On the past of a mathematical object

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Postprint. July 2024

Abstract

This chapter tackles the question of big pictures by taking as a case study the reconstruction of the past of given mathematical objects, the genera of algebraic curves, based on a 1865 definition of them provided by Alfred Clebsch. It is shown that two essentially independent narratives can be used to gain perspective on the event constituted by this definition, one focusing on Abelian functions, the other on algebraic curves. Particular attention is paid to the methodological issues involved, relating both to the formation of corpora and to the description of their content. Finally, a section compares the characteristics of the two narratives and offers some thoughts on what a big picture should, or can, be in this case.

1. Introduction

"We 'big picture' people rarely become historians," proclaims the 6-year-old boy Calvin after answering his history test question on the significance of the Erie Canal: "In the cosmic sense, probably nil." [Watterson 1996, p. 13]. Choosing an appropriate spatiotemporal scale to put an historical event in perspective is indeed a smart way to escape a sticky school situation, but more virtuous reasons may also lead one to try to embed an event of the past into a bigger picture.

For instance, let me consider the issue of investigating the history of a mathematical object. A common approach is to start with what can be deemed, more or less naively, the first definition of the object, a definition considered as the temporal reference point. Drawing a bigger picture, then, may mean figuring out why and how this object was defined at the time. It may also mean understanding how it circulated afterwards or from what it originated, two avenues which come within the scope of reconstituting its future and its past.

In such processes of reconstruction of circulation or origination, the issue of identifying the object under scrutiny is crucial. The act of recognizing it,

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possibly under the garments of earlier or later versions, and of establishing connections between such versions, indeed, is what founds the writing of the past and the future of the object. The pitfall of anachronistic identifications must therefore be avoided, so as not to produce pictures whose temporal extent and historical consistency would be dubious, a particular danger being to end up with illusory long-term histories.[1]

An obvious and customary safeguard against such anachronistic identifications is to rely on traces left by past mathematicians. Nevertheless, as obvious as it may be, it still requires careful reflections. In the case of the future of an object, a way of coping with the identification problem is to track this object thanks to features such as its name in the natural language, its usual notation, or the references to the author who is credited for having defined it. On the contrary, one has to proceed a bit differently when trying to look backwards in time, since the issue is to find footprints of something that does not exist yet in the strict sense of the term.

This does not mean that investigating the past of an object is a vain question, nor that it is a necessary impasse. Mathematical objects do not come from nowhere. They are often introduced to address specific problems which have their own histories, they enter into preexisting theories, they can be the outcome of mechanisms of technical refinement or adjustment of other objects, and their name and notation may have already been used in related situations. Accounting for the past of the object consists precisely in locating and comprehending such features, so as to write a corresponding narrative in a sensible and controlled way.[2]

My aim in this chapter is to illustrate and discuss concretely these considerations of how to account for the past of mathematical objects in one specific case, namely the genera of algebraic curves.

The chosen point of reference consists in the two definitions given by the German mathematician Alfred Clebsch in two papers published in 1865 in the same volume of *Journal für die reine und angewandte Mathematik*, [Clebsch 1865a,b]. One of these definitions[3] reads as follows:

> The class of Abelian functions that is connected with an algebraic curve

[1]On this issue, see [Goldstein 1995, 2019], from which many of my reflections stem. In particular, as will be seen, my conclusions echo in great part those of [Goldstein 1995], which concerns the history of a theorem. A theorem is a mathematical entity of another nature than an object (such as a number or a set), which leads to variations in the way of investigating their respective identities. In any case, whatever the forms it takes, this identity issue is what lies at the core of such inquiries. Among other works on the history of objects or theorems that influenced me, see [Sinaceur 1991; Brechenmacher 2010; Ehrhardt 2012].

[2]Although the difference is not decisive for my purpose, I distinguish between a narrative, which refers to the concrete written piece of historical work, and a picture, which is the mental image that rests on a narrative.

[3]Although they are not formulated in the exact same way, the two definitions accord perfectly with one another. Taking into consideration the two papers containing the definitions will be useful to gather a corpus of investigation.

> of the nth order is determined by the number $p = \frac{n-1 \cdot n-2}{2}$ if the curve has no double point and no cusp, and, in the 63rd volume of this journal, I have given a number of results which rest on this remark. [...] Instead of classifying the algebraic curves in orders, and making subdivisions in them according to the number of double points and cusps that they contain, one can classify them into *genera* according to the number p; in the first genus are thus the curves for which $p = 0$, in the second one those for which $p = 1$, etc. Hence the different orders appear reciprocally as subdivisions of the genera [...].[4] [Clebsch 1865a, p. 43]

The genera introduced by Clebsch are thus categories in a classification of algebraic curves, i.e. curves that can be defined by a polynomial equation. They are defined through the value of a number p coming from the theory of Abelian functions and given by the formula $p = \frac{(n-1)(n-2)}{2} - d$, where n is the order of the considered curve and d is an integer whose value depends on the singularities of the curve.[5]

Before proceeding, let me recall that only a few years after 1865, Clebsch and his contemporaries designated the number p itself as the genus of an algebraic curve, a practice that has persisted until today.[6] However, such a shift in meaning is not easy to locate with precision and does not seem to have been accompanied by any explicit (re)definition. I will thus neglect this phenomenon, my aim being anyway to move back in time from 1865.

As is clear from the above quote, Clebsch's 1865 definition of the genera involves two mathematical domains, which appear in the quote together with indirect hints at some pieces of their history: on the one hand, the theory of algebraic curves, associated with the traditional classification into orders, and, on the other hand, the theory of Abelian functions, the past of which surfaces through a citation to an earlier paper of Clebsch, [Clebsch 1864b]. In what follows, we will see that these two facets correspond to two historical threads that remained essentially distinct from one another before 1865, in the sense that they involved different mathematicians and different mathematical contents, and had their own timelines as well as their own kinds of historical continuities and coherence.

[4] "Die Classe von Abelschen Functionen, mit welcher eine algebraische ebene Curve n^{ter} Ordnung zusammenhängt, wird durch die Zahl $p = \frac{n-1 \cdot n-2}{2}$ bestimmt, wenn die Curve keine Doppel- und Rückkehrpunkte besitzt, und ich habe im 63ten Bande dieses Journal pag. 189 eine Reihe von Resultaten angeführt, welche sich auf diese Bemerkung stützen. [...] Statt die algebraischen Curven nach Ordnungen einzutheilen, und in diesen Unterabtheilungen zu machen nach der Anzahl der Doppel- und Rückkehrpunkte, welche dieselben aufweisen, kann man dieselben in Geschlechter eintheilen nach der Zahl p; zu dem ersten Geschlecht also alle diejenigen für welche $p = 0$, zum zweiten diejenigen, für welche $p = 1$, u.s.w. Dann erscheinen umgekehrt die verschiedenen Ordnungen als Unterabtheilungen in den Geschlechtern".

[5] The order of a curve is the degree of its defining equation. As can be seen in the above quote, the term "order" also designated the category made of the curves of a given degree. Moreover, even though the formula $p = \frac{(n-1)(n-2)}{2} - d$ does not appear as such in the above quote, it is contained in Clebsch's papers of the time.

[6] Examples are provided in [Lê 2020, pp. 90–101].

More remarkable, perhaps, is the fact that these two threads also correspond to two manners of investigating the past of the genera. Indeed, one usual way of gaining a reasoned understanding of this past is to start with Clebsch's two 1865 articles, consider the publications that are cited therein, select those where anything related to the genera appears, and start the process again with these publications. This allows seeing p in a few texts published between 1865 and 1857, which corresponds to Bernhard Riemann's famous memoir on Abelian functions, [Riemann 1857]. It also allows recognizing other versions of this number in earlier publications, in a sense that will be explained below. All these texts relate to the topic of Abelian functions, and, while only a handful of them involve algebraic curves, none mention any object called "genus."

An alternative path consists in starting from a large set of publications on algebraic curves dated before 1865, and searching for elements there that are relevant for our purpose. In doing so, a situation opposite to the previous one arises, since several notions of genera of algebraic curves appear while nothing connected to the number p can be detected.

Let me note that, similarly, one could try to start with a corpus on Abelian functions and find publications that involve the number p. While this exercise yields the majority of the texts found by the citation process, it also suggests new ones, whose interpretation with respect to our issue, however, turns out to be quite perilous. Indeed, as the texts from the citation process show, mathematicians of the time sometimes worked with what they saw as particular values of (earlier versions of) p, such as 1, which corresponds to the case where the Abelian functions are elliptic integrals. This raises the question of whether to select earlier texts where the number 1 could be seen retrospectively as what would become p, even though these texts are not connected to the others by links such as the citations. Added to this is the problem that no appellation in the natural language can be used to detect occurrences of 1 that would be relevant for our purpose: the number p did not bear any name before being called "genus" in the second half of the 1860s. The pervasiveness of the number 1 in mathematical texts thus suggests how delicate the task of avoiding historical artifacts may be in such a selection operation.[7] Faced with such difficulties, I chose to stick to recognition processes supported by the explicit traces that are the citations and the identity of the name "genus," thus guaranteeing more solid foundations for the subsequent inquiry.

The next two sections are devoted to each of the mentioned historical threads, which correspond to tracking backwards the number p and the curve category called "genus," respectively. In both cases, I first make more explicit the processes of corpus formation. Then I present the corresponding narratives

[7] This issue will be commented on further at the end of the next section. The point, in general, is that searching for earlier versions of objects is an enterprise that must always be handled with caution, especially because the link between such versions and the reference point is likely to become more and more blurred as one keeps going back in time.

and analyze them in regard to the “big picture” issue, insisting in particular on the historical intricacy of each situation and on related historiographical questions. The concluding section finally compares these two pictures and reflects on what would be one unified big picture of the past of the genera of curves.

2. Tracking a number

2.1 Formation of the corpus

As just explained, the corpus considered in this section is obtained by selecting the papers that are cited in the two 1865 articles by Clebsch and where p can be found, and by repeating the operation on the references obtained in this manner. I performed this process three times. The reason for this is that a fourth layer only adds texts that deal exclusively with elliptic functions, a situation which is more complicated to handle with respect to the issue of recognizing p through particular values.

