduction modulo 2 (see the proof of theorem 2, $n = 3$). It is then clear that : $\pi_0(I_{TR}) \simeq \mathit{ker}\ l_\sharp \simeq 2\mathbb{Z}$. According to [13], one has $\pi_0(E(\mathbb{S}^3, \mathbb{C}^3)) \simeq \mathbb{Z}$, so : $\pi_0(E_{TR}(\mathbb{S}^3, \mathbb{C}^3)) \simeq \mathbb{Z} \oplus 2\mathbb{Z}$.

The projective spaces $\mathbb{R}P^3$ and $\mathbb{R}P^7$. – The projective spaces $\mathbb{R}P^n$ that admit a totally real embedding into $\mathbb{C}^n$ are $\mathbb{R}P^1$, $\mathbb{R}P^3$ and $\mathbb{R}P^7$ (see. [3]). One can find an explicit totally real embedding of $\mathbb{R}P^3$ into $\mathbb{C}^3$ in [8]. Similarly to the case of $\mathbb{S}^3$ we get : $\pi_0(E_{TR}(\mathbb{R}P^3, \mathbb{C}^3)) \simeq \pi_0(E(\mathbb{R}P^3, \mathbb{C}^3)) \oplus 2\mathbb{Z}$. For $\mathbb{R}P^7$, $\pi_0(E(\mathbb{R}P^7, \mathbb{C}^7)) \simeq H_1(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}_2$ and $I^0_{TR} = I_{TR} = \tilde{K}^{-1}(\mathbb{R}P^7) = \mathbb{Z}$ (see [2]) so : $\pi_0(E_{TR}(\mathbb{R}P^7, \mathbb{C}^7)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}$.

The product of spheres $\mathbb{S}^3 \times \mathbb{S}^{n-3}$ ($n = 3 + 4k, k \geq 1$). – It was first shown in [21] that $M^n = \mathbb{S}^3 \times \mathbb{S}^{n-3}$ admits a totally real emdedding into $\mathbb{C}^n$. This manifold is parallelizable (thus $I^0_{TR} = I_{TR}$) and simply connected (thus $\pi_0(E) = \{0\}$). Therefore : $\pi_0(E_{TR}) = \pi_0(I^0_{TR}) = \pi_0(I_{TR}) = \tilde{K}^{-1}(M^n)$ i. e. $\pi_0(E_{TR}(\mathbb{S}^3 \times \mathbb{S}^{n-3}, \mathbb{C}^n)) = \{0\} \oplus \mathbb{Z}^2$.

The torus $\mathbb{T}^n$ ($n = 3 + 4k$, $k \geq 0$). – If $k \geq 1$, $I^0_{TR} = I_{TR}$ and we immediatly get : $\pi_0(E_{TR}(\mathbb{T}^n, \mathbb{C}^n)) = \mathbb{Z}^n \oplus \mathbb{Z}^{2^{n-1}}$. If $k = 0$, $\pi_0(I_{TR}) \simeq [\mathbb{T}^3, U(3)] \simeq [\mathbb{T}^3, \mathbb{S}^1] \times [\mathbb{T}^3, SU(3)] \simeq \mathbb{Z}^3 \times \mathbb{Z}$. With consistent identifications : $\pi_0(E_{TR}(\mathbb{T}^3, \mathbb{C}^3)) \simeq \pi_0(E(\mathbb{T}^3, \mathbb{C}^3)) \oplus (\mathbb{Z}^3 \times 2\mathbb{Z})$.

The cardinal $N(\epsilon, \iota)$ can be strictly greater than 1. When $n = 2$, such phenomenon was first noticed by T. Fiedler [6] and by L. Polterovich [20].

Theorem 3 . – *Let $M^n = \mathbb{T}^n$ with $n \geq 4$ even and let ι, ϵ be the components of I^0_{TR} and E containing the Lagrangian natural inclusion $\mathbb{T}^n \subset \mathbb{C}^n$. Then $N(\epsilon, \iota)$ is not finite.*

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2. Reduction to a problem of homotopy theory

2.1. **A theorem from Convex Integration Theory.** If E is a bundle, we denote by $\Gamma(E)$ the space of sections of E. Let X be the product $M^n \times W^{2n}$, $X^{(1)}$ be the 1-jet space of the bundle $X \to M^n$ and $\mathcal{R} \subset X^{(1)}$ be the total reality condition.

Theorem. – *Let $f_t : M^n \to W^{2n}$ be an isotopy of embeddings joining two totally real embeddings f_0 and f_1, and let $H_{t,s} : TM^n \to TW^{2n}$ be a homotopy of bundle homomorphisms such that :*

$$\begin{array}{ll} H_{t,0} = df_t & \forall\, t \in [0,1] \\ H_{0,s} = df_0 & \forall\, s \in [0,1] \\ H_{1,s} = df_1 & \forall\, s \in [0,1] \\ H_{t,1} \in \Gamma(\mathcal{R}) & \forall\, t \in [0,1]. \end{array}$$

Then there exists an isotopy of embeddings $F_{t,s}$ such that :

$$\begin{array}{ll} F_{t,0} = f_t & \forall\, t \in [0,1] \\ F_{0,s} = f_0 & \forall\, s \in [0,1] \\ F_{1,s} = f_1 & \forall\, s \in [0,1] \end{array}$$

and $F_{t,1}$ is a totally real isotopy.

This theorem is a parametric version of a theorem of M. Gromov ([9], p. 192, see also [22], p. 164). The transition from a h-principle to a parametric h-principle is classical in convex integration theory. (See the remark p. 17 of [9] or [22] p. 139-144).

2.2. **Sketch of the proofs.** Let j_0 and j_1 be two totally real embeddings of $E_{TR}(\epsilon, \iota)$, f_t an (ordinary) isotopy joining j_0 and j_1 and g_t a regular totally real homotopy joining j_0 and j_1. One can build with f_t and g_t a loop $\gamma(f,g)$ in I by putting :

$$\gamma(f,g)(t) = \begin{cases} f_{2t} & \text{if } t \leq 1/2, \\ g_{2(1-t)} & \text{if } t \geq 1/2. \end{cases}$$

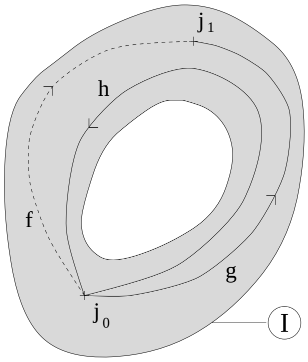
According to the theorem of the preceding section, if $[\gamma(f,g)]$ is zero in $\pi_1(I, j_0)$ then there exists a totally real isotopy j_t joining j_0 and j_1. This is trivially the case when $\pi_1(I, j_0) = 0$.

