# Growth bijections for oriented planar maps

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Joint work with Jérémie Bettinelli and Baptiste Louf

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# Growth bijections

For $\mathcal{A} = \cup_n \mathcal{A}_n$ a combinatorial class, with $a_n = |\mathcal{A}_n|$

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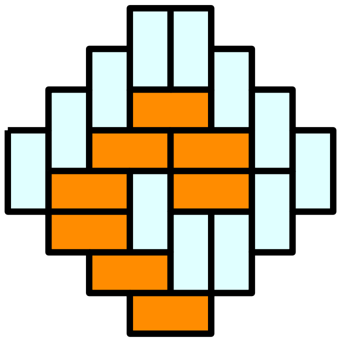
Can be applied to:

- trees [Rémy'85], [Marckert'22]
- planar maps [Bettinelli'14,'20], [Louf'19], [Schabanel'25]
- oriented planar maps [Bettinelli-Louf-F'24]

and other classes, e.g. domino tilings



$n = 4$

$$a_n = 2^{\binom{n+1}{2}}$$

$$\frac{a_n}{a_{n-1}} = 2^n$$

[Elkies et al'92]

# Trees

# Rémy's bijection

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explains $(n+1)\,\mathrm{Cat}_n = 2(2n-1)\,\mathrm{Cat}_{n-1}$

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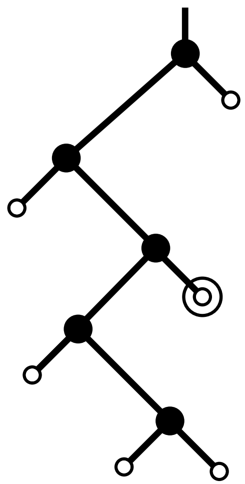

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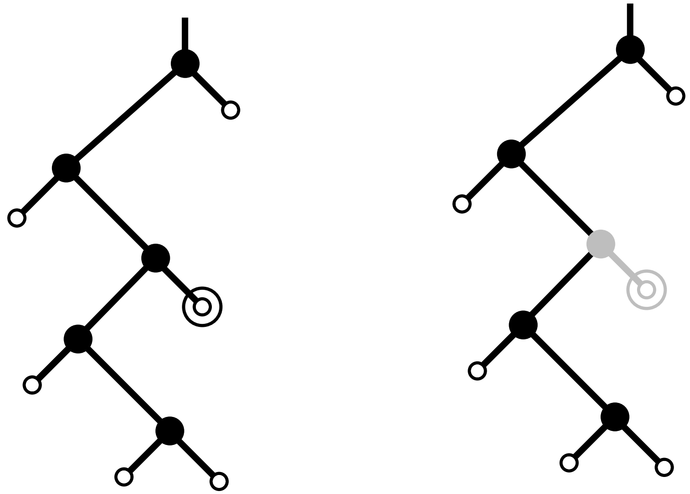

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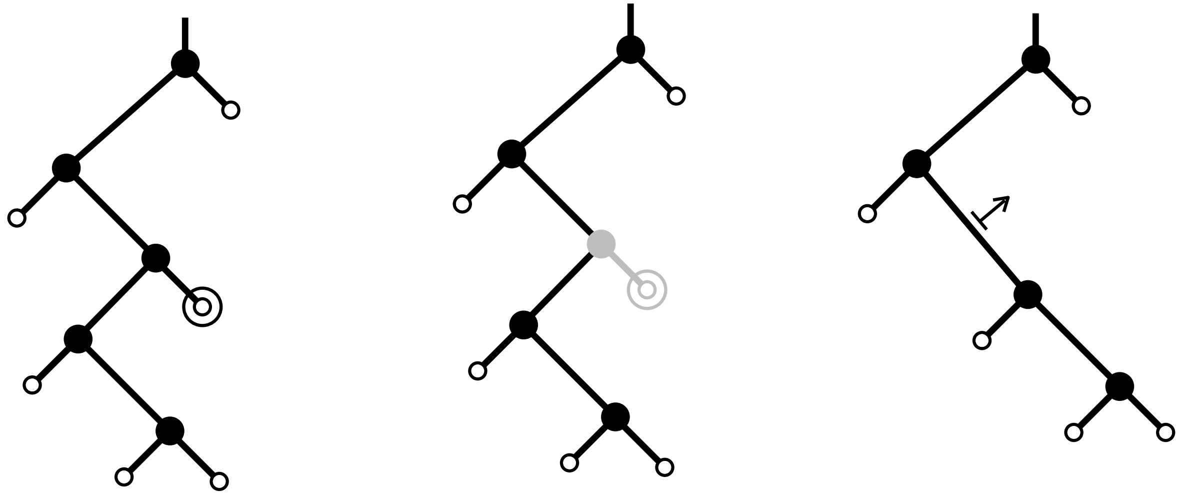

