The Gluck and Ziller Problem with the Euclidean Metric

Vincent Borrelli

August 31, 2004

Résumé. – Le problème de Gluck et Ziller est la recherche des champs de vecteurs unitaires de plus petit volume de la sphère $\mathbb{S}^{2m+1}$. L'image d'un champ de vecteurs unitaires est une sous-variété du fibré tangent unitaire, si ce fibré tangent est muni d'une métrique riemannienne, le *Volume* d'un champ de vecteurs unitaires est le volume de son image pour la métrique induite. Jusqu'à présent le problème de Gluck et Ziller a été essentiellement étudié dans le cas où le fibré tangent unitaire est muni de la métrique de Sasaki, mais il existe une autre métrique naturelle pour ce fibré –que nous appelons *la métrique euclidienne*– et il n'y a aucune de raison de préférer l'une plutôt que l'autre. Dans cet article nous montrons que pour les sphères de dimension impaire, les champs de Hopf (c'est-à-dire, les champs de vecteurs unitaires tangents à une fibration de Hopf) sont critiques pour la fonctionnelle volume avec la métrique euclidienne mais ils ne sont pas toujours *stables*. De manière analogue au cas où le fibré tangent est muni de la métrique de Sasaki, la stabilité dépend du rayon r de la sphère : à chaque dimension impaire n correspond un "rayon critique" qui est tel que, si r est plus petit que ce rayon, les champs de Hopf sont stables sur $\mathbb{S}^n(r)$ et réciproquement.

Abstract. – The Gluck and Ziller Problem is the one of finding unit vector fields of minimum volume on the standard round sphere $\mathbb{S}^{2m+1}$. The image of a smooth unit vector field is a submanifold of the unit tangent bundle. If this unit tangent bundle is endowed with a Riemannian metric, the *Volume* of a unit vector field is the volume of its image for the induced metric. Up to now the Gluck and Ziller problem was mainly studied in a unit tangent bundle endowed with the Sasaki metric, yet there is another natural metric on this unit tangent bundle –that we call the *Euclidean metric*– and there is no obvious reason to give a better place to the Sasaki metric rather than the Euclidean one. In this article, we show that, for odd-dimensional spheres, Hopf vector fields (that is, unit vector fields tangent to the fiber of any Hopf fibration) are critical for the volume functional with the Euclidean metric, but they are not always *stable*. Similarly to the case where the tangent bundle is endowed with the Sasaki metric, stability depends on the radius r of the sphere : for every odd dimension n there exists a "critical radius" such that, if r is lower than this radius, Hopf fields are stable on $\mathbb{S}^n(r)$ and conversely.

2000 Mathematics Subject Classification : 53C20
Keywords and phrases : Volume, Vector field, Stability.

Acknowledgement. – I would like to thank Olga Gil-Medrano for reading the manuscript and for suggesting several corrections.

1 General Introduction

In an article published in 1986 [6], H. Gluck and W. Ziller set the following problem : find the unit vector fields of the standard round sphere $\mathbb{S}^{2m+1}$ of minimum volume, that is, the smooth vector fields $V : \mathbb{S}^{2m+1} \longrightarrow T^1\mathbb{S}^{2m+1}$ such that the volume of their images $V(\mathbb{S}^{2m+1})$ is the lowest possible. In the article, they settled the question for the 3-sphere : the unit vector fields of minimum volume on $\mathbb{S}^3$ are the Hopf vector fields and no others. A Hopf vector field is any unit vector field tangent to the fiber of a Hopf fibration : $\mathbb{S}^1 \longrightarrow \mathbb{S}^{2m+1} \longrightarrow \mathbb{C}P^m$. Unfortunately, the method they used fails for the other dimensions, but it is quite natural to think that their result could hold *mutatis mutandis* on the higher dimensions. Nevertheless a new phenomenon occurs which makes the things more delicate. A necessary condition for the Hopf fields to be minimizers of the volume functional is to be critical and stable. It turns out that they actually are critical but the stability question depends on the radius r of the sphere. Precisely, stability occurs if and only if $r \leq \frac{1}{\sqrt{n-4}}$ where $n = 2m + 1$ is the dimension of the sphere. In particular, the Hopf vector fields can not be minimizers of the Volume if r do not satisfy this inequality [1]. This complication is probably part of the reasons why, eighteen years after, the Gluck and Ziller problem is still unsolved (see [3],[4] for an overview on this problem).