Suppose now that $[\gamma(f,g)]$ is not zero but there exists $[h] \in \pi_1(I_{TR}, j_0)$ such that $l_\sharp([h]) = [\gamma(f,g)]^{-1}$ ($l_\sharp$ is the homomorphism induced at the π_1 level by the inclusion

$I_{TR} \subset I$). Replacing g by $\tilde{g}$:

$$\tilde{g}(t) = \begin{cases} h_{2t} & \text{if } t \leq 1/2, \\ g_{2t-1} & \text{if } t \geq 1/2, \end{cases}$$

one obtains a new element $[\gamma(f, \tilde{g})]$ which is null-homotopic.



Similarly, one can perform the same trick if there exists $[h] \in \pi_1(E, j_0)$ such that $i_\sharp([h]) = [\gamma(f, g)]$ (this time $i_\sharp$ is the homomorphism induced at the π_1 level by the inclusion $E \subset I$). This technique leads to theorems 1 and 2.

2.3. **Proof of theorem 1.** Assume that $|\pi_1(I, j)| = n$ and that $N(\epsilon, \iota) > n$. Let $x_0 = [j], x_1[j_1], ..., x_n = [j_n]$ be $n + 1$ distinct elements of $\pi_0(E_{TR}(\epsilon, \iota))$. For each couple (x_i, x_k) with $i < k$, there is a loop $\gamma_{ik} = \gamma(f_{ik}, g_{ik})$ where $f^t_{ik} \in \epsilon$ is an isotopy joining j_i and j_k and $g^t_{ik} \in \iota$ is a regular totally real homotopy joining j_i and j_k (by convention, $j_0 = j$). This loop determines an element $[\gamma_{ik}]$ in $\pi_1(I, j)$. If $i > k$, one defines γ_{ki} to be $\gamma(g_{ik}, f_{ik})$ and thus $[\gamma_{ki}] = -[\gamma_{ik}]$. By the Convex Integration Theorem, since j_i and j_k belong to two different arcwise connected components of $E_{TR}(\epsilon, \iota)$, the element $[\gamma_{ik}]$ is not zero in $\pi_1(I, j)$. We encode all these data in a complete directed $(n + 1)$-graph K_{n+1} with vertex set $V(K_{n+1}) = \{x_0, ..., x_n\}$. In this way, the elements $[\gamma_{ij}]$ appear as the values of a weight function f defined on the set of directed edges $E(K_{n+1})$:

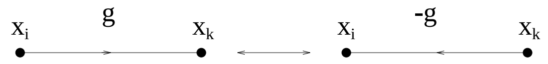
$$\begin{array}{cccc} f: & E(K_{n+1}) & \longrightarrow & G^* = \pi_1(I, j_0) \setminus \{0\} \\ & x_i x_k & \longmapsto & [\gamma_{ik}], \end{array}$$

and satisfying the property (P) $\forall i \neq k, \ : \ f(x_i x_k) = -f(x_k x_i)$. We now need a purely combinatorial result.

Lemma A. – *Let G be a group of order n, K_{n+1} be the complete directed $(n+1)$-graph and $f : E(K_{n+1}) \longrightarrow G^* = G \backslash \{0\}$ be a weight function satisfying the property (P). Then there exists a triple (x_i, x_j, x_k) of distinct vertices of K_{n+1} such that : $f(x_i x_j) + f(x_j x_k) = 0$.*

We postpone the proof of this lemma. Let (x_i, x_j, x_k) be a triple satisfying $f(x_i x_j) + f(x_j x_k) = [\gamma_{ij}] + [\gamma_{jk}] = 0$ with loops $\gamma_{ij} = \gamma(f_{ij}, g_{ij})$ and $\gamma_{jk} = \gamma(f_{jk}, g_{jk})$. Then the loop $\gamma(f_{ij} + f_{jk}, g_{ij} + g_{jk})$ is contractible and thus by the Convex Integration Theorem, j_i and j_k belong to the same arcwise connected component of $E_{TR}(\epsilon, \iota)$. This is a contradiction. $\square$

Proof of the lemma A. – One can visualize the situation with a tournament (an oriented K_{n+1}) provided with a weight function (see the figure above). Nevertheless, this representation is not unique. Indeed, the property (P) allows us to make the following change in the graph :

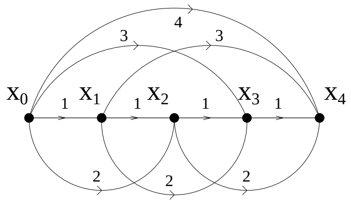


By applying several changes of this kind, one can assume that the tournament representing all the data is in fact a total order on the set $V(K_{n+1})$.
Let x be the minimal element. We can assume $x = x_0$. Since $|G^*| = n - 1$, the n-tuple $(f(x_0x_1), ..., f(x_0x_n))$ contains at least two elements $f(x_0x_p)$ and $f(x_0x_q)$ which are identical. Then (x_p, x_0, x_q) is the desired triple. $\square$

Remark. – In the preceding result, the number of vertices on the complete graph cannot be reduced even if one replaces the conclusion of the lemma A by the following weaker conclusion :

There exists a m-tuple $(x_{i_1}, ..., x_{i_m})$ such that :
1) the loop $(x_{i_1}, ..., x_{i_m})$ contains no cycle,
2) $f(x_{i_1}x_{i_2}) + ... + f(x_{i_{m-1}}x_{i_m}) = 0$.

that also suits to obtain a contradiction. Here is an example in which $G = \mathbb{Z}_5$, $f(x_ix_{i+k}) = k$ and the complete graph K_5 does not satisfy the conditions 1 and 2.



Let i be the natural inclusion $E \subset I$ and $i_\sharp$ the induced map at the π_1-level. We shall need the following result to prove theorem 2.

Lemma B. – *If $\pi_1(I, j)/i_\sharp(\pi_1(E, j))$ has a finite number of elements then $N(\epsilon, \iota) \leq |\pi_1(I, j)/i_\sharp(\pi_1(E, j))|$.*

The proof of lemma B is similar to one of lemma A. Note that $\pi_1(I, j_0)$ is Abelian and so the quotient makes sense. We shall also need the same kind of result for the natural inclusion $l : I_{TR} \to I$.

Lemma B'. – *If $\pi_1(I, j)/l_\sharp(\pi_1(I_{TR}, j))$ has a finite number of elements then $N(\epsilon, \iota) \leq |\pi_1(I, j)/l_\sharp(\pi_1(I_{TR}, j))|$.*

3. Proof of theorem 2 for $n \geq 4$

We first recall the results of A. Haefliger and M. Hirsch [11],[12],[14],[15],[16], on the classification of embeddings and immersions of a n-dimensional closed connected manifold M^n into $\mathbb{R}^m$. As we are only interested in the n-codimensional case, we take $m = 2n$.