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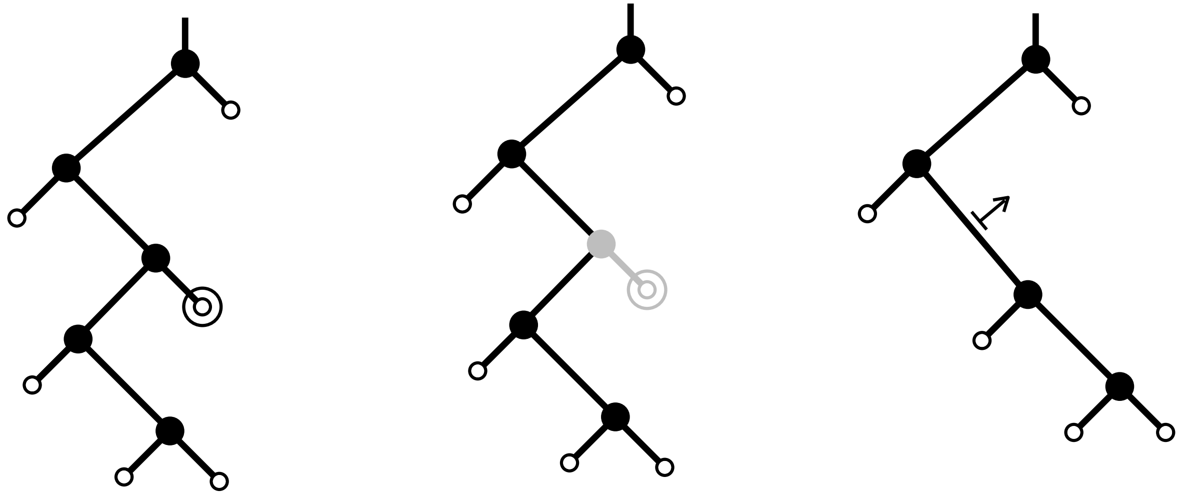
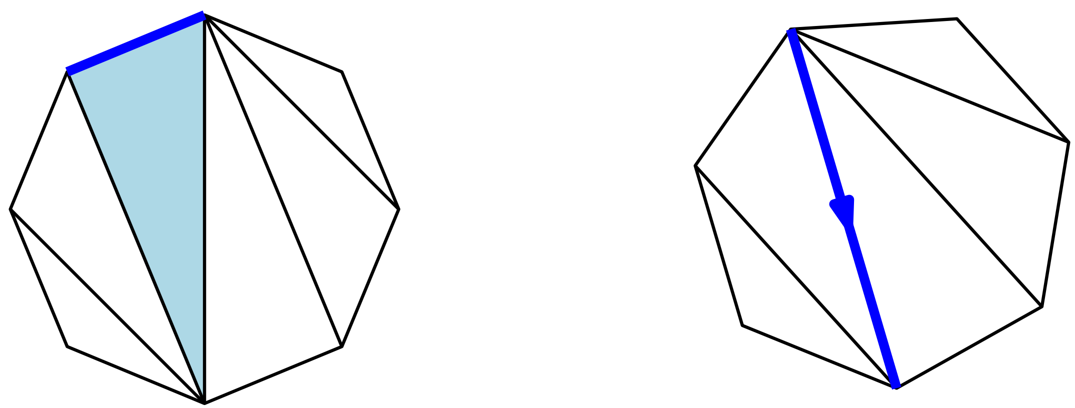