This article is concerned with the study of the stability question of Hopf fields but with the Euclidean metric. Up to now, we have been imprecise in the definition of the volume of a vector field $V : \mathbb{S}^{2m+1} \longrightarrow T^1\mathbb{S}^{2m+1}$ since we have defined it as the volume of $V(\mathbb{S}^{2m+1})$ but we have not mentionned the metric we consider in the unit tangent bundle. The point is that they are *two* "natural" metrics on $T^1\mathbb{S}^{2m+1}$ and these two metrics are going to give two different volumes. The first one is the Sasaki metric. If (M, g) is a Riemannian manifold, the Sasaki metric g^{Sas} on TM (and thus on T^1M) is defined by :

$$\forall \tilde{X}, \tilde{Y} \in TTM : \;\; g^{Sas}(\tilde{X}, \tilde{Y}) = g(d\pi(\tilde{X}), d\pi(\tilde{Y})) + g(K(\tilde{X}), K(\tilde{Y}))$$

where $\pi : TM \longrightarrow M$ is the projection and $K : TTM \longrightarrow TM$ is the connector of the Levi-Civita connection ∇_g of g. In the case where M is the sphere $\mathbb{S}^n(1)$, then g^{Sas} is the $SO(n+1)$-invariant metric on $T^1\mathbb{S}^{2m+1} =$

$SO(n+1)/SO(n-1)$ coming from the bi-invariant metric g^{bi} on $SO(n+1)$ defined by :

$$\forall A, B \in \mathfrak{so}(n+1), \quad g^{bi}_{id}(A,B) = tr({}^tAB).$$

The second metric, that we call the *Euclidean metric* and we denote by g^{Eu}, comes from the natural embedding :

$$\begin{array}{ccc} T^1\mathbb{S}^n & \longrightarrow & (\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}, <.,.>) \\ v_x & \longmapsto & (x,v). \end{array}$$

It is defined as the pull-back of the Euclidean metric $<.,.>$ of $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ by the above map. If $\mathbb{S}^n$ is a sphere of radius 1, then the image of $T^1\mathbb{S}^n(1)$ is the Euclidean manifold $V_{2,n+1}$ of orthonormal 2-frames of $\mathbb{R}^{n+1}$ and these two metrics are also the natural metrics on it.

Let us see quickly that the two metrics are different. Consider the following commutative diagram where p is the obvious projection and $\tilde{\pi}$ the projection on the first two factors of $(\mathbb{R}^{n+1})^{n+1}$.

$$\begin{array}{ccc} SO(n+1) \subset (\mathbb{R}^{n+1})^{n+1} & \xrightarrow{\tilde{\pi}} & \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} \\ p \downarrow & & || \\ SO(n+1)/SO(n-1) & \xrightarrow{\pi} & \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}. \end{array}$$