Let SM be the tangent sphere bundle of M^n. Every immersion $f : M^n \longrightarrow \mathbb{R}^{2n}$ gives rise to a $\mathbb{Z}_2$-equivariant map :

$$\begin{array}{rccc} D(f): & SM & \longrightarrow & \mathbb{S}^{2n-1} \\ & U_x & \longmapsto & \frac{df_x(U_x)}{\|df_x(U_x)\|}. \end{array}$$

If $n \geq 2$, there is a one-to-one correspondence between the set of regular homotopy classes of immersions $\pi_0(I(M^n, \mathbb{R}^{2n}))$ and the set of $\mathbb{Z}_2$-equivariant homotopy classes of $\mathbb{Z}_2$-equivariant maps of SM into $\mathbb{S}^{2n-1}$. Let $PM = SM/\mathbb{Z}_2$ be the tangent projective bundle. There is a natural bijection between $\mathbb{Z}_2$-equivariant maps of SM into $\mathbb{S}^{2n-1}$ and the sections of the bundle $SM \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1} \longrightarrow PM$. Let $[PM, SM \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1}]$ denote the set of homotopy classes of the sections of this bundle. As the fiber $\mathbb{S}^{2n-1}$ is $(2n-2)$-connected, one has :

$$\pi_0(I(M^n, \mathbb{R}^{2n})) \simeq [PM, SM \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1}] \simeq H^{2n-1}(PM, \mathbb{Z}).$$

This last cohomology group is isomorphic to $\mathbb{Z}$ if n is even and to $\mathbb{Z}_2$ if n is odd.

Let ΔM be the diagonal of $M^n \times M^n$. Every embedding $f : M^n \to \mathbb{R}^{2n}$ gives rise to a $\mathbb{Z}_2$-equivariant map :

$$\begin{array}{ccc} M^n \times M^n - \Delta M & \longrightarrow & \mathbb{S}^{2n-1} \\ (x,y) & \longmapsto & \frac{f(x)-f(y)}{\|f(x)-f(y)\|}. \end{array}$$

If $n \geq 4$, there is a one-to-one correspondence between the set of isotopy classes of embeddings $\pi_0(E(M^n, \mathbb{R}^{2n}))$ and the set of $\mathbb{Z}_2$-equivariant homotopy classes of $\mathbb{Z}_2$-equivariant maps of $M^n \times M^n - \Delta M$ into $\mathbb{S}^{2n-1}$. Let $M^* = (M^n \times M^n - \Delta M)/\mathbb{Z}_2$ be the reduced symmetric product of M^n. There is a natural bijection between $\mathbb{Z}_2$-equivariant maps of $M^n \times M^n - \Delta M$ into $\mathbb{S}^{2n-1}$ and the sections of the bundle $(M^n \times M^n - \Delta M) \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1} \longrightarrow M^*$. The fiber is still $\mathbb{S}^{2n-1}$ and one has :

$$\pi_0(E(M^n, \mathbb{R}^{2n})) \simeq [M^*, P] \simeq H^{2n-1}(M^*, \mathbb{Z}),$$

where we have written P for the total space $(M^n \times M^n - \Delta M) \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1}$. The cohomology group $H^{2n-1}(M^*, \mathbb{Z})$ is isomorphic to $H^{n-1}(M^n, \mathbb{Z}_2)$ if M^n is orientable and n is even or M^n is non orientable and n is odd. It is isomorphic to $H^{n-1}(M^n, \mathbb{Z})$ if M^n is orientable and n is odd and to $\mathbb{Z} \oplus ker\ Sq^1$ if M^n is non orientable and n is even ($Sq^1 : H^{n-1}(M^n, \mathbb{Z}_2) \longrightarrow H^n(M^n, \mathbb{Z}_2)$).

The tangent sphere bundle SM is diffeomorphic to the boundary of a tubular neighborhood of the diagonal ΔM in $M^n \times M^n$. In this way, the projective tangent bundle PM can be seen as siting inside M^*. An immersion $f : M^n \longrightarrow \mathbb{R}^{2n}$ ($n \geq 3$) is regularly homotopic to an embedding if and only if the section $\tilde{f}$ of $SM \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1} \longrightarrow PM$ induced by $D(f)$ can be extended all over M^*. Assume that f is an embedding. If $n \geq 4$, there is a one-to-one correspondence between $\pi_1(I(M^n, \mathbb{R}^{2n}), E(M^n, \mathbb{R}^{2n}), f)$ and the set $[M^*, P; \tilde{f}]$ of rel. PM homotopy classes of extensions of $\tilde{f}$. Thus, one has :

$$\pi_1(I(M^n,\mathbb{R}^{2n}),E(M^n,\mathbb{R}^{2n}),f) \simeq [M^*,P;\tilde{f}].$$

Since $H^{2n}(M^*,PM,\mathbb{Z}_2)=\mathbb{Z}_2$ is non zero, the computation of $[M^*,P;\tilde{f}]$ is more complicated. Let η be the line bundle over M^* associated with $M\times M-\Delta M \longrightarrow M^*$ and $\xi=\eta\oplus ...\oplus\eta=2n\eta$. There is a filtration

$$[M^*,P;\tilde{f}]=F_0\supset F_1\supset F_2=0$$

with

1) $F_0/F_1=H^{2n-1}(M^*,PM,\mathbb{Z}[w_1(\xi)])$,
2) $F_1=$ Coker$(\Theta: H^{2n-2}(M^*,PM,\mathbb{Z}[w_1(\xi)])\longrightarrow H^{2n}(M^*,PM,\mathbb{Z}_2))$.

The map Θ is the composition $\alpha\circ\rho_2$ where the map ρ_2 is the reduction modulo 2 : $H^{2n-2}(M^*,PM,\mathbb{Z}[w_1(\xi)])\to H^{2n}(M^*,PM,\mathbb{Z}_2)$ and the map $\alpha: H^{2n}(M^*,PM,\mathbb{Z}_2)\to H^{2n}(M^*,PM,\mathbb{Z}_2)$ is defined by $\alpha(x)=Sq^2x+Sq^1(x)\cup w_1(\xi)+x\cup w_1(\xi)^2+x\cup w_2(\xi)$ (see [18] for example). Let $v=w_1(\eta)$, note that $w_1(\xi)=\binom{2n}{1}v=0$ and $w_2(\xi)=\binom{2n}{2}v^2$. Thus in our case, the cohomology is untwisted and $\Theta(x)=Sq^2\rho_2(x)+nv^2\rho_2(x)$.