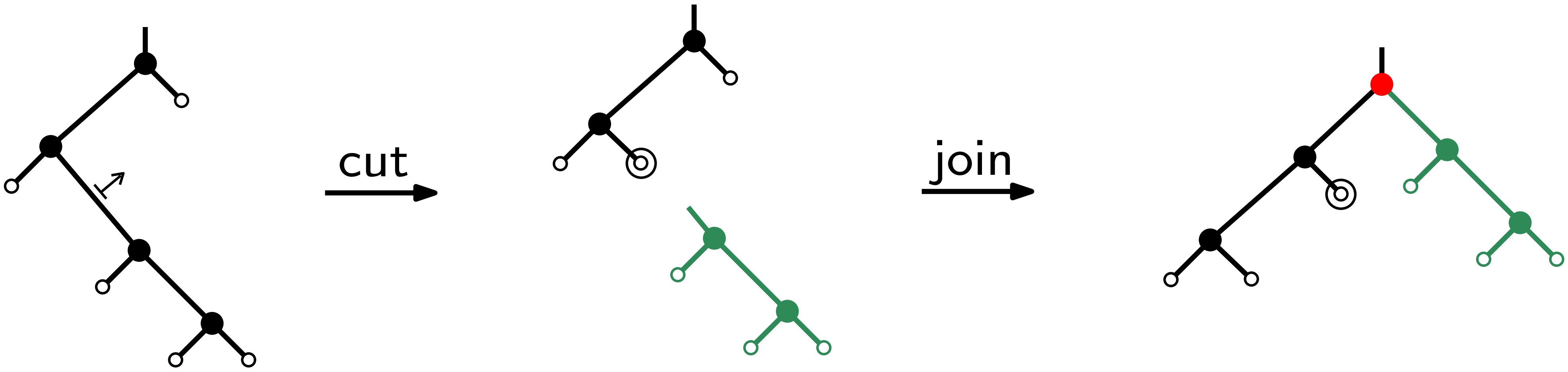
also [Rodrigue'1868] via $\mathrm{Cat}_n = \#$ triangulations of $n+2$-gon

$$\mathrm{Cat}_n = \frac{2(2n-1)}{n+1}\mathrm{Cat}_{n-1}$$



# Other growth bijection for binary trees

[Marckert'22]



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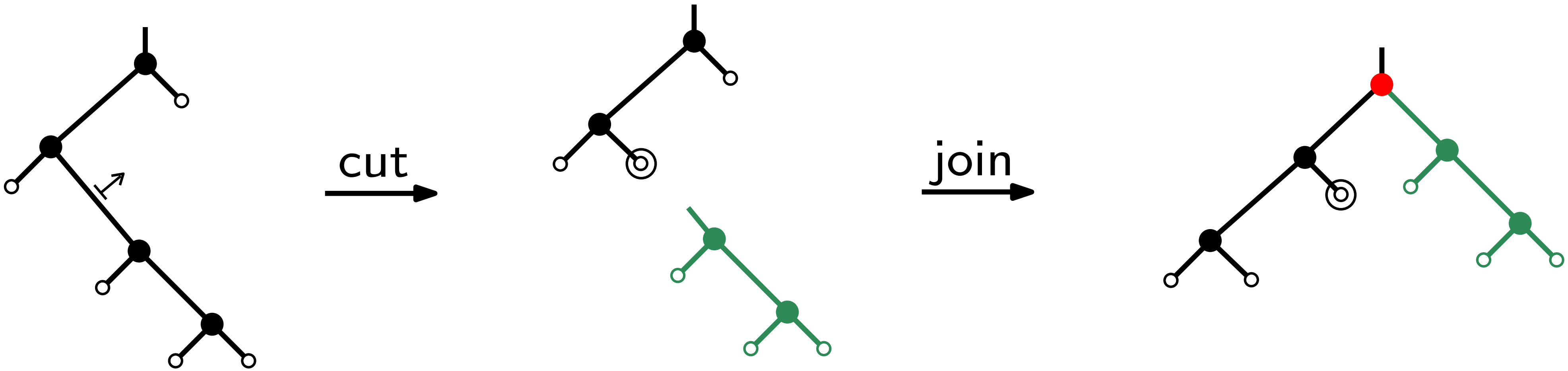
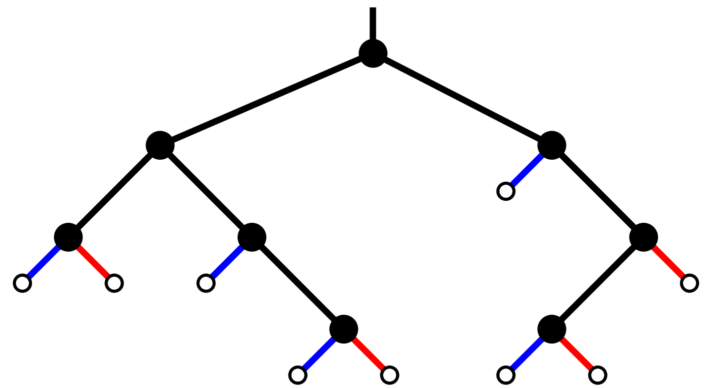