The Lie group $SO(n+1)$ is endowed with the bi-invariant metric g^{bi} and the homogeneous space $SO(n+1)/SO(n-1)$ with the $SO(n+1)$-invariant metric so that p is a Riemannian submersion. Let (x_0, v_0) be the point of $T^1\mathbb{S}^n(1) \subset \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ with coordinates :

$$x_0 = (1, 0, ..., 0), \quad v_0 = (0, 1, 0, ..., 0).$$

The differential of $\tilde{\pi}$ at the point id is the map :

$$\begin{array}{ccc} \mathfrak{so}(n+1) & \longrightarrow & T_{(x_0,v_0)}(T^1\mathbb{S}^n(1)) \\ X = \begin{pmatrix} 0 & x_{12} & \dots & & x_{1,n+1} \\ -x_{12} & 0 & & & \vdots \\ \vdots & & \ddots & & \\ & & & 0 & x_{n,n+1} \\ -x_{1,n+1} & \dots & & -x_{n,n+1} & 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & x_{12} \\ -x_{12} & 0 \\ -x_{13} & -x_{23} \\ \vdots & \vdots \\ -x_{1,n+1} & -x_{1,n+2} \end{pmatrix}. \end{array}$$

Hence :

$$< d\tilde{\pi}(X), d\tilde{\pi}(X) >= 2x_{12}^2 + \sum_{j>2}^{n+1} (x_{1j}^2 + x_{2j}^2).$$

Thus, the Euclidean metric differs from the Sasaki metric on a factor 2 in the direction given by :

$$X = \left(\begin{array}{cc|c} 0 & x_{12} & 0 \\ -x_{12} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right),$$

that is, in the direction $(-v_0, x_0)$. In the orthogonal directions, the two metrics are the same.

In their pionnering work, H. Gluck and W. Ziller solved, in dimension 3, the question of the minimizers of the volume for both metrics. Nevertheless, in the works that have followed, the Gluck and Ziller problem was mainly studied for the Sasaki metric. This is the case, in particular, of the work [1] which studies the question of the stability of the Hopf fields. The aim of this article is to come back to the stability problem but for the Euclidean metric.

We denote by H the Hopf vector field defined by :

$$\forall x \in \mathbb{S}^n(r) \subset \mathbb{R}^{n+1}, \quad H(x) = \frac{1}{r} Jx$$

where J is the standard complex structure of $\mathbb{R}^{n+1} = \mathbb{C}^{\frac{n+1}{2}}$ and $n = 2m+1$. The image submanifold :

$$H(\mathbb{S}^n) = \{(x, \frac{1}{r} Jx) \ : \ x \in \mathbb{S}^n(r)\} \subset \mathbb{S}^{2n+1}(\sqrt{1+r^2})$$

is a round sphere of radius $\sqrt{1+r^2}$. Indeed, if Π is the $(n+1)$-plane of $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ defined by :

$$\Pi = \{x_1 = ry_2\} \cap \{x_2 = -ry_1\} \cap ... \cap \{x_n = ry_{n+1}\} \cap \{x_{n+1} = -ry_n\}.$$

where $(x_1, ..., x_{n+1}, y_1, ..., y_{n+1})$ denotes the coordinates in $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$, then :

$$H(\mathbb{S}^n) = \mathbb{S}^{2n+1}(\sqrt{1+r^2}) \cap \Pi.$$

Since $H(\mathbb{S}^n)$ is a great subsphere of $\mathbb{S}^{2n+1}(\sqrt{1+r^2})$, it is a minimal submanifold of $T^1\mathbb{S}^n(r) \subset \mathbb{S}^{2n+1}(\sqrt{1+r^2})$ and thus H is critical for the volume functional of unit vector fields.

The stability question is more involved and similarly to the case where the Sasaki metric is considered, it depends on the radius, excepted for the 3-dimensional case.

Stability Theorem with the Euclidean metric. – *Let* $n \geq 5$, *Hopf fields of* $\mathbb{S}^n(r)$ *are stable if and only if* $r \leq \frac{1}{\sqrt{n-3}}$ *(or equivalently, if the* *curvature $k = r^{-2}$ satisfies $k \geq n-3$).*

In particular, Hopf fields can not achieve the minimum of the volume if $k < n-3$. The remainder of this article is devoted to the proof of this theorem.