Lemma C.– *Let $n\geq 4$. The cohomology group $H^{2n-1}(M^*,PM,\mathbb{Z})$ is isomorphic to :*

$$\begin{cases} H^{n-1}(M^n,\mathbb{Z})\oplus\mathbb{Z}_2 & \textit{if } M^n \textit{ is orientable and } n \textit{ is even,}\\ H^{n-1}(M^n,\mathbb{Z}_2) & \textit{if } M^n \textit{ is non orientable and } n \textit{ is even,}\\ H^{n-1}(M^n,\mathbb{Z}) & \textit{if } M^n \textit{ is orientable and } n \textit{ is odd,}\\ \textit{ker } Sq^1\oplus\mathbb{Z}_4 & \textit{if } M^n \textit{ is non orientable and } n \textit{ is odd.}\end{cases}$$

Proof of lemma C.– The proof of this lemma is a computation using [17] and [23] (see the appendix). □

Lemma D.– *Let $n\geq 4$; Coker Θ is $\{0\}$ in the following cases :*
1) $n\equiv 3$ mod 4,
2) n is even and M^n is non orientable,
3) M^n is orientable and Coker $Sq^2\circ\rho_2=\{0\}$ where ρ_2 is the reduction modulo 2 and $Sq^2\circ\rho_2: H^{n-2}(M^n,\mathbb{Z})\longrightarrow H^{2n}(M^n,\mathbb{Z}_2)$.

Proof of lemma D.– We suppose that the reader is familiar with the notations and results of [17] and [23] (if not, see the appendix).

1) $n\equiv 3$ mod 4. Let $\beta_2: H^{2n-3}(M^*,PM,\mathbb{Z}_2)\longrightarrow H^{2n-2}(M^*,PM,\mathbb{Z})$ be the Bockstein operator. Then :

$$\begin{aligned}\Theta(\beta_2(v^{n-3}\Lambda\rho_2 M)) &= (Sq^2+v^2)\rho_2\beta_2(v^{n-3}\Lambda\rho_2 M)\\ &= (Sq^2+v^2)Sq^1(v^{n-3}\Lambda\rho_2 M)\\ &= (Sq^2+v^2)(v^{n-2}\Lambda\rho_2 M)\\ &= v^n\Lambda\rho_2 M+Sq^2Sq^{n-2}(\Lambda\rho_2 M)\\ &= v^n\Lambda\rho_2 M+Sq^{n-1}Sq^1\Lambda\rho_2 M\\ &= v^n\Lambda\rho_2 M\neq 0.\end{aligned}$$

Therefore, Coker $\Theta=\{0\}$.

2) n is even and M^n is non orientable. Let $M' \in H^{n-1}(M^n, \mathbb{Z}_2)$ such that $Sq^1 M' = \rho_2 M$. One has : $Sq^1(\Lambda M' \Lambda M') = 0$. The Bockstein sequence

$$H^{2n-2}(X, \mathbb{Z}_2) \xrightarrow{\beta_2} H^{2n-1}(X, \mathbb{Z}) \xrightarrow{\times 2} H^{2n-1}(X, \mathbb{Z}) \xrightarrow{\rho_2} H^{2n-1}(X, \mathbb{Z}_2)$$

for $X = (\Lambda^2 M, \Delta M)$ shows that ρ_2 is a monomorphism since $H^{2n-1}(X, \mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}_2)$ by lemma C. Thus :

$$Sq^1(\Lambda M' \Lambda M') = \rho_2 \beta_2 (\Lambda M' \Lambda M') = 0 \Longrightarrow \beta_2(\Lambda M' \Lambda M') = 0$$

i.e. $\Lambda M' \Lambda M' \in Ker\ \beta_2 = Im\ (\rho_2 : H^{2n-2}(X, \mathbb{Z}) \longrightarrow H^{2n-2}(X, \mathbb{Z}_2))$. On the other hand,

$$\alpha(\Lambda M' \Lambda M') = Sq^2(\Lambda M' \Lambda M') = \Lambda Sq^1 M' \Lambda Sq^1 M' = \Lambda \rho_2 M \Lambda \rho_2 M \neq 0$$

and so Coker $\Theta = \{0\}$.

3) M^n is orientable and Coker $Sq^2 \circ \rho_2 = \{0\}$. By the assumption, there exists $x \in H^{n-2}(M, \mathbb{Z})$ such that $Sq^2 \rho_2(x) = M$. Note that, since Λx and ΛM belong to the twisted cohomology groups $H^{n-2}(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ and $H^n(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$, the product $\Lambda x \Lambda M$ is inside $H^{2n-2}(\Lambda^2 M, \Delta M, \mathbb{Z})$. We have :

$$\begin{aligned} \Theta(\Lambda x \Lambda M) &= (Sq^2 + nv^2)\rho_2(\Lambda x \Lambda M) = (Sq^2 + nv^2)(\Lambda \rho_2 x \Lambda \rho_2 M) \\ &= \Lambda Sq^2 \rho_2 x \Lambda \rho_2 M \neq 0. \end{aligned}$$

and so Coker $\Theta = \{0\}$. $\square$

We are now ready to prove theorem 2 for $n \geq 4$. Let $j : M^n \to \mathbb{R}^{2n}$ be a totally real embedding and let $E \xrightarrow{i} I \xrightarrow{h} (I, E)$ be the natural inclusions. Our main tool here is the lemma B, so we have to compute the order of $\pi_1(I, j)/i_\sharp(\pi_1(E, j))$. The study of the long exact homotopy sequence of the pair (I, E) :

$$\ldots \longrightarrow \pi_1(E, j) \xrightarrow{i_\sharp} \pi_1(I, j) \xrightarrow{h_\sharp} \pi_1(I, E, j) \xrightarrow{\partial_\sharp} \pi_0(E) \xrightarrow{J} \pi_0(I).$$

leads to an upper bound of this number. Indeed, $\pi_0(E)$, $\pi_0(I)$ and $\pi_1(I, E, j)$ are known and the arrow $J : \pi_0(E) \longrightarrow \pi_0(I)$ is always the zero map except for the case n even and M^n non orientable. In this case $J : \mathbb{Z} \oplus ker\ Sq^1 \longrightarrow \mathbb{Z}$ is given by $(a, b) \longmapsto 2a$ (see [23]). It is then straightforward to obtain the results of the array in theorem 2.

4. Proof of theorem 2 for $n = 3$

Let $j : M^3 \longrightarrow \mathbb{C}^3$ be a totally real embedding. By lemma B', it suffices to show that $\pi_1(I, j)/l_\sharp(\pi_1(I_{TR}, j))$ has only one element if M^3 is orientable and at most two elements if M^3 is not orientable.