extends to $k$-ary trees $$a_n^{k+1} = \frac{1}{kn+1}\binom{kn+1}{n}$$

# Growth bijection for Narayana numbers

The Narayana number is $\boxed{\mathrm{Nar}_{a,b} = \frac{1}{n}\binom{n}{a}\binom{n}{b}}$, where $n = a + b - 1$

it counts binary trees with $a$ left leaves and $b$ right leaves



$a = 5$ left leaves

$b = 4$ right leaves

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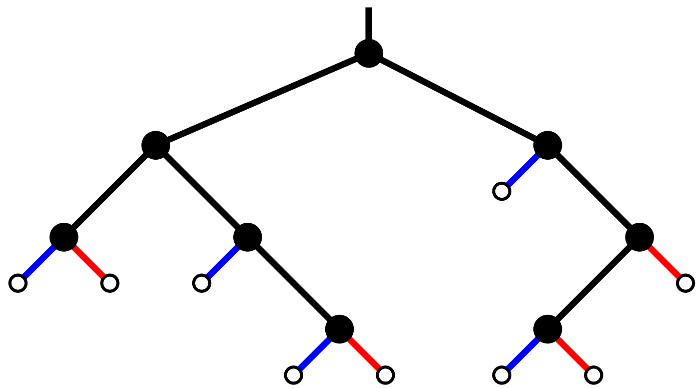

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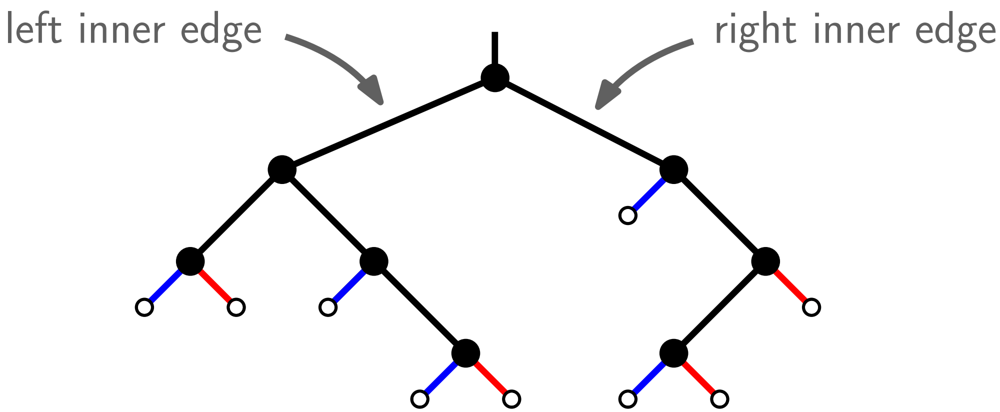


$a = 5$ left leaves
4 right inner edges

$b = 4$ right leaves
3 right inner edges

**Rk:** $a-1$ right inner edges and $b-1$ left inner edges

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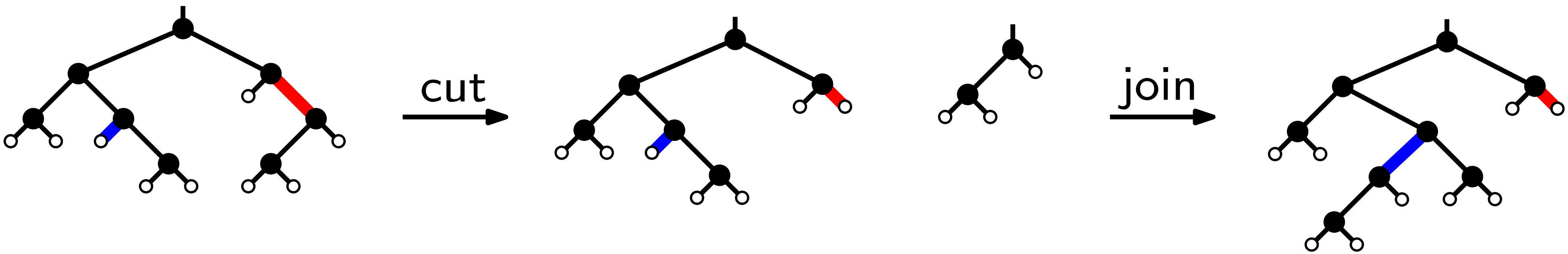
explained by cut-and-join operations [Bettinelli-F-Louf'24, ?]