The corpus thus gathered is made of 29 texts by 13 authors. These texts have been published over a century, between 1766 and 1865, although most of them date from after the mid-1820s. They are represented, together with their explicit[8] citation links, in Figure 1.

As this graph immediately suggests, the situation is quite entangled and cannot be reduced to a linear sequence of texts nicely organized with one another and going from one chronological boundary to the other. But there exist other features which contribute to the complexity of the picture and which cannot be guessed from the appearance of the graph. On the one hand, the citations themselves are of different kinds, some being precise technical borrowings while others acknowledge previous works that must be surpassed, for example,[9] and do not necessarily concern p directly. On the other hand, delving into the texts and focusing on how the number p is involved reveals a variety of mathematical themes that are often technically independent from one another – in the sense that the associated theorems and proofs do not rely on each other – but may still coexist in some papers. Further, these themes do not necessarily accord well with the citations, some of them being addressed in certain references without being taken up by the texts that cite these references.

For reasons of space, I cannot describe all the texts of the corpus and all their links. I will thus confine myself to a selection that illustrates the above-mentioned entanglement of the situation. This selection is made of texts which form an apparently continuous chain of citations with a limited number

[8] Implicit references are important to take into account, too, but may be much more difficult to grasp. An example will be seen below, in the case of [Jacobi 1832].

[9] On the need to examine carefully the role of the citations and, more generally, on their use in the history of mathematics, see [Goldstein 1999].

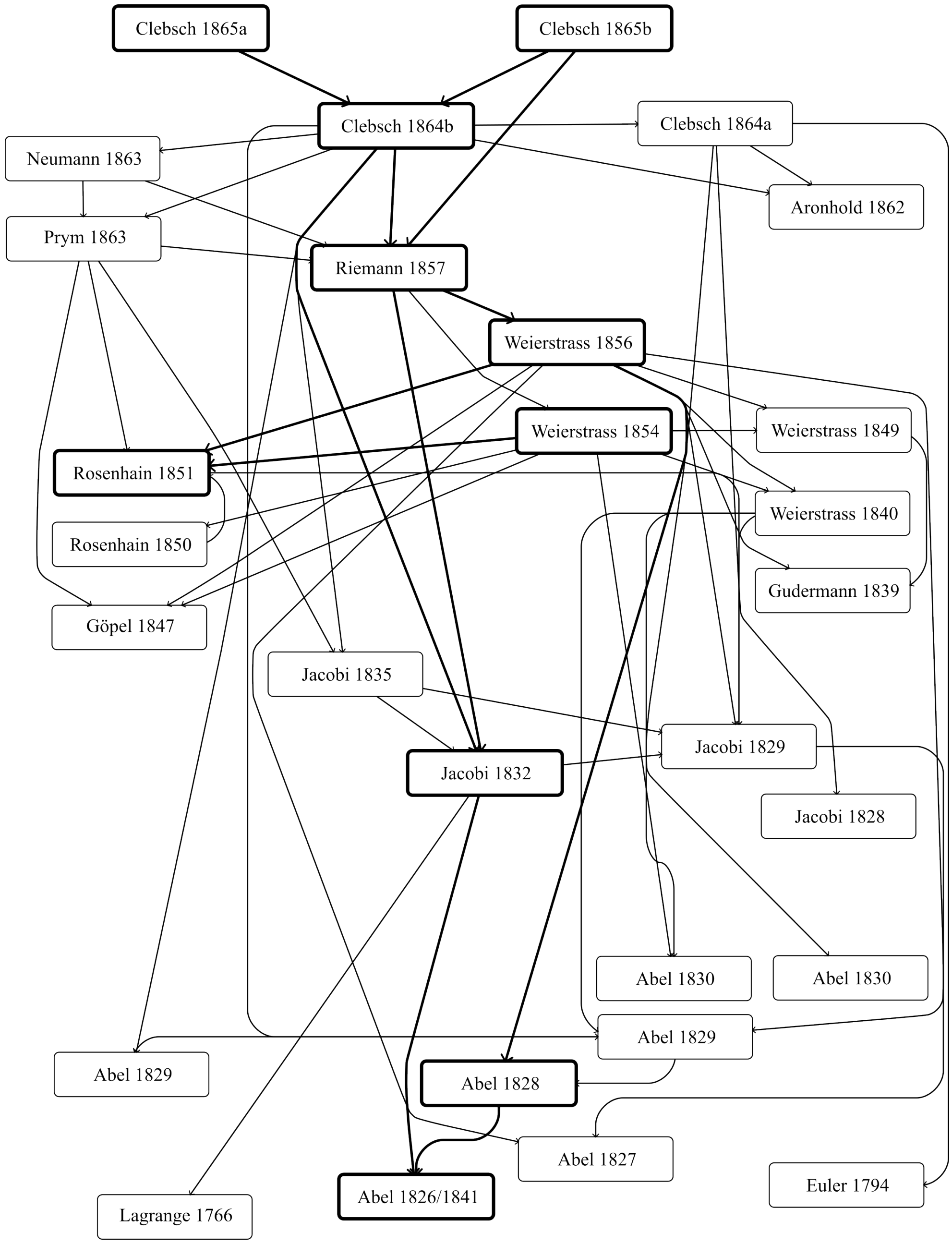

Figure 1: Graph of explicit citations between the texts of the corpus on Abelian functions. These texts are arranged roughly chronologically along the vertical. The bold boxes and arrows correspond to the chosen chain of texts.

of ramifications, in comparison with other possibilities. It contains the two 1865 papers by Clebsch where the genera of curves are defined and the 1864 memoir by the same author that is cited therein, [Clebsch 1864b, 1865a,b], Riemann's memoir on Abelian functions, [Riemann 1857], as well as texts by Carl Weierstrass [1854, 1856], Georg Rosenhain [1851], Carl Gustav Jacob Jacobi [1832], and Niels Henrik Abel [1826, 1828]. I will describe these texts chronologically, concentrating on the occurrences of (earlier versions of) p, on Abel's addition theorem and the inversion problem, and on the connections between the texts.[10]

2.2 Local descriptions...

The first text in our chain is the long memoir that Abel wrote and sent to the French Academy of Science in 1826, but was published only posthumously, in 1841, [Abel 1826]. It is part of the corpus because both Abel himself and Jacobi mentioned the 1826 version in published papers. The memoir was devoted to what would later be called Abelian functions, that is, functions "of which the derivatives can be expressed by the means of algebraic equations, all the coefficients of which are rational functions of one and the same variable."[11] Such a function was denoted by

$$\psi(x) = \int f(x, y)\, \mathrm{d}x,$$

where f is a rational function and y is a function of x defined implicitly by a polynomial equation $\chi(x, y) = 0$. Abel presented these functions as generalizations of rational and elliptic integrals. These integrals, indeed, correspond to the case where $\chi(x, y) = y^2 - p_0(x)$ for a polynomial p_0 of degree 1 or 2, or 3 or 4, respectively. They were taken as examples in the last section of the memoir, together with the case where p_0 is of degree 5 or 6, which leads to hyperelliptic integrals.[12]

The main result in Abel's memoir was the addition theorem. It was introduced as encompassing the known theorems according to which the sum of rational integrals is a rational integral, and that the sum of elliptic integrals is an elliptic integral whose argument is determined algebraically as a function of

[10] Almost all the selected texts have been analyzed in other historical works, without however focusing on the facets that interest me here. See for instance [Gray 1989] or [Houzel 2002], as well as other references that will be given below. Furthermore, my main objective here is historiographical, which is not the case of such past research.

[11] "[Les fonctions] dont les dérivées peuvent être exprimées au moyen d'équations algébriques, dont tous les coefficients sont des fonctions rationnelles d'une même variable" [Abel 1826, pp. 176–177].

[12] More generally, hyperellptic integrals correspond to $\chi(x, y) = y^2 - p_0(x)$ with p_0 of degree greater than 4. Abel did not use the phrase "hyperelliptic integrals." The terminology was not settled at the time, and the corpus even allows to follow part of its evolution. I will nevertheless elude this issue for the sake of simplicity.

the given data.[13] One form of the addition theorem given by Abel states that if $\psi(x_1), \dots, \psi(x_\alpha)$ are α values of an Abelian function ψ, there exist an integer μ as well as algebraic functions $x_{\alpha+1}, \dots, x_\mu$ and an algebraic-logarithmic function v of quantities associated with $x_1, \dots, x_\alpha$, such that[14]

$$\psi(x_1) + \cdots + \psi(x_\alpha) = v - (\psi(x_{\alpha+1}) + \cdots + \psi(x_\mu)).$$

The number $\mu - \alpha$ corresponds to what would coincide with p decades later, at least under some conditions on the equation $\chi(x, y) = 0$. Abel highlighted that this number is "very remarkable,"[15] and he devoted many pages to investigate it. In particular, in the final examples of the paper, Abel found that $\mu - \alpha = 0$, 1 or 2 for rational, elliptic or hyperelliptic integrals for which p_0 is of degree 5 or 6, respectively, [Abel 1826, pp. 256–260].

The lack of response from the French Academy of Sciences to the submission of his 1826 memoir prompted Abel to write an article that would be published in Crelle's *Journal für die reine und angewandte Mathematik*, [Abel 1828]. In the introduction, Abel first stated the addition theorem for Abelian functions, referring to his 1826 manuscript. Then he explained that his present aim was to prove the theorem in the case of hyperelliptic integrals

$$\psi(x) = \int \frac{r(x)}{\sqrt{R(x)}}\, \mathrm{d}x,$$

where r is a rational function and R a polynomial.[16] In fact, for some intermediary results, Abel was led to consider integrals having the specific form

$$\int \frac{\delta_0 + \delta_1 x + \cdots + \delta_{m-2} x^{m-2}}{\sqrt{R(x)}}\, \mathrm{d}x,$$

with $R(x)$ of degree $2m - 1$ or $2m$. Retrospectively, one can see in this formula the general expression of the integrals of the first kind associated with $\chi(x, y) = y^2 - R(x)$, that is, integrals that remain finite everywhere – a related fact is that the maximal number of linearly independent such integrals is $m - 1$, since the numerator of the integrated fraction depends on $m - 1$ coefficients.