We first briefly recall the theory of the classification of totally real immersions. Let $V_{2n,n}(M^n)$ be the associated bundle to the frame bundle of M^n whose fiber is the Stiefel manifold $V_{2n,n}$ of n-frames of $\mathbb{R}^{2n}$ with $Gl(n, \mathbb{R})$ acting in the natural way. Any bundle monomorphism gives rise to a section of $V_{2n,n}(M^n)$ and there is a weak homotopy equivalence between the space $I(M^n, \mathbb{C}^n)$ and the space $\Gamma(V_{2n,n}(M^n))$ of the sections of $V_{2n,n}(M^n)$. Let $V_{TR} \subset V_{2n,n}$ be the submanifold of n-frames of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ such that the n-planes they span are totally real. We denote by $V_{TR}(M^n)$ the obvious subbundle of $V_{2n,n}(M^n)$ with fiber V_{TR}. (Since M^n admits a totally real embedding into $\mathbb{C}^n$, the bundle $V_{TR}(M^n)$ is trivial). According to M. Gromov and J. A. Lees [9],[19], there is a weak homotopy equivalence between the spaces I_{TR} and $\Gamma(V_{TR}(M^n))$ and one has the following commutative diagram :

$$\begin{array}{ccc} I_{TR} & \xrightarrow{l} & I \\ df\downarrow & & \downarrow df \\ \Gamma(V_{TR}(M^n)) & \xrightarrow{l} & \Gamma(V_{2n,n}(M^n)). \end{array}$$

Let $p : M^3 \times \mathbb{S}^1 \to \mathbb{S}^1$ be the projection, we denote by $V_{6,3}(M^3 \times \mathbb{S}^1)$ (resp. $V_{TR}(M^3\times\mathbb{S}^1)$) the bundle $p^*V_{6,3}(M^3)$ (resp. $p^*V_{TR}(M^3)$) A totally real embedding $j : M^3 \longrightarrow \mathbb{C}^3$ gives rise to a section dj of $V_{TR}(M^3) = V_{TR}(M^3\times\{0\})$ and a section $\overline{dj} = l\circ dj$ of $V_{6,3}(M^3\times\{0\})$. We have :

$$\begin{array}{rcl} \pi_1(I,j) & = & [M^3\times\mathbb{S}^1, V_{6,3}(M^3\times\mathbb{S}^1); \overline{dj}] \\ \pi_1(I_{TR},j) & = & [M^3\times\mathbb{S}^1, V_{TR}(M^3\times\mathbb{S}^1); dj], \end{array}$$

where $[X,Y,f]$ denote the set of homotopy classes of extensions over X of a partial section $f : A \to Y$. Let u (resp. v) be the trivial extension over $M^3\times\mathbb{S}^1$ of dj (resp. $\overline{dj}$) given by $u(x,\theta) = dj(x)$ (resp. $v(x,\theta) = \overline{dj}(x)$) for every $(x,\theta)\in M^3\times\mathbb{S}^1$.

We now adapt the General Classification Theorem for sections to our case (see [4] p. 276). For short we put $X = M^3\times\mathbb{S}^1$ and $A = M^3\times\{0\}$ and we denote by E (resp. E') the total space of the bundle $V_{6,3}(M^3\times\mathbb{S}^1)$ (resp. $V_{TR}(M^3\times\mathbb{S}^1)$). We still write $l : E' \to E$ for the natural inclusion. Let $X^0 \subset X^1 \subset ... \subset X^4$ be the successive skeletons of X, $i_k : X_k = X^k \cup A \to X$ be the natural inclusion and E_k be the bundle i_k^*E. The restriction map $X_n \longrightarrow X_{n-1}$ induces an arrow $j_n : [X_n, E_n; v] \longrightarrow [X_{n-1}, E_{n-1}; v]$, (here, of course, $[X,E;v]$ denotes the set of homotopy classes of extensions of $v_{|A}$ over X). We get a filtration :

$$(*) \qquad [X,E;v] \xrightarrow{j_4} [X_3,E_3;v] \xrightarrow{j_3} ... \xrightarrow{j_1} [X_0,E_0;v] = \{0\}.$$

Put $v_n = v_{|X_n}$ and $J_n(X,A,v) = j_n^{-1}([v_{n-1}])$. If $n\geq 2$, the set $J_n(X,A,v)$ has an Abelian group structure with $[v_n]$ as neutral element. There exists a spectral sequence $(E_r^{**}(X,A,v), d_r(v)) = (E_r^{**}, d_r)$ such that :

1) d_r has bidegree $(r,-1)$,
2) $E_r^{p,q} = 0$ for $p<1$ and $q<0$,
3) $E_2^{p,q} = H^p(X_{p+q}, A, \{\pi_{p+q}(F)\})$ for $p+q\geq 2$ and $p\geq 1$,
4) $E_n^{n,0} = J_n(X,A,v)$.

This spectral sequence is natural, that is, in the following situation

$$\begin{array}{ccc} E' & \xrightarrow{l} & E \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

the map l induces a homomorphism (l^r) between the spectral sequences :

$$E_r^{'**}(X,A,u) \xrightarrow{l^r} E_r^{**}(X,A,v = l\circ u).$$

Moreover, for $r=2$, this homomorphism is induced by the coefficient homomorphism $\pi_i(F') \longrightarrow \pi_i(F)$.

In our case, the fiber of E is $V_{6,3}$ which is a 2-connected space with $\pi_3(V_{6,3}) = \mathbb{Z}_2$ and $\pi_4(V_{6,3}) = 0$, thus the filtration $(*)$ reduces to :

$$[X,E;v] \overset{id}{\simeq} [X_3,E_3;v] \xrightarrow{j_3} \{0\},$$

and therefore : $[X,E;v] = J_3(X,A,v) = E_3^{3,0} = E_2^{3,0} = H^3(X,A;\mathbb{Z}_2)$.

The bundle E' is trivial since M^3 admit a totally real immersion and its fiber is $Gl(3,\mathbb{C})$ which is homotopy equivalent to $U(3)$. Since $\pi_4(U(3)) = 0$ and $\pi_3(U(3)) = \mathbb{Z}$ we have : $[X,E';u] \supset J_3'(X,A,u) = E_3^{'3,0} = E_2^{'3,0} = H^3(X,A;\mathbb{Z})$. We denote by $k : J_3'(X,A,u) \longrightarrow [X,E;v] = J_3(X,A,v)$ the restriction of $l_\sharp$ on $J_3'(X,A,u)$. By the naturality of the spectral sequence, this map is the induced map given by the coefficient homomorphism $h : \pi_3(U(3)) \longrightarrow \pi_3(V_{6,3})$.

Let us see what h is. The inclusion $U(3) \longrightarrow V_{6,3}$ is obtained via the composition $U(3) \to SO(6) \to SO(6)/SO(3)$. Since $\pi_3(SO(6)/SO(3)) = 0$, the arrow $\pi_3(U(3)) \to \pi_3(SO(6)) = \mathbb{Z}$ is an isomorphism and since $\pi_2(SO(3)) = 0$ the map $\pi_3(SO(6)) \to \pi_3(V_{6,3}) = \pi_3(SO(6)/SO(3)) = \mathbb{Z}_2$ is the modulo 2 reduction map. Thus the composition h is still the modulo 2 reduction map and so is k on cohomology.