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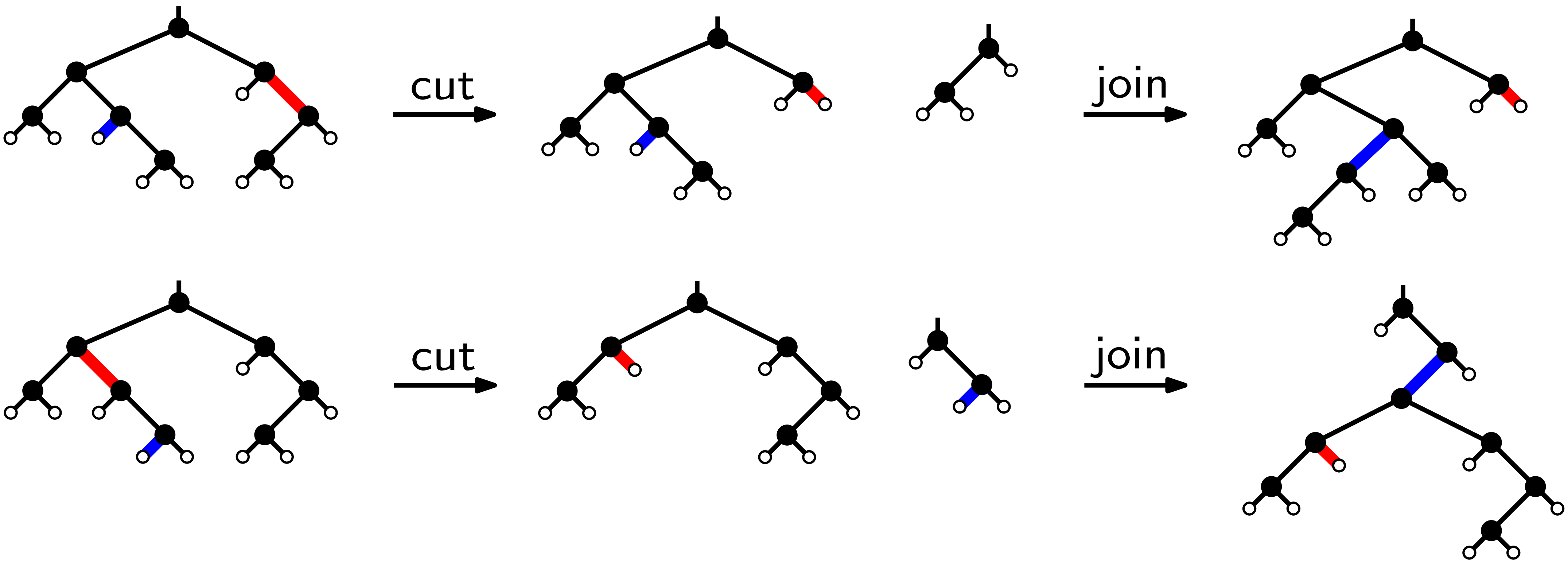

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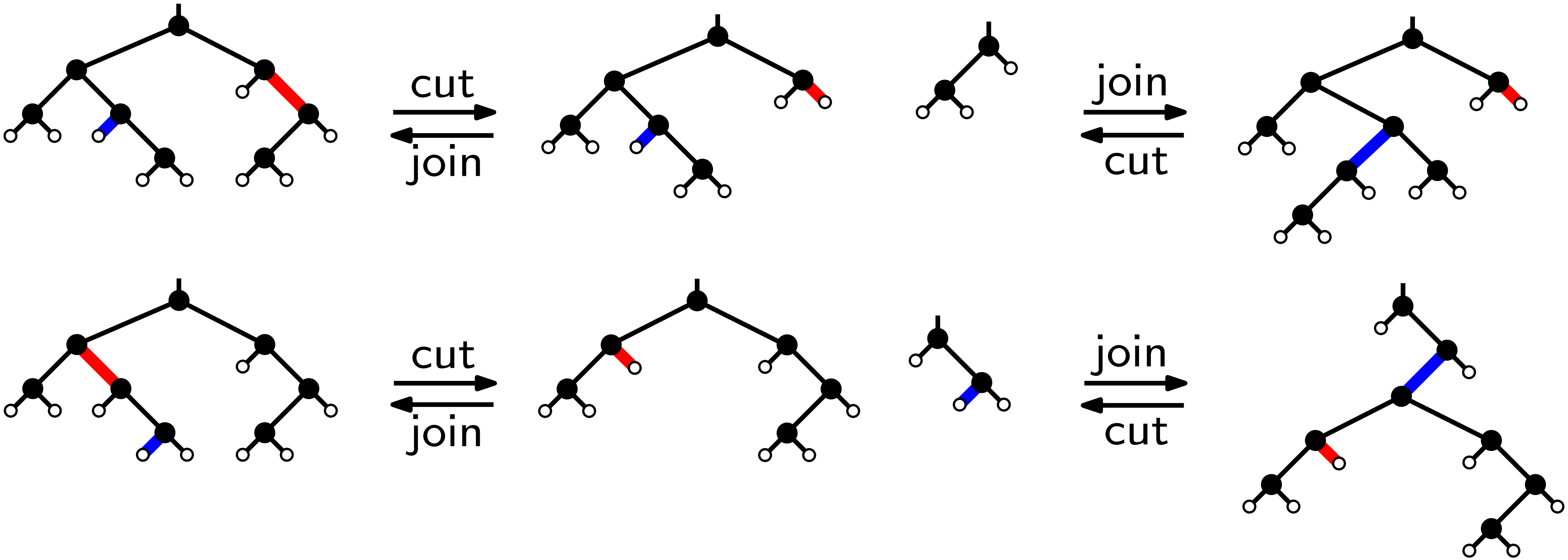

