

# THOMPSON'S GROUP $F$ AND EXTENSIVE AMENABILITY

MIKAEL DE LA SALLE

ABSTRACT. I rewrite the proof of *Thompson's group  $F$  is nonamenable* [6] in a vocabulary that I like better.

Let  $\mathcal{F}$  denote the set of all continuous maps  $u : [0, 1] \rightarrow [0, 1]$  that are piecewise dyadic affine: there are  $0 = s_0 < s_1 < \dots < s_N = 1$  and  $n_1, \dots, n_N \in \mathbb{Z}$  and  $b_1, \dots, b_N \in \mathbb{Z}[\frac{1}{2}]$  such that  $u(t) = 2^{n_i}t + b_i$  on  $[s_{i-1}, s_i]$ . It is a monoid for composition, maybe it should be called Thompson's monoid  $\mathcal{F}$ . Every element of  $\mathcal{F}$  is injective by definition. The subset of surjective maps is Thompson's group  $F$ . We denote by  $\mathcal{F}^0$  its ideal consisting of  $u$  such that  $0 < u(0)$  and  $u(1) < 1$ .

**Theorem 1.**  *$F$  is not amenable. The action of  $F$  on  $[0, 1]$  is not even extensively amenable [3].*

The next lemma expresses the transitivity properties of the left-action of  $F$  on  $\mathcal{F}^0$ . For example, it allows to identify  $\mathcal{F}^0$ , as an  $F$ -set, with  $F/\text{Fix}(I)$ , for any dyadic interval  $I \subset (0, 1)$ .

**Lemma 2.** *For every  $s, u \in \mathcal{F}^0$ , there is  $g \in F$  such that  $gs = u$ .*

*For every  $s, t, u, v \in \mathcal{F}^0$  satisfying  $s(1) < t(0)$  and  $u(1) < v(0)$ , there is  $g \in F$  such that  $(u, v) = (gs, gt)$ .*

*Proof.* This follows by realizing elements of  $F$  and  $\mathcal{F}$  as dyadic rearrangements (see [1, Theorem 1.1.2]): a dyadic partition of a closed interval  $I \subset [0, 1]$  is a finite subdivision into intervals of the form  $[k2^{-n}, (k+1)2^{-n}]$ . Note that  $I$  admits a dyadic partition if and only if  $\inf I, \sup I \in \mathbb{Z}[\frac{1}{2}]$ , and that a dyadic partition into  $N$  pieces can be refined into a dyadic partition into  $N'$  pieces, for every  $N' \geq N$ . If  $I$  and  $J$  have dyadic partitions into the same number of pieces, there is a unique map  $I \rightarrow J$  that maps affinely the  $i$ -th piece in  $I$  onto the  $i$ -th piece in  $J$ . Then  $\mathcal{F}$  is the set of all such dyadic rearrangements with  $I = [0, 1]$  and  $J \subset [0, 1]$ , and  $F$  is its subset with  $J = [0, 1]$ . Note that different pairs of dyadic partitions of  $I, J$  can give rise to the same dyadic rearrangement. With this representation, the lemma is clear.  $\square$

Fix once and for all a pair  $a, b \in \mathcal{F}^0$  with  $a(1) < b(0)$ , for example  $a(t) = \frac{t}{4} + \frac{1}{8}$  and  $b(t) = \frac{t}{4} + \frac{1}{2}$ . The following is a general criterion for using  $F$  and  $\mathcal{F}$  for finding fixed points for Lipschitz maps on Hilbert spaces.