Further, these integrals helped Abel prove the addition theorem for hyperelliptic integrals: if R is of degree $2m - 1$ or $2m$, and if $x_1, \dots, x_{\mu_1}$, $x'_1, \dots, x'_{\mu_2}$ are any variables, there exist algebraic functions $y_1, \dots, y_{m-1}$ of these variables, such that

$$\psi(x_1) + \cdots + \psi(x_{\mu_1}) - \psi(x'_1) - \cdots - \psi(x'_{\mu_2}) = v + \varepsilon_1 \psi(y_1) + \cdots + \varepsilon_{m-1} \psi(y_{m-1}),$$

[13] The theorem on elliptic integrals is usually attributed to Leonhard Euler, including by several authors in the corpus, as will be exemplified below.

[14] Abel also showed how to replace the sign $-$ by a sign $+$, and how to extend the theorem so as to treat linear combinations of values $\psi(x_i)$ with rational coefficients. On this addition theorem and on other results that have been called "Abel's theorem," see [Kleiman 2004].

[15] "Dans cette formule le nombre des fonctions $[\psi(x_{\alpha+1}), \dots, \psi(x_\mu)]$, est très-remarquable." [Abel 1826, p. 210].

[16] With the 1826 notations, it corresponds to $\chi(x, y) = y^2 - R(x)$ and $f(x, y) = \frac{r(x)}{y}$.

where v is an algebraic-logarithmic function and the ε are ± 1. In particular, and although Abel did not write it explicitly, comparing this formulation of the theorem with the 1826 one shows that $\mu - \alpha = m - 1$ in the present case.

While the 1826 memoir could not be read by most mathematicians before 1841, the version published in Crelle's journal did circulate. Even if Jacobi did not cite the latter explicitly in his paper belonging to our sub-corpus, the many attributions to Abel of results that are contained in it leaves little doubt of the fact that he drew upon it.[17] Jacobi's general framework was hyperelliptic integrals, which he presented based on known results on rational and elliptic integrals. In particular, Jacobi underscored the importance of the addition theorem, which he explicitly attributed to Abel and presented as the generalization of Euler's theorem on elliptic integrals

$$\Pi(x) = \int_0^x \frac{\mathrm{d}x}{\sqrt{X}},$$

where X is a fourth-degree polynomial. After having stated Euler's theorem, Jacobi turned to particular hyperelliptic integrals

$$\Pi(x) = \int_0^x \frac{A + A_1 x}{\sqrt{X}}\,\mathrm{d}x,$$

with X of degree 5 or 6, and then to general hyperelliptic integrals

$$\Pi(x) = \int_0^x \frac{A + A_1 x + \cdots + A_{m-2} x^{m-2}}{\sqrt{X}}\,\mathrm{d}x,$$

with X of degree $2m - 1$ or $2m$. For these last integrals, the addition theorem was stated under the form that for any variables $x, x_1, \ldots, x_{m-1}$, there exist algebraic functions $a, a_1, \ldots, a_{m-2}$ of these variables, such that

$$\Pi(x) + \Pi(x_1) + \cdots + \Pi(x_{m-1}) = \Pi(a) + \Pi(a_1) + \cdots + \Pi(a_{m-2}).$$

Jacobi then highlighted that this theorem implies that the sum of any number of $\Pi(x_i)$ can be expressed by a sum of $m - 1$ values $\Pi(b_i)$: this result corresponds to the addition theorem contained in [Abel 1828] and stresses the role of the number $m - 1$, for which Jacobi used the same notation as Abel.

The number $m - 1$ also appeared in original results by Jacobi, including the statement of the so-called inversion problem. The initial question was to find the reciprocal function of $\Pi(x)$, i.e. to find $\lambda(u)$ such that $u = \Pi(x)$ if and only if $x = \lambda(u)$. Jacobi first recalled that it had already been solved for rational and elliptic integrals. Moving on to hyperelliptic integrals and using the addition theorem, he explained that the adequate way to tackle the

[17] On the other hand, Jacobi explicitly mentioned the 1826 memoir in a footnote, explaining that it had been sent to the Academy of Sciences and that it should be published, [Jacobi 1832, p. 397].

inversion was to introduce several variables. Thus, if X is of degree 5 or 6, instead of considering the single equation $u = \Pi(x)$, the idea was to invert the system

$$\begin{cases} u = \Phi(x) + \Phi(y) \\ v = \Phi_1(x) + \Phi_1(y), \end{cases}$$

where $\Phi(x) = \int \frac{\mathrm{d}x}{\sqrt{X}}$ and $\Phi_1(x) = \int \frac{x\,\mathrm{d}x}{\sqrt{X}}$. Similarly, Jacobi indicated that the inversion problem for hyperelliptic integrals with X of degree $2m-1$ or $2m$ consisted in inverting a system of $m-1$ equations in $m-1$ unknowns.

If Jacobi indicated the right way of tackling the inversion problem, the effective solution of the latter was carried out by some of his successors. In particular, it was tackled in the case of hyperelliptic integrals corresponding to a polynomial X of degree 5 or 6 in a paper that Rosenhain sent to the French Academy of Sciences in 1846 at the occasion of the *concours* of that year. The paper was successful and was published five years later, [Rosenhain 1851]. Jacobi's name was associated with the inversion problem only through vague references; explicit citations concerned mostly the *Fundamenta nova*, [Jacobi 1829], from which Rosenhain took the idea to consider θ-functions to solve the problem. Jacobi, indeed, had used θ-functions of one variable to study elliptic functions, and Rosenhain defined θ-functions of two variables to solve the inversion system with two unknowns and two equations. Hence one sees, both in the data of the inversion problem and in the definition of the new θ-functions, the importance of the number 2, a number that can be recognized as a special value of $m-1$ in the light of Abel's and Jacobi's works. Rosenhain, however, never mentioned this number $m-1$, nor did he explicitly comment on the significance of 2 in the problem he dealt with.[18]

Later, Weierstrass investigated and solved the inversion problem for general hyperelliptic integrals. The main lines of the results were first indicated in a short paper, [Weierstrass 1854], before a more complete version was published, [Weierstrass 1856]. While acknowledging Rosenhain's contribution, Weierstrass explained that his predecessor's approach could not be generalized adequately, and thus that he developed a new way to handle the problem. Jacobi's name was mentioned a number of times, most often without explicit reference; on the contrary, Abel's 1828 paper was cited for the proof of the addition theorem for hyperelliptic integrals.

Interestingly, Weierstrass did not adopt Abel's or Jacobi's notations. He fixed a polynomial $R(x) = A_0(x-a_1)(x-a_2)\cdots(x-a_{2\rho+1})$ for a given integer

[18] This example also suggests why the situation can much more complicated when it comes to recognizing the numbers 1 or 2 as particular values of $m-1$ (or p) in works that appeared before, say, 1826.

ρ, the inversion problem being then encapsulated in the differential system

$$\begin{cases} \mathrm{d}u_1 = \frac{1}{2}\frac{P(x_1)}{x_1-a_1} \cdot \frac{\mathrm{d}x_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2-a_1} \cdot \frac{\mathrm{d}x_2}{\sqrt{R(x_2)}} + \cdots + \frac{1}{2}\frac{P(x_\rho)}{x_\rho-a_1} \cdot \frac{\mathrm{d}x_\rho}{\sqrt{R(x_\rho)}} \\ \mathrm{d}u_2 = \frac{1}{2}\frac{P(x_1)}{x_1-a_2} \cdot \frac{\mathrm{d}x_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2-a_2} \cdot \frac{\mathrm{d}x_2}{\sqrt{R(x_2)}} + \cdots + \frac{1}{2}\frac{P(x_\rho)}{x_\rho-a_2} \cdot \frac{\mathrm{d}x_\rho}{\sqrt{R(x_\rho)}} \\ \quad\vdots \\ \mathrm{d}u_\rho = \frac{1}{2}\frac{P(x_1)}{x_1-a_\rho} \cdot \frac{\mathrm{d}x_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2-a_\rho} \cdot \frac{\mathrm{d}x_2}{\sqrt{R(x_2)}} + \cdots + \frac{1}{2}\frac{P(x_\rho)}{x_\rho-a_\rho} \cdot \frac{\mathrm{d}x_\rho}{\sqrt{R(x_\rho)}}, \end{cases}$$

where $P(x) = (x-a_1)(x-a_2)\cdots(x-a_\rho)$. The number ρ thus appears both as the number of equations and variables in the inversion problem, and as the number of constants in the polynomials $\frac{P(x)}{x-a_j}$. In other words, it corresponds to what was denoted by $m-1$ in Abel [1828] and in Jacobi [1832]. Incidentally, this change of notation may be seen as a trace of a shift of focus: since this number played an important role in these matters, Weierstrass adapted the symbols so that it appeared as an entity in its own right, denoted by the single letter ρ.[19]

Riemann's paper, whose framework was completely different, was published only one year after Weierstrass's, [Riemann 1857]. Like Abel, Riemann studied general Abelian functions, that is, primitives of algebraic functions s defined implicitly by a polynomial equation $F(s, z) = 0$. The investigation made use of the surfaces that would soon bear Riemann's name.[20] Riemann studied these surfaces first in a setting independent of Abelian functions. In particular, he characterized them with their connectivity order, a surface being $(n+1)$-ply connected if it can be separated in two pieces by making n cross-cuts. In the special case of a surface associated with an algebraic function s, Riemann then showed that the number n is necessarily even, the connectivity order being thus of the form $2p+1$. The number p thus introduced is the one which would eventually be used to define the genera in Clebsch's papers.