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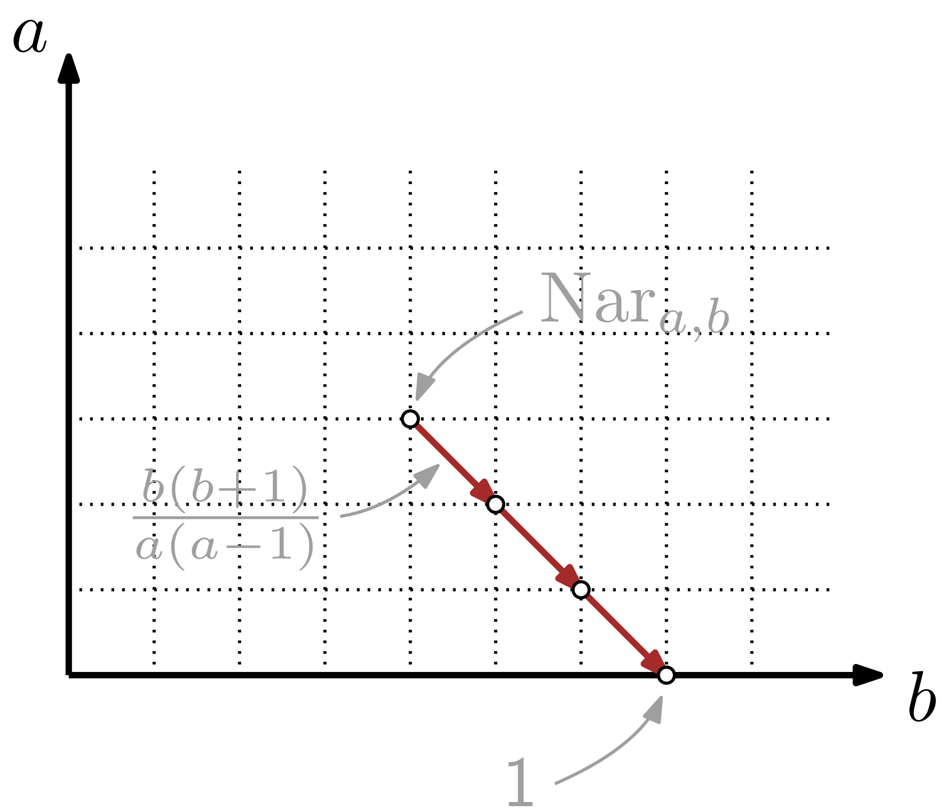
**Rk:** Identity (& boundary case $\mathrm{Nar}_{1,b} = 1$) yields formula for $\mathrm{Nar}_{a,b}$