**Lemma 3.** *Let  $(\pi, \mathcal{H})$  be a unitary representation of  $F$ ,  $K \subset \mathcal{H}$  a convex subset and  $f : K \rightarrow K$  a Lipschitz function with Lipschitz constant  $< \sqrt{2}$ . Assume that there is a function  $X : \mathcal{F} \rightarrow K$  satisfying the following properties:*

- *$X$  is  $F$ -equivariant:  $X_{gs} = \pi(g)X_s$  for every  $g \in F, s \in \mathcal{F}$ ;*
- *for every  $s \in \mathcal{F}$ ,  $X_s = f\left(\frac{X_{sa} + X_{sb}}{2}\right)$ .*

*Then  $f$  has a fixed point in  $K$ , and more precisely  $X$  is constant equal to a fixed point of  $f \cup \pi(F)$ .*

*Proof.* By Lemma 2, and the equivariance, there are  $\rho, \delta, \alpha \in \mathbb{R}$  such that  $\|X_s\| = \rho$  for every  $s \in \mathcal{F}^0$ ,  $\|X_s - X_t\| = \delta$  and  $\langle X_s, X_t \rangle = \alpha$  for every  $s, t \in \mathcal{F}^0$  with  $s(1) < t(0)$ . By the second hypothesis, we have

$$\delta = \|X_s - X_t\| \leq \text{Lip}(f) \left\| \frac{X_{sa} + X_{sb}}{2} - \frac{X_{ta} + X_{tb}}{2} \right\| = \frac{\text{Lip}(f)}{\sqrt{2}} \delta.$$

The last equality is because  $\xi = X_{sa} - X_{ta}$  and  $\eta = X_{sb} - X_{tb}$  are two orthogonal vectors of norm  $\delta$ , so their average has norm  $\delta/\sqrt{2}$ . Since  $\text{Lip}(f) < \sqrt{2}$ , we deduce  $\delta = 0$ . This implies that  $X$  is constant on  $\mathcal{F}^0$ , say equal to  $x \in K$ . Now if  $s \in \mathcal{F}$ , we have  $sa, sb \in \mathcal{F}^0$  so  $X_s = f(x)$ , so  $X$  is actually constant on  $\mathcal{F}$ , equal to  $f(x)$ . In particular  $x = f(x)$ .  $\square$

**Remark 4.** Consider the following situation:  $(\pi, \mathcal{H})$  is a unital representation by contractive operators of the monoid  $\mathcal{F}$ ,  $K \subset H$  is a convex  $\pi(\mathcal{F})$ -invariant subset,  $f : K \rightarrow K$  is  $L < \sqrt{2}$ -Lipschitz and  $\pi(\mathcal{F})$ -equivariant and there is  $p \in K$  such that  $p = f(\frac{\pi(a)p + \pi(b)p}{2})$ . Then the hypotheses of the lemma hold with  $X_s = \pi(s)p$ , and the lemma implies that  $p$  is invariant by  $f \cup \pi(\mathcal{F})$ .

We recall a notion from [3]. A mean on a set  $X$  is a finitely additive probability measure defined on all subsets of  $X$ ; it is the same as a state on  $\ell_\infty(X)$ . A mean on the set  $\mathcal{P}_f(X)$  of all finite subsets of  $X$  is cofinal if for every finite subset  $A$ , it gives full weight to the collection of subsets containing  $A$ . Following [3], we say that an action of a group  $G$  on a set  $X$  is extensively amenable if there is a  $G$ -invariant cofinal mean on  $\mathcal{P}_f(X)$ . It is almost obvious [3, Lemma 2.1] that a group is amenable if and only if all its actions are extensively amenable. Sometimes however (e.g. [4, 3]), when the action is sufficiently rich, extensive amenability implies the amenability of the group and is accessible; that is the reason why this property was introduced.

**Lemma 5.** Assume that the action of  $F$  on  $[0, 1]$  is extensively amenable. Then for every nonempty bounded convex subset  $B$  in a Hilbert space  $\mathcal{H}$ , and every Lipschitz function  $f : B \rightarrow B$  has almost fixed points:  $\inf_{x \in B} \|f(x) - x\| = 0$ .

*Proof.* Let  $\mathbb{P}$  be a cofinal invariant mean on  $\Omega := \mathcal{P}_f(I)$ , and  $\mathbb{E} : \ell_\infty(\Omega) \rightarrow \mathbb{C}$  to corresponding state. Let  $\mathcal{H}, B, f$  as in the lemma. Replacing  $f$  by  $\lambda f + (1 - \lambda)\text{id}$  for some small  $\lambda$  (whose Lipschitz constant is  $\leq \lambda \text{Lip}(f) + (1 - \lambda)$ ), we can assume that the Lipschitz constant  $L$  of  $f$  is  $< \sqrt{2}$ . I thank Yash Lodha for pointing out this reduction to me.