By using the Universal Coefficient Theorem we get that k is onto if M^3 is orientable and $\pi_1(I,j)/Im\ k = \mathbb{Z}_2$ if not. $\square$

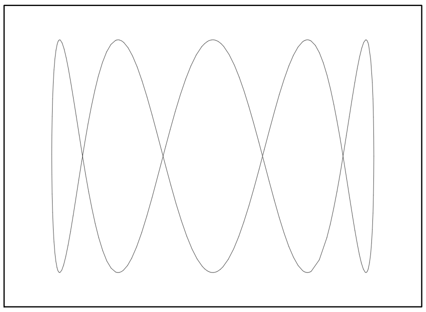
5. Some comments on $l_\sharp : \pi_1(I_{TR},j) \to \pi_1(I,j)$

The reader may wonder why we do not study the map $l_\sharp : \pi_1(I_{TR},j) \to \pi_1(I,j)$ induced by the inclusion $I_{TR} \subset I$ for $n \geq 4$. A first answer is that the author is unable to compute it completely. Nevertheless, for many important cases, the image of this map is very small. For instance if M^n is an even-dimensional parallelizable manifold, then $|Im\ l_\sharp| \leq 4$ and thus $l_\sharp$ will not give us a significant improvement of the theorem 2.

6. Proof of theorem 3.

We denote by $i_n : \mathbb{T}^n \subset \mathbb{C}^n$ the natural Lagrangian inclusion. By pertubing i_n we are going to build sufficiently many distinct totally real embeddings to prove theorem 3.

Let $N \geq 1$ be an odd integer and $\gamma_N : \mathbb{S}^1 \to \mathbb{C}$ be the curve $t \mapsto (\cos t, \sin Nt)$. This curve has $N-1$ double points and is regularly homotopic to the inclusion $i_1 : \mathbb{S}^1 \subset \mathbb{C}$. (See the figure below for $N = 5$).



We set :

$$\begin{array}{cccc} g_\eta : & \mathbb{T}^{n-1} & \longrightarrow & \mathbb{C}^{n-1} \\ & (t,\theta_1,...,\theta_{n-2}) & \longmapsto & (\gamma_N(t), h(t,\theta_1)e^{i\theta_1},...,h(t,\theta_{n-2})e^{i\theta_{n-2}}) \end{array}$$

where $h(t,\theta) = 1 + \eta \sin t \sin\theta$ and $\eta > 0$. We take η small enough so that g_η is a C^∞-close to the Lagrangian immersion $g_0 = \gamma_N \times i_1 \times ... \times i_1$ and thus g_η is a totally real immersion. Then $s \mapsto g_{s\eta}$ is a homotopy in $I^0_{TR}(\mathbb{T}^{n-1},\mathbb{C}^{n-1})$ joining g_η to g_0. The set of double points of g_η :

$$\{g_\eta(\pm\frac{k\pi}{N},0,...,0), g_\eta(\pm\frac{k\pi}{N},\pi,...,0),..., g_\eta(\pm\frac{k\pi}{N},\pi,...,\pi) \mid k = 1,..,N-1\}$$

has cardinality $P = 2^{n-2}(N-1)$. Moreover, all the double points are transversal. Finally, we define a totally real immersion f of $\mathbb{T}^n = \mathbb{T}^{n-1} \times \mathbb{S}^1$ into $\mathbb{C}^n$ by putting $f(x,\varphi) = (g_\eta(x), e^{i\varphi})$. From f, by applying the techniques of [5] §3.1 and §3.3, we get a family

$$\{j_p : \mathbb{T}^n \to \mathbb{C}^n\}_{p\in\{-P/2,...,P/2\}}$$

of totally real embeddings such that $< \sigma_{n-1}(j_p, v), [\mathbb{T}^{n-1}] >= 2p$, where $\sigma_{n-1} \in H^{n-1}(\mathbb{T}^n,\mathbb{Z})$ is the linking class of f associated to a non-vanishing vector field v tangent to the last factor $\mathbb{S}^1$. Since the linking class is invariant by totally real isotopy, we obtain that the j_p represent $P + 1$ distinct components of E_{TR}. By construction, all the j_p are obtained by a C^∞-small pertubation of f, thus they all belong to the same component $\iota = [f]$ of I^0_{TR}. But $f = g_\eta \times i_1$ is regularly homotopic among totally immersions to $g_0 \times i_1$, which is itself regularly homotopic among totally immersions to the standard Lagrangian inclusion $i_n = i_1 \times ... \times i_1$. Thus $\iota = [i_n]$. Since $\mathbb{T}^n$ is parallelizable and n even, theorem 3 of [5] states that two totally real embeddings of $\mathbb{T}^n$ into $\mathbb{C}^n$ that belong to the same component of I^0_{TR} belong to the same component of E if and only if the reduction modulo 2 of their linking classes is the same. Let us denote by E_q the $(n-1)$-cycle $[\mathbb{S}^1 \times ... \times \{0\} \times ... \times \mathbb{S}^1]$ where the $\{0\}$ is the q-th factor. We already have $< \sigma_{n-1}(j_p, v), E_n >= 2p$ and it is not difficult to check that :

$$< \sigma_{n-1}(j_p, v), E_q >= 0, \quad \text{for every} \quad 1 \le q \le n-1.$$

Thus $\rho_2\sigma_{n-1}(j_p, v) = 0$ and all the j_p belong to the same component $\epsilon \subset E$. Of course $\sigma_{n-1}(i_n, v) = 0$, hence $\epsilon = [i_n]$. So, given any odd integer $N \ge 1$ we have shown that : $N(\epsilon,\iota) \ge 2^{n-2}(N-1) + 1$. □

7. Appendix

We include this appendix for the sake of completness. All the results of subsections 7.1 and 7.2 are from [17], [18] and [23]. Section 7.3 is devoted to the proof of lemma C.

7.1. **Cohomology in $\mathbb{Z}_2$ of the pair** (M^*, PM)**.** Let $\Lambda^2 M = (M \times M)/\mathbb{Z}_2$ be the 2-fold symmetric product of M and N be a tubular neighborhood of ΔM in $\Lambda^2 M$. The natural inclusions $(M^*, PM) \subset (\Lambda^2 M, N)$ and $(\Lambda^2 M, \Delta M) \subset (\Lambda^2 M, N)$ induce isomorphisms on cohomology groups. To save space, we sometimes denote by X the pair $(\Lambda^2 M, \Delta M)$. Let $\pi : (M^2, \Delta M) \longrightarrow (\Lambda^2 M, \Delta M)$ be the double covering, v be the first Stiefel-Whitney class of this covering and

$$\to H^k(X, \mathbb{Z}_2) \xrightarrow{\pi^*} H^k(M^2, \Delta M, \mathbb{Z}_2) \xrightarrow{\theta} H^k(X, \mathbb{Z}_2) \xrightarrow{\cup v} H^{k+1}(X, \mathbb{Z}_2) \to$$

be the exact Thom-Gysin sequence of this covering. It is built in [17] a monomorphism $\Lambda : \tilde{H}^k(M, \mathbb{Z}_2) \longrightarrow H^k(\Lambda^2 M, \Delta M, \mathbb{Z}_2)$ and Λx satisfies : $\pi^* \Lambda x = x \otimes 1 + 1 \otimes x$ ($\tilde{H}^*$ denotes the reduced cohomology). Moreover, if $\Delta(x, y) = \Lambda x \Lambda y + \Lambda(xy)$, then : $\pi^* \Delta(x, y) = x \otimes y + y \otimes x$.