This number was pivotal in Riemann's whole theory of Abelian functions. For instance, Riemann proved that p is equal to the maximal number of linearly independent integrals of the first kind. This resulted from writing such an integral as

$$\int \frac{\varphi(s,z)}{\frac{\partial F}{\partial s}}\,\mathrm{d}z,$$

where φ is a polynomial having p independent coefficients – this encompasses the elliptic and hyperelliptic integrals considered by Abel, Jacobi, and Weierstrass, for which the number of coefficients was denoted by $m-1$ and ρ respectively. However, contrary to these numbers, the number p cannot be read off the

[19] It is more difficult to make such a comparison with Abel's 1826 notation $\mu - \alpha$, since the kind of Abelian functions is not the same.

[20] On Riemann surfaces, see for instance [Scholz 1980]. As is well known, these surfaces were first introduced in Riemann's doctoral dissertation, [Riemann 1851].

expression of the polynomial $F(s,z)$. Instead, Riemann established a formula expressing p as a function of numbers associated with F: if n is the degree of F with respect to s and if w is the number of pairs (s,z) such that $F(s,z) = \frac{\partial F}{\partial s}(s,z) = 0$ and $\frac{\partial F}{\partial z}(s,z) \neq 0$, then

$$p = \frac{w}{2} - (n-1),$$

provided a supplementary technical condition is satisfied.

As for Jacobi's inversion problem, Riemann first stated that Weierstrass had solved it in the hyperelliptic case. Nevertheless, he added that his knowledge of Weierstrass's works consisted mostly in the sketch given in the 1854 paper, and thus that the correspondence "not only in the results but also in the methods leading to them will for the most part only be revealed by the promised detailed presentation" given in [Weierstrass 1856].[21] Riemann did not restrict himself to hyperelliptic integrals: he expressed the inversion problem as the issue of finding, for each system of complex numbers $(e_1, \dots, e_p)$, some values $\eta_1, \dots, \eta_p$ such that

$$(e_1, \dots, e_p) \equiv \left(\sum_{\nu=1}^{p} u_1(\eta_\nu), \dots, \sum_{\nu=1}^{p} u_p(\eta_\nu) \right),$$

where $u_1, \dots, u_p$ form a maximal system of linearly independent integrals of the first kind, and where the congruence sign refers to their periods. The problem was then solved with the help of θ-functions of p variables, the theory of which was developed in the same paper, without mentioning Rosenhain's works where θ-functions of two variables had been introduced.

Finally, Abel's addition theorem also appeared in Riemann's paper, but there was no question in proving it again. Instead, Riemann reformulated it and used it, "following Jacobi [1832], for the integration of a system of differential equations."[22]

A number of Riemann's results were taken up by Clebsch in his 1864 memoir devoted to the application of Abelian functions to geometry, [Clebsch 1864b]. Clebsch explained that Riemann's paper contained everything that was needed for such applications, but that difficulties in reading it had prevented mathematicians from doing so. At the same time, he seemed to be opposed to

[21] "[I]n wie weit zwischen den späteren Theilen dieser Arbeiten und meinen hier dargestellten eine Uebereinstimmung nicht bloss in Resultaten, sondern auch in den zu ihnen führenden Methoden stattfindet, wird grossentheils erst die versprochene ausführliche Darstellung derselben ergeben können." [Riemann 1857, p. 116]. The exact reference to Weierstrass's 1856 paper was given in Riemann's paper but, oddly enough, at the end of the sentence where the 1854 one (which was also cited a few lines before) was described. That Riemann wrote that he would compare the methods and results later seems to indicate that he added the reference quite late, or even that the journal editor added the exact bibliographic data shortly before publication.

[22] "Ich benutze nun nach Jacobi (dieses Journals Bd. 9 Nr. 32 §. 8) das Abel'sche Additionstheorem zur Integration eines Systems von Differentialgleichungen". [Riemann 1857, p. 137]. This refers to a theorem of Jacobi's that I chose not to mention above for the sake of brevity.

using Riemann surfaces to conceptualize integrals, and instead adopted the viewpoint of Jacobi [1832].

Clebsch was first and foremost interested in the study of algebraic curves. At the beginning of his work, he considered a plane algebraic curve defined by an equation $f(x_1, x_2, x_3) = 0$ of degree n between homogeneous coordinates of the plane, which he saw as the locus of the intersection points of the lines of two pencils whose parameters s and z are linked by an equation $F(s, z) = 0$.[23] The similarity with Riemann's notations is no coincidence: it prepared the effective mathematical transfer, especially the transformation of the formula $p = \frac{w}{2} - (n-1)$. Indeed, having linked the curve $f = 0$ to an equation $F(s, z) = 0$ allowed Clebsch to interpret w as the class of the curve, that is, as the number of tangents that can be drawn to it from any (generic) point of the plane. The combination of the well-known equality $w = n(n-1) - d$, where d is the number of double points of $f = 0$,[24] with Riemann's formula eventually yielded

$$p = \frac{(n-1)(n-2)}{2} - d.$$

The number p can be seen in many other places in Clebsch's paper. For instance, it was involved in the homogeneous expression of the integrals of the first kind:

$$\int \Theta \cdot \frac{\sum \pm c_1 x_2 \, \mathrm{d}x_3}{c_1 \frac{\partial f}{\partial x_1} + c_2 \frac{\partial f}{\partial x_2} + c_3 \frac{\partial f}{\partial x_3}},$$

where $\sum \pm c_1 x_2 \, \mathrm{d}x_3$ is the determinant whose elements are the c_i, x_j, and $\mathrm{d}x_k$, and where Θ is a polynomial having p independent coefficients. Just like in Riemann, this expression was associated with the fact that p is the maximal number of independent integrals of the first kind, a system of which is denoted by $u_1, \dots, u_p$.

On the other hand, contrary to Riemann, Clebsch did not deal with Jacobi's inversion problem. Instead, he presented one his main theorems as a consequence of Abel's addition theorem:[25] mn points $x_1, \dots, x_{mn}$ of the curve $f = 0$ are the intersection points of this curve with a curve of order m if and only if

$$\begin{cases} u_1(x_1) + \cdots + u_1(x_{mn}) \equiv 0 \\ \quad \vdots \qquad\qquad\qquad \vdots \\ u_p(x_1) + \cdots + u_p(x_{mn}) \equiv 0 \end{cases}$$

[23] For more details on Clebsch's research on this point, see [Lê 2020, pp. 79–84].

[24] This expression of w had been proved by Poncelet in the 1820s. We will encounter the definition of this concept of class in the next section. In the case studied by Clebsch, its validity comes from Riemann's supplementary condition mentioned above, which amounts to the fact that the curve $f = 0$ has only nodes as singularities.

[25] The reference given by Clebsch was [Abel 1829], and not [Abel 1826] or [Abel 1828]. The cited paper contains what Steven L. Kleiman [2004] called "Abel's elementary function theorem."

modulo the periods of the Abelian functions. As can be seen, p occurs here as the number of equations, which corresponds to the number of integrals of the first kind.

We finally arrive at Clebsch's publications of 1865 where the genera of curves are defined, [Clebsch 1865a,b]. The main reference for this definition was the memoir on the application of Abelian functions to geometry, a reference that was used above all for the formula $p = \frac{(n-1)(n-2)}{2} - d$. This formula was taken as such, without being reworked, and served to gather algebraic curves into genera, as has been seen in our introduction. Riemann's memoir was also cited in [Clebsch 1865b], when Clebsch presented the invariance of p under birational transformations of a curve as "just another clothing of Riemann's theorem."[26] This invariance was used to derive results on algebraic curves, which I will not mention here.

2.3 … of a global picture

In the above narrative, the number p, or some of its earlier versions, can be recognized at each step, whether in the very writing of the hyperelliptic integrals, the addition theorem or the inversion problem. However, the identification of these versions and the establishment of links between them takes different forms according to the cases. In the passage from Riemann to Clebsch, the comparison is made quite easy by a range of features: Clebsch adopted Riemann's notations for p, $F(s, z)$, w, etc., cited him explicitly for the formula for p, and worked at the same degree of generality for Abelian functions, in the sense that the grounding equation $F = 0$ was given by a polynomial of any form and any degree. On the contrary, no such elements are available to connect explicitly Riemann with his predecessors, or to connect some of them with one another. In particular, as mentioned, the fact that the numbers $\mu - \alpha$, $m - 1$, ρ, and p do not bear any name in the natural language contributes to the difficulty of the situation. Hence it is the responsibility of the historian to identify Riemann's p and Abel's $\mu - \alpha$, for instance.

In general, such an identification process is often performed via the understanding of the mathematical content, and it may be more or less immediate and explicit in the historian's mind. This is where features such as citations between texts, even if vague or not directly linked to the investigated object, are useful safeguards. It is with this in mind that I warned against the hazard of recognizing the number 1 as a particular value of p in texts that would deal with elliptic functions all the while being disconnected from sources where p or its earlier versions appear explicitly.[27]

[26] "In der That ist dieser Satz nur eine andere Einkleidung desjenigen, welchen Herr Riemann dieses Journal Band 54 pag. 133 gegeben hat." [Clebsch 1865b, p. 98].

[27] The case of $p = 0$, which is associated with rational (or trigonometric) functions, would be even more problematic because of the greater number of possibly relevant texts.

My analysis also shows that the citations between our selection of texts reflect a spectrum of positions, from the plain adoption to the claim of breaks in approach, by way of ambivalent attitudes, with one author refusing a part of a work while still borrowing some of its results. Moreover, the addition theorem and the inversion problem are examples of specific questions that are important to take into account to understand the evolution of p but have their own dynamics in terms of occurrences, proofs, uses, and reformulations. The succession of the chosen texts, finally, does not reflect any process such as a progressive rise in generality or in rigor.

All this contributes to drawing a bigger picture into which the episode of Clebsch's definition of the genera fits, and which must be thought of with the necessary nuances and caution. In particular, even if one could have thought that extracting a continuous chain of texts would lead to an easy narrative, it is not the case: there is no direct, obvious historical path that would echo a linear process of conceptualization of p and of its occurrences in various problems.