# Planar maps

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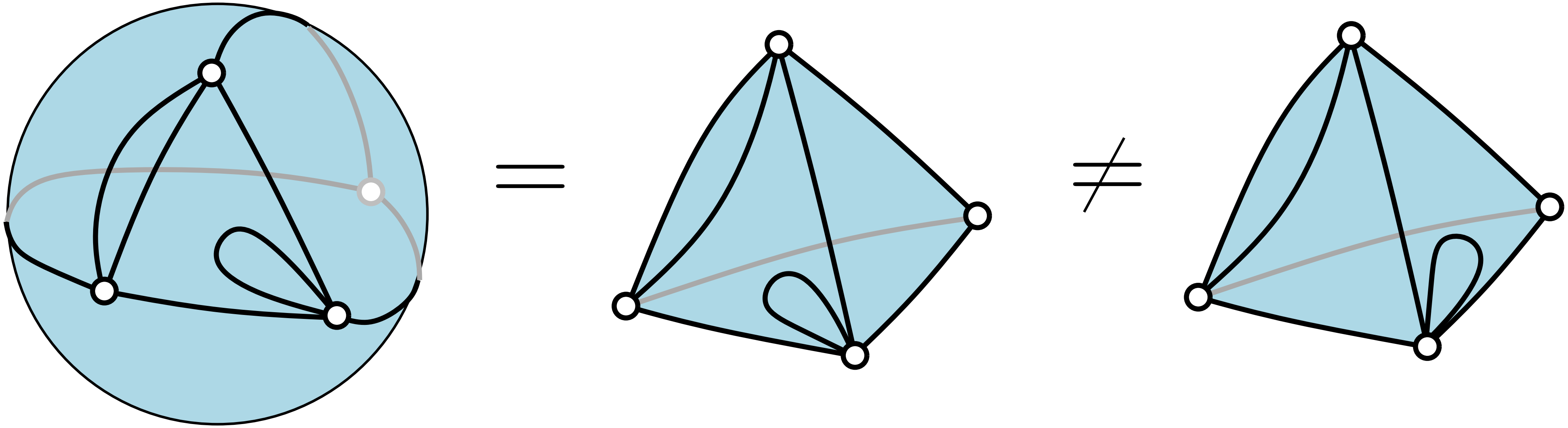
**Def.** Planar map = connected graph embedded on the sphere



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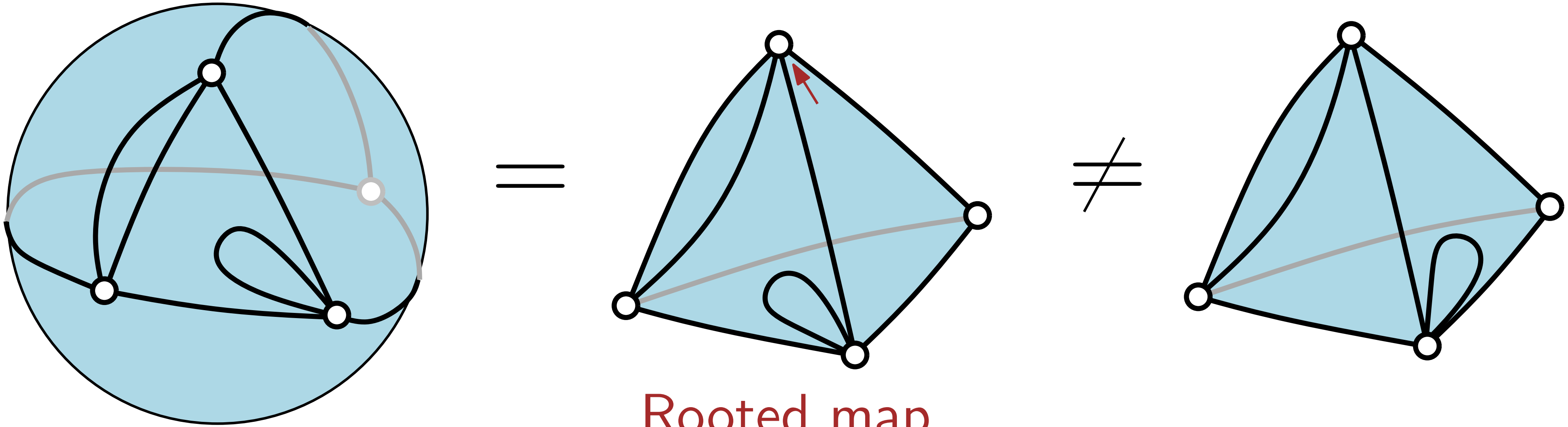