By [17] theorem 11, as an algebra over $\mathbb{Z}_2[v]$, $H^*(\Lambda^2 M, \Delta M, \mathbb{Z}_2)$ is generated by all Λx for $x \in \tilde{H}^*(M, \mathbb{Z}_2)$ subject to the relations :

i) $\Lambda(x + y) = \Lambda x + \Lambda y$ for all $x, y \in \tilde{H}^*(M, \mathbb{Z}_2)$,
ii) $v \Lambda x \Lambda y = 0$ for all $x, y \in \tilde{H}^*(M, \mathbb{Z}_2)$,
iii) $\Lambda x \Lambda y \Lambda z = \Lambda x \Lambda(yz) + \Lambda y \Lambda(xz) + \Lambda z \Lambda(xy)$ for all $x, y, z \in \tilde{H}^*(M, \mathbb{Z}_2)$,
iv) $(\Lambda x)^2 = \sum_{i=0}^{k} v^{k-i} \Lambda^i x$ for all $x \in \tilde{H}^k(M, \mathbb{Z}_2)$, $k > 0$.

It is also worth noting that :

$$Sq^k \Lambda x = \sum_{i=0}^{k} v^{k-i} \Lambda Sq^i x \text{ for } x \in H^n(M, \mathbb{Z}_2) \text{ and } k \geq 0.$$

As an immediate application, one obtains :

$$\begin{array}{lcl} H^{2n}(\Lambda^2 M, \Delta M, \mathbb{Z}_2) & = & \mathbb{Z}_2 < \Lambda \rho_2 M \Lambda \rho_2 M > \\ H^{2n-1}(\Lambda^2 M, \Delta M, \mathbb{Z}_2) & = & \quad \{\Lambda x \Lambda \rho_2 M \mid x \in H^{n-1}(M, \mathbb{Z}_2)\} \\ & & \oplus\ \mathbb{Z}_2 < v^{n-1} \Lambda \rho_2 M > \\ H^{2n-2}(\Lambda^2 M, \Delta M, \mathbb{Z}_2) & = & \quad \{\Lambda x \Lambda \rho_2 M \mid x \in H^{n-2}(M, \mathbb{Z}_2)\} \\ & & \oplus \{\Lambda x \Lambda y \mid x, y \in H^{n-1}(M, \mathbb{Z}_2)\} \\ & & \oplus\ \mathbb{Z}_2 < v^{n-2} \Lambda \rho_2 M > . \end{array}$$

where $\rho_2 : H^n(M, \mathbb{Z}) \longrightarrow H^n(M, \mathbb{Z}_2)$ is the reduction modulo 2 and M is a generator of $H^n(M, \mathbb{Z})$ ($= \mathbb{Z}$ or $\mathbb{Z}_2$ depending on the orientability of M).

7.2. **Cohomology in $\mathbb{Z}[v]$ of the pair** (M^*, PM)**.** There is still a monomorphism $\Lambda : \tilde{H}^k(M, \mathbb{Z}) \longrightarrow H^k(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ and a bilinear map $\Delta : \tilde{H}^k(M, \mathbb{Z}) \times \tilde{H}^k(M, \mathbb{Z}) \longrightarrow H^k(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ and a long exact sequence

$$(*) \quad \to H^k(X, \mathbb{Z}[v]) \xrightarrow{(\pi^*)^T} H^k(M^2, \Delta M, \mathbb{Z}) \xrightarrow{\theta} H^k(X, \mathbb{Z}) \xrightarrow{\cup V} H^{k+1}(X, \mathbb{Z}[v])$$

with $(\pi^*)^T \Lambda x = x \otimes 1 - 1 \otimes x$ and $(\pi^*)^T \Delta(x, y) = x \otimes y - (-1)^{rs} y \otimes x$ ($x \in H^r(M, \mathbb{Z})$ and $y \in H^s(M, \mathbb{Z})$). Moreover $V = \beta_2^T(1)$ with $1 \in H^0(X, \mathbb{Z}_2)$ where $\beta_2^T : H^{k-1}(M, \mathbb{Z}_2) \longrightarrow H^k(M, \mathbb{Z}[v])$ is the twisted Bockstein operator. In particular, $\rho_2^T(V) = v$ ($\rho_2^T : H^k(X, \mathbb{Z}[v]) \longrightarrow H^k(X, \mathbb{Z}_2)$ is the twisted reduction modulo 2). Note also that $\rho_2^T(\Lambda x) = \Lambda(\rho_2 x)$ and $\rho_2^T(\Delta(x,y)) = \Delta(\rho_2 x, \rho_2 y)$ for $x \in H^k(M,\mathbb{Z})$, $y \in H^r(M,\mathbb{Z})$ (see [23], lemma 1.5).

In [23], the cohomology groups $H^{2n}(X,\mathbb{Z}[v])$ and $H^{2n-1}(X,\mathbb{Z}[v])$ are computed. The result depends on four cases :

(i) n is even and M is orientable,
(ii) n is even and M is non orientable,
(iii) n is odd and M is orientable,
(iv) n is odd and M is non orientable.

Precisely, $H^{2n}(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ is

$$\begin{array}{llll} (i) & \mathbb{Z}_2 < \beta_2^T(v^{n-1}\Lambda\rho_2 M) > & (ii) & \mathbb{Z}_2 < \beta_2^T(v^{n-1}\Lambda\rho_2 M) > \\ (iii) & \mathbb{Z} < \Delta(M,M) > & (iv) & \mathbb{Z}_2 < \beta_2^T(\Lambda M'\Lambda\rho_2 M) > \end{array}$$

and $H^{2n-1}(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ is

$$\begin{array}{llll} (i) & G & (ii) & K \oplus \mathbb{Z}_4 < \frac{1}{2}\beta_2^T(v^{n-1}\Lambda M') > \\ (iii) & G \oplus \mathbb{Z}_2 < \beta_2^T(v^{n-2}\Lambda\rho_2 M) > & (iv) & K \oplus \mathbb{Z}_2 < \beta_2^T(v^{n-2}\Lambda\rho_2 M) > \end{array}$$

where

$$\begin{array}{rl} G = & \{\Delta(x,M) \mid x \in H^{n-1}(M,\mathbb{Z})\} \simeq H^{n-1}(M,\mathbb{Z}), \\ K = & \{\beta_2^T(\Delta(M',\rho_2 x)) \mid x \in H^{n-1}(M,\mathbb{Z})\} \simeq Im\ \rho_2. \end{array}$$

7.3. **Cohomology in $\mathbb{Z}$ of the pair (M^*, PM).** It is easy to see that $H^{2n}(\Lambda^2 M, \Delta M, \mathbb{Z})$ is $\mathbb{Z}$ in the case (i) and $\mathbb{Z}_2$ in the other cases. We denote by H the cohomology group $H^{2n-1}(\Lambda^2 M, \Delta M, \mathbb{Z})$. The result still depends on the four cases (i) - (iv).