Of course, an even bigger picture could be obtained by taking into account elements that have been ruled out above, such as entire texts, or topics that are connected to p and are tackled in the publications considered above, as is illustrated by Riemann's counting of $3p-3$ constants on which depends the birational classes of equations $F(s,z)=0$ for which $p>1$. Such avenues, which will not be explored here, illustrate the richness of the situation and, consequently, the obstacles with which one is confronted when searching for a global description. To employ a mathematical metaphor, the situation at hand – like many other situations, presumably – is locally linear (and has been locally depicted), but globally much more intricate.

3. Tracking a name

I now turn to the nominal path, associated with the name "genus" and with algebraic curves.

3.1 Formation of the corpus

Let me first recall that none of the references given in Clebsch's 1865 papers introducing the genera of curves deal with a notion bearing the same name. This is why the angle of attack must be changed, for instance by starting from a large corpus related to algebraic curves and then searching in it for texts dealing with any notion of genus.

Because the introduction of algebra in geometry made by René Descartes and Pierre Fermat greatly changed methods of dealing with curves, I chose as the lower bound of my time interval the year 1637, when *La Géométrie* was published and *Ad locos pianos et solidos isagoge* was (approximately) written. The corpus thus extends over almost two centuries and a half, from 1637 to

1865.

To collect publications on algebraic curves, I first used the *Catalogue of scientific papers*, which enables surveying the period 1800–1865. Specifically, the *Catalogue* contains a section devoted to "Algebraic Curves and Surfaces of degree higher than the second," and I considered all the papers referenced in the "General" subsection and in the subsections on plane curves, which represents 368 articles. For earlier publications, I drew, on the one hand, upon Jeremias David Reuss' *Repertorium commentationum* [1808], of which the chapter devoted to mathematics contains sections on algebraic curves: 74 articles, published between 1694 and 1795, are thus retained. To complete the corpus, I eventually gathered all the primary references listed in Carl B. Boyer's *History of analytic geometry* and dated between 1637 and 1799, which adds 80 new references.[28]

Then I investigated these 522 texts, searching in particular for any notion called *Geschlecht*, which is the German original word employed by Clebsch. In order not to restrict myself to German publications, words used at different times as explicit foreign equivalents of *Geschlecht* have also been taken into account: the Latin *genus*, the French *genre*, the Italian *genere*, the English "genus," as well as a few other ones.[29]

The examination of these words reveals two important points. First, they all refer to categories of curves, and prior to Clebsch's 1865 definition there were four main notions of genera successively proposed by Descartes (1637), Newton (1704), Euler (1748) and Cramer (1750).[30] Second, the context in which genera of curves are introduced and used is always that of curve classifications, and, except for the case of Descartes, they are accompanied by considerations of other taxa of curves, such as orders, classes, and species. Although special attention is paid to genera, to include these categories in the study thus helps build a more coherent narrative and understand phenomena that concern genera. I will elaborate on this issue after describing the most salient points of

[28]Only 5 references appear both in Reuss and in Boyer. Moreover, a historical work not reduced to being a topical bibliography, Boyer's book, [Boyer 1956], is different in nature than the *Catalogue* and Reuss' *Repertorium*. The main reason that I chose to rely on it is that Boyer used bibliographies made by mathematicians of the past to which I could not access. Of course, my narrative has been constructed independently from Boyer's, who, besides, dealt with a different historical issue.

[29]Let me recall that the investigation follows here a nominal way. Thus the question is not to locate concepts which would correspond mathematically to genera but would have a completely unrelated name. That said, it turns out that the corpus does not bear any trace of such concepts. As these observations suggest, I carried out a systematic reading of the texts, and not just a textual search for a few key words, precisely to detect possible problems. There are about 30 texts where the name "genus" or one of its foreign equivalents appear.

[30]A few other technical or semi-technical meanings of the genus of curves can be found, and will be accounted for below. Non-technical meanings are not retained as relevant for the present study: for instance, the French *genre* is used in phrases such as "*ce genre de considérations*," which can be translated by "this kind of considerations," or "this type of considerations."

this narrative.[31]

3.2 Classifications of curves

The first genera that can be found in our corpus are defined by René Descartes in *La Géométrie*, [Descartes 1637]. In the second book of this text, Descartes proposed to "distinguish [curves] by order in certain genera"[32] by using the equations defining them. More precisely, if $n > 1$, the nth genus encompassed the curves whose equation is of degree $2n-1$ or $2n$. As for the first genus, it consisted only of curves with an equation of the second degree because Descartes did not recognize straight lines as curves – significantly, he called the latter "curved lines."

Descartes justified this way of classifying curves only by alluding to the fact that the "difficulties" of the fourth degree can be "reduced" to the third degree, and that those of the sixth degree can similarly be reduced to the fifth degree.[33] This justification referred to the possibility of reducing equations with one unknown of degree 4 to equations of degree 3, a possibility that Descartes apparently believed to be true also for higher degrees and for equations with two unknowns.[34]

The classification proposed by Descartes was received in different ways in the 17th century. Fermat, for one, criticized it in his *Dissertatio tripartita* and preferred to classify curves degree by degree, calling "species" the families of curves having an equation of the same degree.[35] Another mathematician who adopted such a negative opinion of Descartes' classification is Jacob Bernoulli [1695], who also preferred grouping curves by degrees. On the other hand, Descartes' genera were fully integrated in the works of other mathematicians, such as Frans van Schooten [1657] and Jacques Ozanam [1687].

In the corpus under scrutiny, only one 17th-century reference contains a notion of genus of curves that is not related to Descartes'. This is a book by John Craig, where genera are used to divide the totality of curves, be they algebraic or not. Specifically, algebraic curves themselves form the first genus, while transcendent curves are classified into the other genera according to

[31] A similar study is proposed in [Lê 2023], although the corpus is a bit different and no special emphasis is made on the genera.

[32] "[D]istinguer [les lignes courbes] par ordre en certains genres" [Descartes 1637, p. 319]. On *La Géométrie*, see [Bos 2001; Serfati 2005; Herreman 2012, 2016].

[33] "[I]l y a règle générale pour réduire au cube toutes les difficultés qui vont au quarré de quarré, & au sursolide toutes celles qui vont au quarré de cube" [Descartes 1637, p. 323].

[34] See for instance [Bos 2001, p. 356]. Another interpretation of Descartes' grouping of curves is given in this reference, and is linked with the issue of constructing curves associated with the problem of Pappus and with the Cartesian classification of geometrical problems.

[35] The *Dissertatio* is edited in [Fermat 1891, pp. 118–132]. According to [Mahoney 1999, p. 130], it has probably written at the beginning of the 1640s. See also a 1657 letter from Fermat to Kenelm Digby where the arguments of the *Dissertatio* are taken up and made more explicit, [Fermat 1999, pp. 491–497].

further criteria, [Craig 1693, p. 42]. Such a definition, however, does not seem to have been taken up by later mathematicians.

This contrasts with the notion introduced by Isaac Newton in his famous *Enumeratio linearum tertii ordinis*, published in 1704 as an appendix to the treatise *Opticks*, [Newton 1704].[36] At the very beginning of this text, Newton asserted that lines can be divided into orders "according to the dimensions of the equation expressing the relation between absciss and ordinate, or, which is the same thing, according to the number of points in which they can be cut by a straight line."[37] The nth order was thus made up of the lines defined by an equation of degree n. Furthermore, echoing the distinction between lines and curves, Newton proposed a parallel classification of curves, the corresponding categories being genera. More precisely, the nth genus of curves was that of the curves defined by an equation of degree $n + 1$, so that "[a] curve of the second genus is the same as a line of the third order."[38]

The title of his work made clear that Newton's aim was to classify the lines of the third order. In fact, past this title and the very first lines, Newton mainly used the vocabulary of curves in the text, and he divided the curves of the second genus into 72 species. Such a terminology thus refers to the usual connection between genera and species, the orders not being common categories of classification at the time.[39]

Many publications from the first half of the 18th century used the same classifying vocabulary. They often involved both lines and curves, even though a certain prevalence of the former can be observed. For instance, James Stirling's *Lineae Tertii Ordinis Neutonianae* only dealt with lines and their orders, [Stirling 1717], while François Nicole also mentioned curves and their genera in his *Traité des lignes du troisième ordre, ou des courbes du second genre*, [Nicole 1731]. Such works were direct continuations of Newton's *Enumeratio*, but orders and genera appeared in other types of publications. One of them is a 1705 book by Nicolas Guisnée, entitled *Application de l'algèbre à la géométrie*, where the double classification was adopted although Newton was not referred to, [Guisnée 1705]. Other examples are Edmund Stone's *New Mathematical Dictionary*, [Stone 1726], and Colin Maclaurin's *Treatise of Algebra*, [MacLaurin 1748], where the English words used to refer to what Newton designated as genera are "genders" and "kinds," respectively. Maria Gaetana Agnesi [1748], for her part, also spoke both about orders of lines and genera of curves, the Italian words being *ordine* and *genere*. But she also made use of the latter term in a less technical sense, like when she described the

[36]On the *Enumeratio*, see [Guicciardini 2009, pp. 109–136].

[37]"Lineae Geometricae secundum numerum dimensionum aequationis qua relatio inter Ordinatas & Abscissas definitur, vel (quod perinde est) secundum numerum punctorum in quibus a linea recta secari possunt, optimè distinguuntur in Ordines." [Newton 1704, p. 139]. The given English translation comes from [Talbot 1860].

[38]"Curva secundi generis eadem cum Linea Ordinis tertii." [Newton 1704, p. 139].

[39]On this point, see [Lê 2023, pp. 96–100].

curve defined by $a^{m-1}x = y^m$ as a "curve of the genus of the parabolas," a phrase which recalls the taxonomic connotation of the word.[40]

No publication, however, used the vocabulary of curves and genera without involving that of lines and orders. Moreover, in many cases the terms related to curves appeared only in the titles of the papers or of their sections, or were hardly used in the statement of theorems and their proofs, unlike "lines" and "orders."[41] Lines and orders are therefore the objects that were mainly used in practice during in the first half of the 18th century.