2 Proof of the stability theorem

It is stated in the work of H. Gluck and W. Ziller [6] that Hopf fields are stable for the two metrics for dimension three, thus we assume from now on that $n \geq 5$. For short we sometime denote by (M, ∇) the sphere $\mathbb{S}^n(r)$ with its Levi-Civita connection, we also denote by D the usual connection of the Euclidean space $\mathbb{R}^{n+1}$ or indifferently, the connection of $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$. We have a sequence of inclusions of Riemannian manifolds with their corresponding Levi-Civita connections :

$$(M' = H(M), \nabla') \subset (\tilde{M} = T^1 M, \tilde{\nabla}) \subset (\overline{M} = \mathbb{S}^{2n+1}(\sqrt{1+r^2}), \overline{\nabla}) \subset (\mathbb{R}^{2n+2}, D).$$

Let $A' : M' \longrightarrow T\tilde{M}$ be a vector field and ξ its normal component, the second variation formula for the volume of the closed minimal submanifold $M' \subset \tilde{M}$ along the direction A' is given by :

$$V''(A') = \int_{M'} \left(\|\nabla^{\perp}\xi\|^2 - \tilde{S}(\xi,\xi) - \|W_\xi\|^2 \right) dvol_{M'}$$

where $\nabla^{\perp}$ is the normal connection, W_ξ the shape operator ($\tilde{\nabla}_Z \xi = -W_\xi(Z) + \nabla^{\perp}_Z \xi$, for any $Z \in TM'$) and :

$$\tilde{S}(\xi,\xi) = \sum_{j=1}^{n} \tilde{R}(\xi, E_j, E_j, \xi)$$

with $E_1, ..., E_n$ a local orthonormal frame of M' and $\tilde{R}$ the curvature tensor of $\tilde{M}$ (see [2] p. 208 for instance). In our case :

$$TM'_{(x,H(x))} = \{(X, \frac{1}{r} JX) \; : \; X \in x^{\perp} \subset \mathbb{R}^{n+1}\}$$

and since our variations derive from variations of a vector field, they have the following form :

$$A'(x, H(x)) = (0, A(x)) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$$

with $A : M \longrightarrow \mathbb{R}^{n+1}$ such that $< A(x), x >=< A(x), H(x) >= 0$ for every $x \in M$. The above vector A' decomposes into :

$$A' = (0, A) = \frac{r^2}{1+r^2} \left((\frac{1}{r} JA, A) + (-\frac{1}{r} JA, \frac{1}{r^2} A) \right)$$

where $\xi = (\frac{1}{r}JA, A)$ and $\zeta = (-\frac{1}{r}JA, \frac{1}{r^2}A)$ are orthogonal. Moreover, ζ is of the form $(X, \frac{1}{r}JX)$ with $X = -\frac{1}{r}JA$, thus $\frac{r^2}{1+r^2}\zeta$ is the tangential part of A' and $\frac{r^2}{1+r^2}\xi$ is its normal part. Hence, the Hopf vector field H is stable if and only if :

$$\int_{M'} \left(\|\nabla^{\perp}\xi\|^2 - \tilde{S}(\xi,\xi) - \|W_{\xi}\|^2 \right) dvol_{M'} \geq 0$$

for any direction $\xi = (\frac{1}{r}JA, A)$.

Proposition 1. – *Let $A' = (0, A)$ with A as above, one has :*

$$V''(A') = \frac{2}{1+k} \int_{M'} \left(\|\nabla A\|^2_{TM,H^{\perp}} + (k-2m)k\|A\|^2 \right) dvol_{M'}.$$

In that formula :

$$\|\nabla A\|^2_{TM,H^{\perp}} = \sum_{i=1}^{n}\sum_{j=1}^{n-1} < \nabla_{e_i} A, e_j >^2$$

where $(e_1, ..., e_{n-1}, e_n)$ any local orthonormal frame of M such that $e_n = \frac{1}{r}Jx$.