(i) The exact sequence $(*)$ becomes :

$$\ldots \longrightarrow G \xrightarrow{(\pi^*)^T} G \oplus G \longrightarrow H \xrightarrow{\cup V} \mathbb{Z}_2 \longrightarrow \mathbb{Z} \longrightarrow \ldots$$

where $(\pi^*)^T$ can be identified with the map $x \longmapsto (x,-x)$. Hence this sequence is equivalent to :

$$0 \longrightarrow G \longrightarrow H \xrightarrow{\cup V} \mathbb{Z}_2 < \beta_2^T(v^{n-1}\Lambda\rho_2 M) > \longrightarrow 0.$$

But

$$\begin{array}{rcl} \rho_2^T(\beta_2(v^{n-2}\Lambda\rho_2 M) \cup V) & = & \rho_2(\beta_2(v^{n-2}\Lambda\rho_2 M)) \cup \rho_2^T(V) \\ & = & Sq^1(v^{n-2}\Lambda\rho_2 M) \cup v \\ & = & v^n\Lambda\rho_2 M. \end{array}$$

This implies that $\beta_2(v^{n-2}\Lambda\rho_2 M)\cup V \neq 0$ and $\sigma : \mathbb{Z}_2 \longrightarrow H$ defined by $\sigma(\beta_2^T(v^{n-1}\Lambda\rho_2 M)) = \beta_2(v^{n-2}\Lambda\rho_2 M)$ is a monomorphism which satisfies $(\cup V) \circ \sigma = id_{\mathbb{Z}_2}$. Thus : $H = G \oplus \mathbb{Z}_2$.

(ii) The exact sequence $(*)$:

$$\ldots \xrightarrow{(\pi^*)^T} (G \otimes \mathbb{Z}_2)^2 \oplus \mathbb{Z}_2 \longrightarrow H \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

tells that H is of the form $\mathbb{Z}_2^{P+1}$ or $\mathbb{Z}_2^P \oplus \mathbb{Z}_4$ with $P \in \mathbb{N}$. Applying the universal coefficient theorem, we get the exact sequence :

$$0 \longrightarrow H \otimes \mathbb{Z}_2 \longrightarrow H^{2n-1}(X,\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

By section 7.2, $H^{2n-1}(X,\mathbb{Z}_2) \simeq H^{n-1}(M,\mathbb{Z}_2) \oplus \mathbb{Z}_2$, applying again the universal coefficient theorem, we have :

$$H^{n-1}(M,\mathbb{Z}_2) \simeq G \otimes \mathbb{Z}_2 \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2^{N+1}$$

where $N = dim_{\mathbb{Z}_2} G \otimes \mathbb{Z}_2$. Hence $H \otimes \mathbb{Z}_2 = \mathbb{Z}_2^{N+1}$ and thus $H = \mathbb{Z}_2^{N+1}$ or $\mathbb{Z}_2^N \oplus \mathbb{Z}_4$. Assume that $H = \mathbb{Z}_2^N \oplus \mathbb{Z}_4$. The Bockstein sequence is then :

$$\begin{array}{ccccccc} H^{2n-2}(X,\mathbb{Z}_2) & \xrightarrow{\beta_2} & H^{2n-1}(X,\mathbb{Z}) & \xrightarrow{(0,\times 2)} & H^{2n-1}(X,\mathbb{Z}) & \xrightarrow{\rho_2} & H^{2n-1}(X,\mathbb{Z}_2) \\ & & \| & & \| & & \| \\ & & \mathbb{Z}_2^N \oplus \mathbb{Z}_4 & & \mathbb{Z}_2^N \oplus \mathbb{Z}_4 & & \mathbb{Z}_2^{N+2} \end{array}$$

Let g be a generator of the factor $\mathbb{Z}_4$. Then Im $\beta_2 = \mathbb{Z}_2^N \oplus \mathbb{Z}_2 < 2g >$ and Im $\rho_2 \circ \beta_2 = \mathbb{Z}_2^N$. On the other hand, $Sq^1 = \rho_2\beta_2$ and it is easy to verify that : $Sq^1(H^{2n-2}(\Lambda^2 M, \Delta M, \mathbb{Z}_2)) \simeq \mathbb{Z}_2^{N+1}$. Contradiction. Thus, $H = \mathbb{Z}_2^{N+1}$.

(iii) One has :

$$... \longrightarrow G \oplus \mathbb{Z}_2 < a > \xrightarrow{(\pi^*)^T} G \oplus G \longrightarrow H \xrightarrow{0} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

with $a = \beta_2^T(v^{n-2}\Lambda\rho_2 M)$. Since $(\pi^*)^T\beta_2^T(v^{n-2}\Lambda\rho_2 M) = \beta_2^T\pi^*(v^{n-2}\Lambda\rho_2 M)$ and $v^{n-2}\Lambda\rho_2 M = v \cup v^{n-3}\Lambda\rho_2 M \in Im\ (\cup v) = Ker\ \pi^*$, we get :$(\pi^*)^T(a) = 0$. Hence, $H \simeq G$.

(iv) The exact sequence $(*)$:

$$... \xrightarrow{(\pi^*)^T} (G \otimes \mathbb{Z}_2)^2 \oplus \mathbb{Z}_2 \longrightarrow H \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

tells that H is of the form $\mathbb{Z}_2^{P+1}$ or $\mathbb{Z}_2^P \oplus \mathbb{Z}_4$ with $P \in \mathbb{N}$. Similarly to the case (ii), one can easily show that $P = N = dim_{\mathbb{Z}_2} G \otimes \mathbb{Z}_2$. Assume that $H = \mathbb{Z}_2^{N+1}$. The Bockstein sequence is then :

$$\begin{array}{ccccccc} H^{2n-2}(X,\mathbb{Z}_2) & \xrightarrow{\beta_2} & H^{2n-1}(X,\mathbb{Z}) & \xrightarrow{0} & H^{2n-1}(X,\mathbb{Z}) & \xrightarrow{\rho_2} & H^{2n-1}(X,\mathbb{Z}_2) \\ & & \| & & \| & & \| \\ & & \mathbb{Z}_2^{N+1} & & \mathbb{Z}_2^{N+1} & & \mathbb{Z}_2^{N+2} \end{array}$$

One has : Im $\beta_2 = \mathbb{Z}_2^{N+1}$ and Im $\rho_2 \circ \beta_2 = \mathbb{Z}_2^{N+1}$. On the other hand :

$$Sq^1(H^{2n-2}(\Lambda^2 M, \Delta M, \mathbb{Z}_2)) \simeq \mathbb{Z}_2^N.$$

Contradiction. Thus, $H = \mathbb{Z}_2^N \oplus \mathbb{Z}_4$.

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