The coexistence of lines and curves was abolished explicitly in two major books published in the middle of that century. One of them is Leonhard Euler's *Introductio ad analysin infinitorum*, whose second volume contained chapters devoted to the theory of algebraic curves [Euler 1748]. Euler first defined the orders of "curved lines" via the degree of the equations. Further, to encompass the case of straight lines, he explained that since it would be inappropriate to qualify the latter as curves, he would only speak about "lines" to refer to both cases, [Euler 1748, p. 26].

Genera in the Newtonian sense thus disappeared together with the distinction between lines and curves, but Euler did introduce genera to divide lines of a given order. For instance, in the chapter devoted to the lines of the third order, Euler first explained that they can be classified into species, according to the number and the nature of their infinite branches. After having brought out 16 such species, he emphasized that they are not the same as Newton's 72 species, and he showed how these 72 can be distributed into his 16. At the end, he added:

> Most of these species are so extensive that they each include quite considerable varieties, if we consider the shape they present in a finite space. It is for this reason that Newton multiplied the number of species, in order to distinguish one from another the curves that offer notable differences in this space. It will therefore be more appropriate to call *Genera* what we have designated as *Species*, and to refer to *Species* the varieties they contain.[42] [Euler 1748, p. 126]

In other words, Euler adjusted the terminology to fit with the great number of categories he had to deal with, and what he had first called "species" were renamed "genera."

A similar situation occurred in Gabriel Cramer's *Introduction à l'analyse*

[40] "$a^{m-1}x = y^m$, curva del genere delle parabole." [Agnesi 1748, p. 940].

[41] This is thus the exact opposite of Newton's case.

[42] "Species autem hae plerumque tam late patent, ut sub unaquaque varietates fatis notabiles contineantur; si quidem ad formam, quam Curvae habent in spatio finito, respiciamus. Hancque ob causam Newtonus numerum specierum multiplicavit, ut eas Curvas, quae in spatio finito notabiliter discrepant, a se invicem secerneret. Expediet ergo has, quas *Species* nominavimus, *Genera* appellare, atque varietates, quae sub unoquoque deprehendantur, ad *Species* referre."

des lignes courbes algébriques, [Cramer 1750].[43] In the preamble of the book, Cramer recalled that "Algebra alone provides the means to distribute Curves into Orders, Classes, Genera & Species" and that "it is to the illustrious Newton that Geometry is most indebted for this distribution."[44] Later, Cramer defined the orders of lines through the degree of their defining equation. He then explicitly recalled the old distinction between lines and curves:

> Mr. Newton distinguishes between the Orders of Lines & the Genera of Curves. Since the first Order contains only the straight Line [...], he calls Curves of the first Genus the Lines of the second Order, Curves of the second Genus the Lines of the third Order, & so on. However reluctant one may be to deviate from the denominations established by this Great Man, it seemed to me that this expression was too cumbersome in terms of expression, & I decided to say indifferently Curves or Lines of the second Order, Curves or Lines of the third Order, &c.[45] [Cramer 1750, p. 53]

Just like in Euler, the notion of genus that Newton had defined thus vanished with the disappearance of the difference between curves and lines. Although neither of these authors explained why they favored lines and orders, it is most likely that these objects survived because their numbering corresponds exactly to the degrees of the equations.

Genera of lines can still be found in Cramer's text, as subcategories of each order that reflect properties related to the infinite branches. In particular, Cramer classified curves of the third order into 14 genera, and he systematically indicated their correspondence to Newton's species. For instance, the curves with three concurrent asymptotic lines were gathered in one genus, which "contains the nine species of redundant hyperbolas whose asymptotes intersect at one point. Newton, Nb. 4."[46]

Comments on the appropriate classification scale appeared when Cramer treated the case of curves of the fourth order. Imitating the approach he used in the third-order case, Cramer was first led to consider eight cases of equations

[43] On this book, see [Joffredo 2017, 2019]. As Cramer explained in the beginning of the *Introduction*, he only became aware of Euler's *Introductio* after he had developed his own ideas. The two mathematicians were in epistolary communication during the 1740s, though. See [Joffredo 2019, pp. 250–258].

[44] "[L]'Algèbre seule fournit le moyen de distribuer les Courbes en Ordres, Classes, Genres & Espèces" and "c'est à l'illustre Newton que la Géométrie est surtout redevable de cette distribution" [Cramer 1750, p. VIII]

[45] "Mr. Newton distingue les Ordres des Lignes & les Genres des Courbes. Comme le premier Ordre ne renferme que la Ligne droite [...], il appelle Courbes du premier Genre, les Lignes du second Ordre, Courbes du second Genre, les Lignes du troisième Ordre, & ainsi de suite. Quelque répugnance qu'on ait à s'écarter des dénominations établies par ce Grand Homme, il m'a paru que cette distinction génoit trop l'expression, & je me suis déterminé à dire indifféremment, Courbes ou Lignes du second Ordre, Courbes ou Lignes du troisième Ordre, &c.".

[46] "Ce Genre contient les neuf espèces d'Hyperboles redondantes dont les trois asymptotes se croisent en un point. Newton, N.º 4." [Cramer 1750, p. 362].

of curves. The three first cases yielded 1, 6, and 9 genera, respectively. But when Cramer treated the next case, he indicated that the number of genera was too great to be completely listed: "It would be impossible to enumerate all the genera of curves included in this IVth Case: but they can be reduced to five Classes."[47] In other words, since the genera became too numerous to be enumerated exhaustively, Cramer decided to introduce a new type of curve categories, situated between the fourth order and its genera.

Strikingly, no technical use of the word "genus" can be found in the texts of our corpus between Euler's and Cramer's books and 1833, when Euler's genera were mentioned again. On the contrary, the orders of lines (or curves, the difference being abandoned) were absolutely commonplace in the texts that were published during this period. Their use caused many comments at the occasion of a specific episode, during which new "classes" of curves were introduced.

This episode is the famous duality controversy, which, at the end of the 1820s, opposed Joseph-Diez Gergonne and Jean-Victor Poncelet.[48] Let me briefly recall that Gergonne's principle consisted in associating a dual theorem with any given theorem, the two being related by the exchange of the words "points" and "lines," and of associated verbs and adjectives. However, Gergonne made the mistake to proceed as though curves of order $n > 1$ were replaced by curves of the same order in this process: this is one of the points that Poncelet used against him. When he reworked his theory, Gergonne [1827] introduced the notion of "class," a curve being of the nth class if n tangents can be drawn to it from a given point of the plane. The principle of duality could then be corrected by making curves of the nth class correspond to curves of the nth order.

Classification issues were part and parcel of Gergonne's and Poncelet's arguments. On the one hand, Gergonne emphasized that orders and classes provided two ways of classifying curves which were dual to one another. On the other hand, Poncelet blamed Gergonne for having "admitted simultaneously two essentially different classifications for curves," and he insisted that he himself did not "shy away from the difficulty of preserving [to the classification] of curves [its] legitimate and universally [accepted] definition."[49] If Poncelet thus refused the classification itself, he was still interested in evaluating the number of tangents that can be drawn to a curve, and he found that this number is $n(n-1) - 2d$ if the curve is of order n and has only d nodes as

[47] "On ne saurait énumérer tous les genres des courbes comprises dans ce IVe Cas : mais on peut les réduire à cinq Classes." [Cramer 1750, p. 379].

[48] On this controversy, see [Lorenat 2015] and the references given on p. 547, as well as [Etwein, Voelke, and Volkert 2019].

[49] "[I]l ne s'est agi de rien moins que de torturer le sens des mots, en admettant simultanément deux classifications essentiellement distinctes pour [les] courbes." Further: "je n'ai pas reculé devant la difficulté de conserver aux classifications des courbes [...] leur définition légitime et universellement admise." [Poncelet 1828, pp. 300, 302].

singularities.

In spite of Poncelet's hostility, the notion of class defined by Gergonne was adopted quite quickly by the 19th-century geometers. In the corpus, the first one to use it is Julius Plücker, who published many papers on algebraic curves during the 1830s,[50] including those where he completed Poncelet's formula for the class-number of a curve.

Genera are mentioned in one of these papers, which is a presentation of the book *System der analytischen Geometrie*, [Plücker 1833, 1835]. As this presentation explained, one of the aims of the book was to rework the classification of the curves of the third order. Plücker rooted the question in Newton's *Enumeratio* and declared that Euler, though having made progress in the theory of infinite branches, did not succeed in settling the issue. The genera that Euler had defined were thus briefly evoked in this discussion – the presentation paper, written in French, used the word *genres*, while the book itself referred to these categories by the term *Geschlechter*. However, no genera were involved in Plücker's own classification, which consisted in dividing the third order into species only.

Only a very few occurrences of genera occur in the corpus between 1833 and 1865. They either correspond to concepts defined by past mathematicians, or concepts that were used in a less technical way, just like with Agnesi's example mentioned above. For instance, Joseph Dienger devoted a paper published in 1847 to a "curve deriving from the ellipse, of the genus Conchoid"[51] and Ernest De Jonquières, in a 1861 article, was led to consider a "curve of the fourth order, of the genus of the *lemniscates*."[52] Finally, a paper published in 1863 by a certain F. Lucas also employed the term "genus" as he referred to Newton's works on the classification of third-order curves, [Lucas 1863].

Such a paucity shows that when Clebsch defined the *Geschlechter* in 1865, the word was, so to speak, available to be endowed with a new technical definition. Moreover, we saw that when Clebsch did so, the idea of introducing a new classification of curves was clear. A notable difference between his *Geschlechter* and the other notions of genus that we encountered is that these genera were not presented as subcategories of orders, quite the opposite. The idea was to divide the totality of algebraic curves into genera, each of them being then subdivided according to the orders. This classification program was thus different from Gergonne's, for whom the distribution into classes was not supposed to replace that into orders, but to offer another way to conceive of the division of curves.

[50]The adoption of the notion of class by Plücker and other mathematicians is described in [Lê 2023, pp. 110–116].

[51]"Note sur une Courbe dérivant d'une ellipse (du genre Conchoïde)", [Dienger 1847, p. 234]. The words that are here between parentheses are in a footnote in the original paper.

[52]"[La courbe...] est une courbe du quatrième ordre, du genre des *lemniscates*." [Jonquières 1861, p. 211].