Corollary 2. – *There is a positive constant C depending only on the curvature k and the dimension n of the sphere such that :*

$$V''(A') = C \int_{\mathbb{S}^n(r)} \left(\|\nabla A\|^2_{TM,H^{\perp}} + (k-2m)k\|A\|^2 \right) dvol_{\mathbb{S}^n(r)}.$$

This corollary is an immediate consequence of proposition 1. Let us put :

$$Q(A) = \int_{\mathbb{S}^n(r)} \left(\|\nabla A\|^2_{TM,H^{\perp}} + (k-2m)k\|A\|^2 \right) dvol_{\mathbb{S}^n(r)}.$$

Obviously, H is stable for the volume of unit vector fields if and only if $Q(A) \geq 0$ for all vectors fields $A : M \longrightarrow TM$ such that $< A, H >= 0$.

Proof of proposition 1. – Let $n(x) = \frac{1}{\sqrt{1+r^2}}(x, \frac{1}{r}Jx)$, $n_1(x) = \frac{1}{\sqrt{1+r^2}}(\frac{1}{r}x, -Jx)$ and $n_2(x) = \frac{1}{\sqrt{1+r^2}}(\frac{1}{r}Jx, x)$. These three orthonormal vectors span the normal bundle $N\tilde{M}$ of $\tilde{M}$ in $\mathbb{R}^{2n+2}$ above the point $(x, H(x))$ of M' : $N\tilde{M}_{(x,H(x))} = \mathbb{R}n(x) \oplus^{\perp} \mathbb{R}n_1(x) \oplus^{\perp} \mathbb{R}n_2(x)$. Indeed :

$$\tilde{M} = \mathbb{S}^{2n+1}(\sqrt{1+r^2}) \cap \{\sum_{j=1}^{n+1} x_j^2 - r^2 y_j^2 = 0, \sum_{j=1}^{n+1} x_j y_j = 0\}$$

and by differentiating the two equations, we obtain two normal vectors and this leads to n_1 and n_2. The last vector n is just the unit exterior normal of $\mathbb{S}^{2n+1}(\sqrt{1+r^2})$. Let $X \in TM$ and $Z = (X, \frac{1}{r}JX) \in TM'$, we thus have :

$$\tilde{\nabla}_Z \xi = D_Z \xi - < D_Z \xi, n > n - < D_Z \xi, n_1 > n_1 - < D_Z \xi, n_2 > n_2$$

and a straightforward computation shows that $\tilde{\nabla}_Z \xi = (\frac{1}{r}JB, B)$ with

$$B = D_X A + \frac{1}{r^2} < A, JX > Jx + \frac{1}{r^2} < A, X > x.$$

Thus $\tilde{\nabla}_Z \xi = \nabla^{\perp}_Z \xi$ and $W_\xi = 0$. Now, using the relation :

$$\nabla_X A = D_X A - < D_X A, N > N$$

with $N = \frac{1}{r}x$, it is readily obtained that $B = \nabla_X A + \frac{1}{r^2} < A, JX > Jx$ and :

$$\|\nabla^{\perp}_Z \xi\|^2 = 2 \left(\|\nabla_X A\|^2 - \frac{1}{r^2} < JA, X >^2 \right).$$

If $(e_1, ..., e_{n-1}, e_n = \frac{1}{r}Jx)$ is a local orthonormal frame of M, we define a local orthonormal frame $(E_1, ..., E_n)$ of M' by putting : $E_j = \frac{r}{1+r^2}(e_j, \frac{1}{r}Je_j)$. We have :

$$\|\nabla^{\perp} \xi\|^2 = \sum_{j=1}^{n} \|\nabla^{\perp}_{E_j} \xi\|^2 = \frac{2r^2}{1+r^2} \left(\|\nabla A\|^2 - \frac{1}{r^2} \sum_{j=1}^{n} < JA, e_j >^2 \right).$$