We consider  $\tilde{\mathcal{H}} := L_2(\Omega, \mathbb{P}; \mathcal{H})$ , the completion of the set of bounded functions  $X : \Omega \rightarrow \mathcal{H}$  for the seminorm  $\mathbb{E}\|X(Q)\|^2$ . It is a Hilbert space, and the  $F$ -invariance of  $\mathbb{P}$  implies that the natural action of  $F$   $\pi(g)X(Q) = X(g^{-1}Q)$  extends to a unitary representation. With Remark 4 in mind, we would like to extend  $\pi$  to a representation of  $\mathcal{F}$ , but, unless I missed something obvious, this would require additional invariance on  $\mathbb{P}$ . But it is still good to have Remark 4 in mind as a guide.

Consider  $K$ , the image in  $L_2(\Omega, \mathbb{P}; H)$  of the set of functions  $\Omega \rightarrow B$ . It is a convex  $\pi(G)$ -invariant set. Moreover the function  $X \mapsto f \circ X$  gives rise to an  $L$ -Lipschitz function  $\tilde{f} : K \rightarrow K$ . Observe that  $\tilde{f}$  has a fixed point in  $K$  if and only if  $\inf_{x \in B} \|f(x) - x\| = 0$ , so our task is to find a fixed point for  $\tilde{f}$ .

We claim that there is a  $p : \Omega = \mathcal{P}_f(X) \rightarrow B$  satisfying

$$p(Q) = f\left(\frac{p(a^{-1}Q) + p(b^{-1}Q)}{2}\right) \text{ if } Q \cap a(I) \neq \emptyset \text{ and } Q \cap b(I) \neq \emptyset.$$

Indeed, we can define  $p$  by induction on  $|Q|$ . Let  $x_0 \in B$  arbitrary. If one of the disjoint sets  $Q \cap a(I), Q \cap b(I)$  is empty, we set  $p(Q) = x_0$ . Otherwise  $a^{-1}(Q)$  and  $b^{-1}(Q)$  both have cardinality  $< |Q|$ , which allows us to run the induction.

From this  $p$ , we define, for  $s \in \mathcal{F}$ ,  $X_s(Q) = p(s^{-1}(Q))$ . Since  $\mathbb{P}$  is cofinal, we have  $X_s(Q) = f\left(\frac{X_{s \circ a}(Q) + X_{s \circ b}(Q)}{2}\right)$  almost everywhere, so in  $\tilde{\mathcal{H}}$ ,  $X_s = \tilde{f}\left(\frac{X_{sa} + X_{sb}}{2}\right)$ . The  $F$ -equivariance is clear. Lemma 3 implies that  $F$  has a fixed point. This concludes the proof of the lemma.  $\square$

Theorem 1 follows immediately by confronting Lemma 5 with the following theorem of Benyamini and Sternfeld.

**Theorem 6.** [2] For every infinite dimensional Banach space with closed unit ball  $B$ , there is a Lipschitz function  $f : B \rightarrow B$  without almost fixed point:  $\inf_{x \in B} \|f(x) - x\| > 0$ .

**AI statement** On October 9, I asked ChatGPT:

Extract from the proof of <https://github.com/openai/math/blob/main/preprints/Thompsons-group-F-is-nonamenable-September-23-2026/paper.pdf> a general criterion for non-amenability.

It provided me with a 2 page pdf that can be read here. No AI was used after this prompt. A first version of this note circulated since October 12. The current one is a major rewriting of it, where I try to isolate the different ingredients more clearly.

**Acknowledgement** Yash Lodha and Sang-hyun Kim [5], as well as Kate Juschenko (personal communication) have also idenpendently written an exposition of [6]. I use one of the simplifications from [5] in the proof of Lemma 5, and I thank them for that.

## REFERENCES

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