3.3 Continuities and discontinuities

If Clebsch's definition of the genera is thus embedded into a narrative that extends over almost two centuries and a half, it is true that there is no direct, effective link made by Clebsch between this episode and the previous ones – his mention of the classification by orders, which has been reported in the introduction of this chapter, appears more as a general mathematical fact than a historical reference *per se*.

More generally, the genera proposed by Descartes, Newton, Euler, Cramer, and Clebsch, have no direct mathematical link to one another. They cannot be seen as different instances of one and the same concept which would correspond to different degrees of generality of the associated algebraic curves, for example. The situation is thus distinct from that of the numbers p, ρ and $m-1$, the first becoming equal to the other two if the framework of Abelian functions in which it is defined is particularized. Nor are they concepts, each of which would have been developed from the previous one following a discussion of its relevance and an attempt to replace it. Significantly, if Cramer did refer explicitly to Newton to explain why the latter's genera should be abandoned, his own genera referred to something else, and the bridge with Newton was made on the level of species.

From this point of view, isolating the works of the five above-mentioned mathematicians, focusing on their definition of genera solely, and juxtaposing their description would yield quite a discontinuous result, both in terms of chronology and intellectual dynamics. Greater coherence is achieved by taking into account both other works that contain a notion of genus (be it a simple adoption of Descartes' notion, for example, or a semi-technical one) and works that involve other categories of curves, namely orders, classes, and species: in doing so, the historian weaves a more complete and continuous fabric related to the classifications of curves, in which the episodes of the definitions of the genera can be advantageously included.

As in the previous section, this bigger picture could be extended or made more complex. For instance, most of the texts that have been kept silent above do not contain any new notion of genera, orders, classes, or species, but they could be used to get a view of the range of the mathematical questions in which these categories have been involved, or to analyze the process of their banalization. Finally, questions revolving around the tension between the categories of curves and the numbers that characterize them, or the mathematical links between these numbers, could also be investigated further.

4. Bigger pictures

The two pictures obtained by following the paths of the number p and of the name "genus" are essentially disjoint in terms of content, which reflects the historical divide between the theory of Abelian functions and the theory of algebraic curves before Clebsch began to systematize their interactions in 1864. Most mathematicians of our corpora, indeed, only contribute to one of these pictures, the three exceptions being Clebsch, Jacobi, and Siegfried Aronhold, among whom only the first has papers that belong to both corpora. The mathematical questions and the techniques that are deployed to address them are also proper to both pictures, except for these three mathematicians' cases – Clebsch's interpretation of the number of ramification points as the class of a curve nicely illustrates this point.[53]

The two narratives have also their own timelines and their own pace, with one extending over almost two centuries and a half while the other is mainly concentrated in forty years. As explained in the introduction, this difference does not merely stem from a convention of corpus formation. It reflects an asymmetry in the process of identifying earlier versions of mathematical objects and in the limits with which the historian is confronted when doing so. In particular, the sameness[54] that underlies each picture is not of the same nature, as is reflected by the differences between the intellectual dynamics that govern the succession of the diverse concepts of genera and that of the numbers $\mu - \alpha$, $m - 1$, ρ, and p. These numbers, indeed, can be seen as being the same, provided the technical frameworks with which they are associated are adjusted to one another. On the contrary, what remains stable in the nominal path – and seems to allow a more straightforward writing of a long-term history – is a framework of curve classifications, a framework that is general enough to give rise to different ways of approaching it, and to answers having little in common apart from the names of the considered categories of curves.

As disconnected as the two pictures that have been obtained can be, both have their own kind of coherence, and there is no question of ranking them, by asserting that one would be more relevant or more significant than the other. Quite simply, they refer to two types of connection of a newly defined object with its past.

This observation takes us eventually back to the issue of drawing one picture of the past of the genera into which the episode of Clebsch's 1865 definition would be embedded.

Considering what has been done in this chapter, I think it would be misguided to aim at writing a final result consisting in one unified narrative, well-ordered chronologically, and integrating all the above elements. In fact, what has been proposed corresponds better to what should be done to draw a bigger picture, in my view: the construction of corpora on the basis of explicit criteria, their systematic study with regards to determined objectives, the comparison of the obtained results, and a reflection on the possibility of

[53] Aronhold and Jacobi mix together elliptic functions and algebraic curves in papers that belong to the corpus on Abelian functions, [Jacobi 1828; Aronhold 1862]. Their works are analyzed and compared to Clebsch's approach, [Clebsch 1864a], in [Lê 2018].

[54] I borrow this term from [Goldstein 2019].

merging them or not. In particular, I do not see what has been presented in the chapter as simply setting two narratives down along side each other, even though these narratives are related to quite autonomous historical situations.

Of course, the result is still a partial picture, since other entire tracks could be followed and yield other pictures with their own content, dynamics, and chronologies. For instance, one could try tracing back the topological interpretation of the number p as the half of the number of sections required to disconnect a surface, or researching the past of the number $\frac{(n-1)(n-2)}{2}$, which would lead to the path of curve singularities. On the other hand, focusing on synchronicity rather than diachronicity, one could try to embed each of the episodes that compose our two narratives into thicker wholes extending even beyond mathematics, for instance by considering the practice of classification in science at different moments between the 17th and the 19th centuries in Europe, or by investigating how the epistemic values shared in given European universities could have impacted their mathematicians' ways of thinking of Abelian functions and curves.

Such possibilities are legion, because any human activity is entangled in various intellectual, social, cultural, and institutional configurations, and because any work of any mathematician comes within the framework of its present and its past. Confronted to such an endless spectrum of tracks, however, one is not doomed to capitulate and stay within the relative safety of local studies.

Facing the fantasy of absolute exhaustiveness, indeed, systematicity presents itself as a more reasonable option for drawing bigger pictures. From the formation of relevant corpora to the treatment of a given question, it enables reaching a form of completeness in the sense that it forces the historian not to set pieces of history aside too rapidly, or even without realizing it. Further, provided methodology is made explicit in the final outcome, it helps the reader appreciate the significance of the constructed narrative, even though this narrative is necessarily constrained by the tension between the need of providing enough information for the historical proof, and that of condensing the written account, for concrete reasons of space, for instance.

Said differently, big pictures are not meant to stay out of range of the historian of mathematics, as long as sound methodological groundings underpin their drawing and significant details are still taken into account to delineate clearly the specificities of each author and each text in relation to the issue at hand. I have tried to illustrate this in presenting and confronting two ways of reconstructing the past of a mathematical object. Considering other objects or other mathematical entities, searching for their future, or trying to thicken the comprehension of the episode of their definition in synchronicity would certainly require proceeding a bit differently. In this respect, this chapter is not to be seen as a way of settling the question by inferring general laws on how to draw bigger pictures in the history of mathematics, or on what such pictures may look like, but rather as a case study aimed at stimulating historiographical and methodological reflection.

Acknowledgement

I would like to thank the anonymous reviewer and Jemma Lorenat for their helpful suggestions and linguistic support. Further thanks are due to Jemma and Catherine Goldstein, who organized the conference on the theme of this book and invited me to take part.

References

Abel Niels Henrik (1826–1841), "Mémoire sur une propriété générale d'une classe très-étendue de fonctions transcendantes", *Mémoires présentés par divers savants à l'Académie royale des sciences* **7**, 176–264 (↑ 7, 8, 13).

— (1828), "Remarques sur quelques propriétés générales d'une certaine sorte de fonctions transcendantes", *Journal für die reine und angewandte Mathematik* **3**, 313–323 (↑ 7–9, 11, 13).

— (1829), "Démonstration d'une propriété générale d'une certaine classe de fonctions transcendentes", *Journal für die reine und angewandte Mathematik* **4**, 200–201 (↑ 13).

Agnesi Maria Gaetana (1748), *Instituzioni analitiche ad uso della gioventù*, Milano: Regia-Ducale corte (↑ 18, 19).

Aronhold Siegfried (1862), "Algebraische Reduction des Integrals $\int F(x,y)\,\mathrm{d}x$ wo $F(x,y)$ eine beliebige rationale Function von x, y bedeutet, und zwischen diesen Grössen eine Gleichung dritten Grades von der allgemeinsten Form besteht, auf die Grundform der elliptischen Transcendenten", *Monatsberichte der Königlichen Preußischen Akademie der Wissenschaften zu Berlin* **(aus dem Jahre 1861)**, 462–468 (↑ 24).

Bernoulli Jacob (1695), "Notae et animadversiones tumultuariae in universum opus", in Descartes René, *Geometria*, trans. by Frans van Schooten, with a comment. by Florimond De Beaune, Frans van Schooten, Jan Hudde, Hendrik van Heuraet, Jan de Witt, and Jacques Bernoulli, Frankfurt am Main: Friderici Knochii, 423–468 (↑ 17).

Bos Henk J. M. (2001), *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction*, New York: Springer (↑ 17).

Boyer Carl B. (1956), *History of Analytic Geometry*, New York: Scripta Mathematica (↑ 16).

Brechenmacher Frédéric (2010), "Une histoire de l'universalité des matrices mathématiques", *Revue de synthèse* **131** (4), 569–603 (↑ 2).