Moreover :

$$\begin{aligned} \|\nabla A\|^2 &= \|\nabla A\|^2_{TM,H^{\perp}} + \sum_{j=1}^{n} < \nabla_{e_j} A, \frac{1}{r}Jx >^2 \\ &= \|\nabla A\|^2_{TM,H^{\perp}} + \frac{1}{r^2} \sum_{j=1}^{n} < A, \nabla_{e_j} Jx >^2 \\ &= \|\nabla A\|^2_{TM,H^{\perp}} + \frac{1}{r^2} \sum_{j=1}^{n} < A, Je_j >^2 . \end{aligned}$$

Therefore :

$$\|\nabla^{\perp} \xi\|^2 = \frac{2r^2}{1+r^2} \|\nabla A\|^2.$$

It remains to compute $\tilde{S}(\xi, \xi)$. Let us write the Gauss equation for $\tilde{M} \subset \overline{M}$. One has :

$$\tilde{R}(\xi, E_j, E_j, \xi) = \overline{R}(\xi, E_j, E_j, \xi) + < h(\xi, \xi), h(E_j, E_j) > - < h(\xi, E_j), h(E_j, \xi) >$$

where h denotes the second fundamental form of $\tilde{M}$ and $\tilde{R}$, $\overline{R}$ the curvature tensors of $\tilde{M}$ and $\overline{M}$ respectively[1]. Since $\overline{M}$ is a round sphere of radius $\sqrt{1+r^2}$, one has : $R(\xi, E_j, E_j, \xi) = \frac{2}{1+r^2}\|A\|^2$. In addition, M' is totally geodesic in $\overline{M}$, thus $h(E_j, E_j) = 0$. It remains to determine $\|h(\xi, E_j)\|$, that is, the length of the normal part of $\overline{\nabla}_{E_j}\xi$. We obtain :

$$\begin{aligned}
\|h(\xi, E_j)\|^2 &= < D_{E_j}\xi, n_1 >^2 + < D_{E_j}\xi, n_2 >^2 \\
&= \frac{1+r^2}{r^4}\left(< A, \frac{r}{\sqrt{1+r^2}} e_j >^2 + < A, \frac{r}{\sqrt{1+r^2}} Je_j >^2\right) \\
&= \frac{1}{r^2}(< A, e_j >^2 + < A, Je_j >^2).
\end{aligned}$$

Finally :

$$\begin{aligned}
\tilde{S}(\xi, \xi) &= \sum_{j=1}^{n} \left(\frac{2}{1+r^2}\|A\|^2 - \frac{1}{r^2}(< A, e_j >^2 + < A, Je_j >^2 \right) \\
&= \frac{2k}{k+1}(n-k-1)\|A\|^2.
\end{aligned}.$$

This ends the proof of proposition 1. □

Proposition 3. − *Let $n \geq 5$. If $k < n-3$ then Hopf vector fields are unstable.*

Proof of proposition 3. − We consider the variation A defined in [5] p. 542. Let $a \in \mathbb{R}^{n+1}$ and put :

$$A = a - \bar{f}_a H - f_a N$$

where $H(x) = \frac{1}{r}Jx$, $N(x) = \frac{1}{r}x$, $\bar{f}_a =< a, H >$ and $f_a =< a, N >$. According to [5] proposition 14, one has :

$$\begin{aligned}
\|\nabla A\|^2_{TM,H^\perp} &= 2mk(\bar{f}_a^2 + f_a^2), \\
\|A\|^2 &= |a|^2 - (\bar{f}_a^2 + f_a^2).
\end{aligned}$$

Thus :

$$Q(A) = \int_M \left((4m-k)(\bar{f}_a^2 + f_a^2) + (k-2m)k|a|^2\right) dvol_M.$$

It is computed in [5] that :

$$\int_M (\bar{f}_a^2 + f_a^2) dvol_M = \frac{|a|^2}{m+1} Vol(M).$$

[1]In the Riemannian manifold (M, g) the notation $R(X, Y, Z, W)$ means $g(\nabla_X \nabla_Y Z, W) - g(\nabla_Y \nabla_X Z, W) - g(\nabla_{[X,Y]} Z, W)$ with $\nabla = \nabla_g$ the Levi-civita connection.

Therefore :
$$Q(A) = \frac{m}{m+1} k(k+2-2m)|a|^2 Vol(M).$$
$\square$

The rest of the proof of the stability theorem is very similar to the proof of Proposition of [1]. We introduce some notations of this article.

Let $W : \mathcal{U} \subset \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ be a vector field, we put $D^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W - J\bar{\nabla}_X W$ and $\bar{D}^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W + J\bar{\nabla}_X W$. Recall that W is *holomorphic* (resp.*anti-holomorphic*) if for all X, $D^{\mathbb{C}}_X W = 0$ (resp. $\bar{D}^{\mathbb{C}}_X W = 0$).

Let $H^{\perp}$ be the distribution $Span(x, Jx)^{\perp}$ on $\mathbb{R}^{n+1} = \mathbb{C}^{m+1} \setminus \{0\}$ and $\pi : T(\mathbb{C}^{m+1} \setminus \{0\}) \to H^{\perp}$ be the natural projections $\{x\} \times \mathbb{C}^{m+1} \to H^{\perp}_x$. We denote by $\|\pi \circ D^{\mathbb{C}} W\|_{H^{\perp}}$ the norm of $\pi \circ D^{\mathbb{C}} W_{|H^{\perp}} : H^{\perp} \to H^{\perp}$ that is :
$$\|\pi \circ D^{\mathbb{C}} W\|^2_{H^{\perp}} = \sum_{j=1}^{2m} \|\pi \circ D^{\mathbb{C}}_{e_j} W\|^2$$
Similarly :
$$\|\pi \circ \bar{D}^{\mathbb{C}} W\|^2_{H^{\perp}} = \sum_{j=1}^{2m} \|\pi \circ \bar{D}^{\mathbb{C}}_{e_j} W\|^2,$$
but in that case
$$\pi \circ \bar{D}^{\mathbb{C}} W_{|H^{\perp}} = \bar{D}^{\mathbb{C}} W_{|H^{\perp}} : H^{\perp} \to H^{\perp}$$
so that :
$$\|\pi \circ \bar{D}^{\mathbb{C}} W\|^2_{H^{\perp}} = \|\bar{D}^{\mathbb{C}} W\|^2_{H^{\perp}}.$$
In [1] the following identities are stated :
$$\int_{\mathbb{S}(r)} \frac{1}{2}\|\pi \circ D^{\mathbb{C}} A\|^2_{H^{\perp}} = \int_{\mathbb{S}(r)} \|\nabla A\|^2 - 3k\|A\|^2 - \|\nabla_H A\|^2 - 2m\sqrt{k} < \nabla_H A, JA >,$$
and
$$\int_{\mathbb{S}(r)} \frac{1}{2}\|\bar{D}^{\mathbb{C}} A\|^2_{H^{\perp}} = \int_{\mathbb{S}(r)} \|\nabla A\|^2 + k\|A\|^2 - \|\nabla_H A\|^2 + 2m\sqrt{k} < \nabla_H A, JA > .$$
These identities, combined with the trivial fact that :
$$\|\nabla A\|^2_{TM,H^{\perp}} = \|\nabla A\|^2_{H^{\perp}} + \|\nabla_H A\|^2$$
lead to the following proposition.