Clebsch Alfred (1864a), “Ueber einen Satz von Steiner und einige Punkte der Theorie der Curven dritter Ordnung”, *Journal für die reine und angewandte Mathematik* **63**, 94–121 (↑ 24).
— (1864b), “Ueber die Anwendung der Abelschen Functionen in der Geometrie”, *Journal für die reine und angewandte Mathematik* **63**, 189–243 (↑ 3, 7, 12).
— (1865a), “Ueber diejenigen ebenen Curven, deren Coordinaten rationale Functionen eines Parameters sind”, *Journal für die reine und angewandte Mathematik* **64**, 43–65 (↑ 2, 3, 7, 14).
— (1865b), “Ueber die Singularitäten algebraischer Curven”, *Journal für die reine und angewandte Mathematik* **64**, 98–100 (↑ 2, 7, 14).
Craig John (1693), *Tractatus mathematicus de figurarum curvilinearum quadraturis et locis geometricis*, London: Smith and Walford (↑ 18).
Cramer Gabriel (1750), *Introduction à l’analyse des lignes courbes algébriques*, Genève: Frères Cramer et Cl. Philibert (↑ 20, 21).
Descartes René (1637), “La géométrie”, in, *Discours de la méthode*, Leyde: Ian Maire, 295–413 (↑ 17).
Dienger Joseph (1847), “Note sur une Courbe dérivant d’une ellipse”, *Nouvelles Annales de mathématiques* **6**, 234–240 (↑ 22).
Ehrhardt Caroline (2012), *Itinéraires d’un texte mathématique : Les réélaborations des écrits d’Évariste Galois au* XIX*e* *siècle*, Paris: Hermann (↑ 2).
Etwein Frank, Voelke Jean-Daniel, and Volkert Klaus (2019), *Dualität als Archetypus mathematischen Denkens – Klassische Geometrie und Polyedertheorie*, Göttingen: Cuvillier Verlag. With contributions by Jean-Pierre Friedelmeyer and Erhardt Scholz (↑ 21).
Euler Leonhard (1748), *Introductio in analysin infinitorum*, vol. 2, Lausanne: Marcum-Michaelem Bousquet (↑ 19).
Fermat Pierre (1891), *Œuvres complètes*, Charles Henry and Paul Tannery (eds.), vol. 1, Paris: Gauthier-Villars (↑ 17).
— (1999), *Œuvres de Pierre Fermat*, trans. by Paul Tannery, with a comment. by Roshdi Rashed, Christian Houzel, and Gilles Christol, vol. 1, La théorie des nombres, Paris: Blanchard (↑ 17).
Gergonne Joseph-Diez (1827–1828), “Rectification de quelques théorèmes énoncés dans les *Annales*”, *Annales de Mathématiques pures et appliquées* **18**, 149–154 (↑ 21).
Goldstein Catherine (1995), *Un théorème de Fermat et ses lecteurs*, Saint-Denis: Presses Universitaires de Vincennes (↑ 2).
— (1999), “Sur la question des méthodes quantitatives en histoire des mathématiques : le cas de la théorie des nombres en France (1870-1914)”, *Acta historiae rerum naturalium nec non technicarum* **3**, 187–214 (↑ 5).
— (2019), “Long-term History and Ephemeral Configurations”, in Boyan Sirakov, Paulo Ney de Sousa, and Marcelo Viana (eds.), *Proceedings of*

the International Congress of Mathematicians: Rio de Janeiro 2018, vol. 1. Plenary lectures, Singapore: World Scientific, 487–522 (↑ 2, 24).

Gray Jeremy (1989), "Algebraic Geometry in the Late Nineteenth Century", in John McCleary and David E. Rowe (eds.), *The History of Modern Mathematics*, vol. 1. Ideas and their Reception, Boston, San Diego, New York: Academic Press, 361–385 (↑ 7).

Guicciardini Niccolò (2009), *Isaac Newton on Mathematical Certainty and Method*, Cambridge, Massachusetts & London, England: The MIT Press (↑ 18).

Guisnée Nicolas (1705), *Application de l'algèbre à la géométrie*, Paris: Boudot et Quillau (↑ 18).

Herreman Alain (2012), "La fonction inaugurale de *La Géométrie* de Descartes", *Revue d'histoire des mathématiques* **18**, 67–156 (↑ 17).

— (2016), "L'introduction des courbes algébriques par Descartes : genèse et inauguration. Complément à la conjecture de H. Bos sur le rôle historique du problème de Pappus", *Revue d'histoire des mathématiques* **22**, 97–137 (↑ 17).

Houzel Christian (2002), *La géométrie algébrique : recherches historiques*, Paris: Albert Blanchard (↑ 7).

Jacobi Carl Gustav Jacob (1828), "Ueber die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie: 'Die Relation zwischen der Distanz der Mittelpuncte und der Radien zweier Kreise zu finden, von denen der eine einem unregelmässigen Polygon eingeschrieben, der andere demselben umgeschrieben ist'", *Journal für die reine und angewandte Mathematik* **3**, 376–389 (↑ 24).

— (1829), *Fundamenta nova theoriae functionum ellipticarum*, Berlin: Bornträger (↑ 10).

— (1832), "Considerationes generales de transcendentibus Abelianis", *Journal für die reine und angewandte Mathematik* **9**, 394–403 (↑ 5, 7, 9, 11–13).

Joffredo Thierry (2017), "Approches biographiques de l'*Introduction à l'analyse des lignes courbes algébriques* de Gabriel Cramer", PhD thesis, Nancy: Université de Lorraine (↑ 20).

— (2019), "Une analyse génétique de l'*Introduction à l'analyse des lignes courbes algébriques* de Gabriel Cramer", *Revue d'histoire des mathématiques* **25**, 235–289 (↑ 20).

Jonquières Ernest de (1861), "Lieu géométrique du sommet d'un angle de grandeur constante circonscrit à une courbe de la classe n", *Nouvelles Annales de mathématiques* **20**, 206–211 (↑ 22).

Kleiman Steven L. (2004), "What is Abel's Theorem Anyway?", in Olav Arnfinn Laudal and Ragni Piene (eds.), *The Legacy of Niels Henrick Abel*, Berlin, Heidelberg, New York: Springer, 395–440 (↑ 8, 13).

Lê François (2018), "Le paramétrage elliptique des courbes cubiques par Alfred Clebsch", *Revue d'histoire des mathématiques* **24**, 1–39 (↑ 24).

Lê François (2020), "'Are the *genre* and the *Geschlecht* one and the same number?' An inquiry into Alfred Clebsch's *Geschlecht*", *Historia Mathematica* **53**, 71–107 (↑ 3, 13).

— (2023), "Des taxons et des nombres : quelques remarques sur les ordres, classes et genres des courbes algébriques", *Revue d'histoire des sciences* **76** (1), 85–134 (↑ 17, 18, 22).

Lorenat Jemma (2015), "Polemics in Public: Poncelet, Gergonne, Plücker, and the Duality Controversy", *Science in Context* **28** (4), 545–585 (↑ 21).

Lucas F. (1863), "Nouvelle théorie des diamètres", *Journal de Mathématiques pures et appliquées* **8**, 145–160 (↑ 22).

MacLaurin Colin (1748), *A Treatise of Algebra*, London: Millar & Nourse (↑ 18).

Mahoney Michael S. (1999), *The Mathematical Career of Pierre de Fermat (1601–1695)*, Princeton: Princeton University Press (↑ 17).

Newton Isaac (1704), "Enumeratio linearum tertii ordinis", in, *Opticks: or, a Treatise of the Reflexions, Refractions, Inflexions and Colours of Light*, London: Smith and Walford, 139–211 (↑ 18).

Nicole François (1731), "Traité des lignes du troisième ordre, ou des courbes du second genre", *Histoire de l'Académie royale des sciences* **année 1729**, 194–224 (↑ 18).

Ozanam Jacques (1687), *Traité des lignes du premier genre*, Paris: Estienne Michallet (↑ 17).

Plücker Julius (1833), "Nachrichten von Büchern", *Journal für die reine und angewandte Mathematik* **10**, 98–100 (↑ 22).

— (1835), *System der analytischen Geometrie*, Berlin: Duncker und Humbolt (↑ 22).

Poncelet Jean-Victor (1828), "Sur la dualité de situation et sur la théorie des polaires réciproques, 2e article en réponse aux observations de M. Gergonne, mentionnées pag. 23, cahier de janvier du présent *Bulletin des Sciences*, et insérées dans le tome XVIII des *Annales de Mathématiques*, p. 125 et suiv.", *Bulletin des sciences mathématiques, astronomiques, physiques et chimiques* **9**, 292–302 (↑ 21).

Reuss Jeremias David (1808), *Repertorium commentationum*, vol. 7, Göttingen: Heinrich Dietrich (↑ 16).

Riemann Bernhard (1851), "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse", PhD thesis, Universität Göttingen (↑ 11).

— (1857), "Theorie der Abel'schen Functionen", *Journal für die reine und angewandte Mathematik* **54**, 115–155 (↑ 4, 7, 11, 12).

Rosenhain Johann Georg (1851), "Mémoire sur les fonctions de deux variables et à quatre périodes, qui sont les inverses des intégrales ultra-elliptiques de la première classe", *Mémoires présentés par divers savants à l'Académie royale des sciences de l'Institut de France* **11**, 361–468 (↑ 7, 10).

Scholz Erhard (1980), *Geschichte des Mannigfaltigkeitsbegriffs von Riemann bis Poincaré*, Boston: Birkhäuser (↑ 11).

Schooten Frans van (1657), *Exercitationes mathematicarum*, Leiden: Johannis Elsevirii (↑ 17).

Serfati Michel (2005), "René Descartes, *Géométrie*, Latin Edition (1649), French Edition (1637)", in Ivor Grattan-Guinness (ed.), *Landmark Writings in Western Mathematics, 1640–1940*, Amsterdam: Elsevier Press, 1–22 (↑ 17).

Sinaceur Hourya (1991), *Corps et modèles*, Paris: Vrin (↑ 2).

Stirling James (1717), *Lineae tertii ordinis Neutonianae*, Oxford: E Theatro Sheldoniano (↑ 18).

Stone Edmund (1726), *A New Mathematical Dictionary*, Londres: Innys, Woodward, Longman & Senex (↑ 18).

Talbot Christopher R. M. (1860), *Sir Isaac Newton's Enumeration of Lines of the Third Order*, London: Bohn (↑ 18).

Watterson Bill (1996), *It's A Magical World: A Calvin and Hobbes Collection*, Kansas City: Andrews and McMeel (↑ 1).

Weierstrass Karl (1854), "Zur Theorie der Abelschen Functionen", *Journal für die reine und angewandte Mathematik* **47**, 289–306 (↑ 7, 10).

— (1856), "Theorie der Abel'schen Functionen", *Journal für die reine und angewandte Mathematik* **52**, 285–379 (↑ 7, 10, 12).