Proposition 4. $-$ *One has :*
$$1)\ Q(A) = \int_M \Big((k - 2m + 2 - m^2)k\|A\|^2 + \|\nabla_H A + \sqrt{k} mJA\|^2 +$$

$$\frac{1}{2}\|\pi \circ D^{\mathbb{C}} A\|^2_{H^\perp}\Big) \, dvol_M$$

$$2)\ Q(A) = \int_M \Big((k - 2m + 2 - m^2)k\|A\|^2 + \|\nabla_H A - \sqrt{k} m J A\|^2 +$$

$$\frac{1}{2}\|\bar{D}^{\mathbb{C}} A\|^2_{H^\perp}\Big) \, dvol_M$$

The Fourier serie of any variation $A : M \longrightarrow H^\perp \subset \mathbb{C}^{m+1}$ converges since A is smooth. Precisely, if we set :

$$A_l(p) = \frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}p)e^{-il\theta} d\theta \in H_p^\perp$$

then :

$$A(p) = \sum_{l \in \mathbb{Z}} A_l(p)$$

for every $p \in \mathbb{S}^n(r) \subset \mathbb{R}^{n+1}$. Since $A_l(e^{i\theta}p) = e^{il\theta}A_l(p)$, we have :

$$\nabla_H A = \bar{\nabla}_H A = \sum_{l \in \mathbb{Z}} i\sqrt{k} l A_l$$

and, if $\mathcal{C}(p)$ denotes the fiber of the Hopf fibration $\mathbb{S}^{2m+1} \to \mathbb{C}P^m$ passing through p :

$$\int_{\mathcal{C}(p)} < A_l, A_q >= 0,$$

if $l \neq q$. We denote by B the bilinear form associated to Q. Using the technics of [1], it is easy to check that $B(A_l, A_q) = 0$ if $l \neq q$, thus :

$$Q(A) = \sum_{l \in \mathbb{Z}} Q(A_l)$$

and the stability question reduces to show that $Q(A_l) \geq 0$ for all $l \in \mathbb{Z}$. If we compute $Q(A_l)$ with the first expression of proposition 4, adding the assumption that $k \geq 2m - 2$, we obtain :

$$Q(A_l) \geq \int_M \left(l(l + 2m)k\|A_l\|^2 + \frac{1}{2}\|\pi \circ D^{\mathbb{C}} A_l\|^2_{H^\perp}\right) dvol_M.$$

Thus $Q(A_l) \geq 0$ if $l \notin \{-2m + 1, ..., -1\}$. If we use the second expression, we obtain :

$$Q(A_l) \geq \int_M \left((l(l - 2m) - 4)k\|A_l\|^2 + \frac{1}{2}\|\bar{D}^{\mathbb{C}} A_l\|^2_{H^\perp}\right) dvol_M.$$

It is readily seen that, if $l \in \{-2m+1, ..., -1\}$, then $(l(l-2m)-4) \geq 2m-3$ and since $n \geq 5$, $2m - 3$ is positive.

References

[1] V. Borrelli and O. Gil-Medrano, *A critical radius for unit Hopf vector fields on spheres*, Preprint.

[2] B. -Y. Chen, *Riemannian submanifolds*, Handbook of Differential Geometry, Vol 1, 2000, Elsevier.

[3] O. Gil-Medrano *Volume and Energy of vector fields on spheres. A survey*, Differential Geometry, Valencia 2001, 167-178, World Sci. Publishing, River Edge, NJ2002.

[4] O. Gil-Medrano *Unit vector fields that are critical points of the volume and the energy : characterization and examples*, to appear in Complex, contact and symmetric spaces: papers in honour of Lieven Vanheche. Progress in Math. Birkhauser.

[5] O. Gil-Medrano and E. Llinares-Fuster, *Second variation of Volume and Energy of vector fields. Stability of Hopf vector fields*, Math. Ann. 320 (2001), 531-545.

[6] H. Gluck and W. Ziller, *On the volume of a unit vector field on the three-sphere*, Comment. Math. Helv. 61 (1986), 177-192.

V. Borrelli
Institut Girard Desargues, Université Claude Bernard, Lyon 1
43, boulevard du 11 Novembre 1918
69622 Villeurbanne Cedex, France
email : borrelli@igd.univ-lyon1.fr