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= map with marked corner

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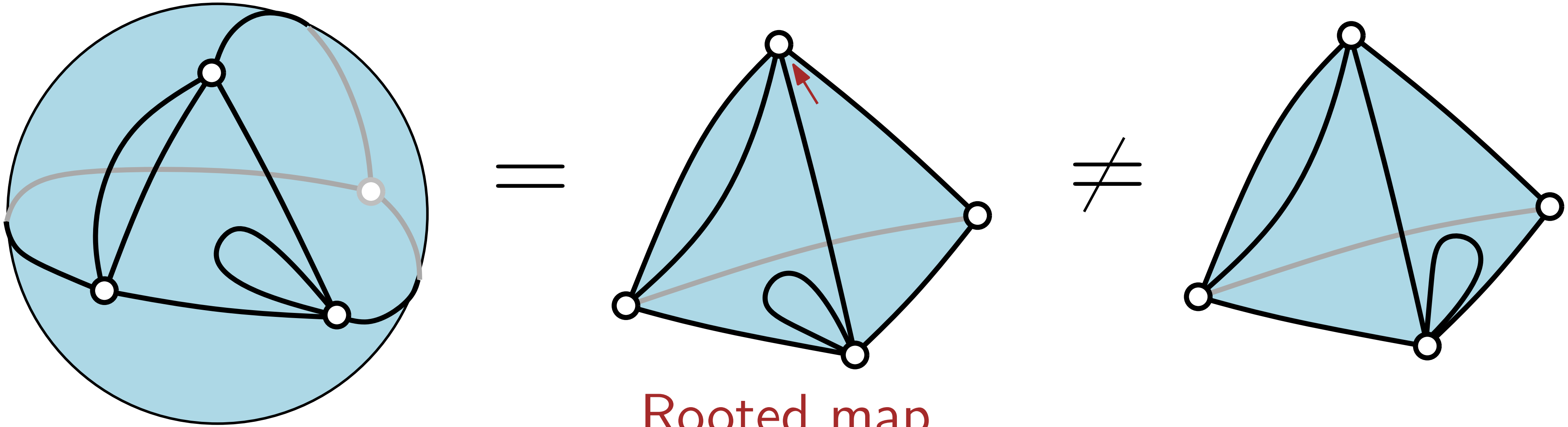
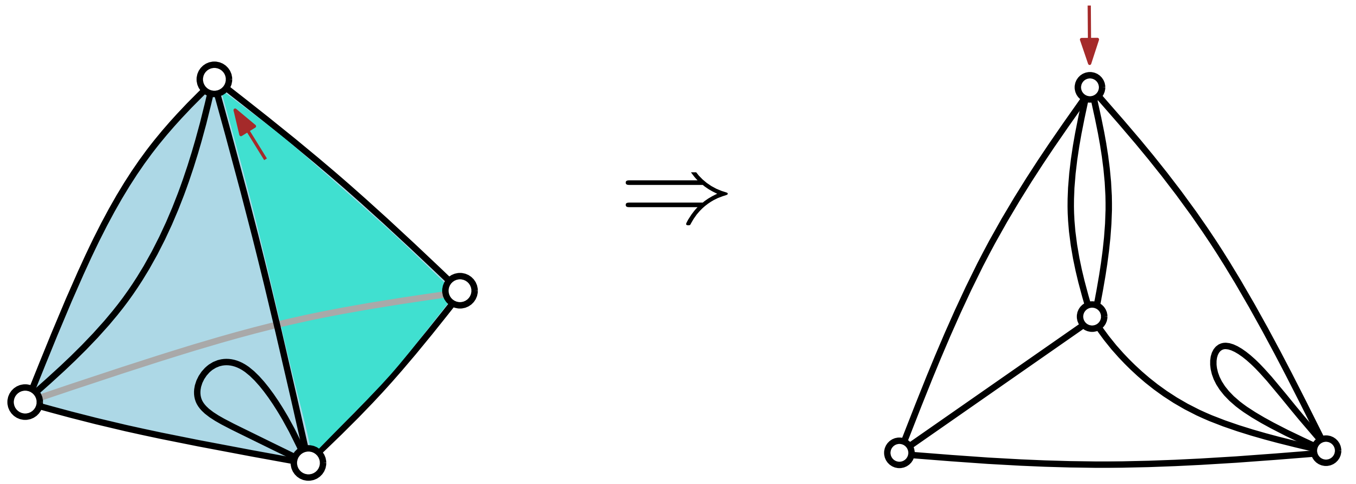


Rooted map
= map with marked corner

Easier to draw in the plane (choosing root-face to be the outer face)



# Counting planar maps

- **Nice counting formulas** [Tutte'62,63]

arbitrary maps $n$ edges

$$\frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}$$

simple quadrangulations $n$ faces

$$\frac{2}{n(n+1)} \binom{3n}{n-1}$$

bipartite maps $n$ edges

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loopless triangulations $2n$ faces

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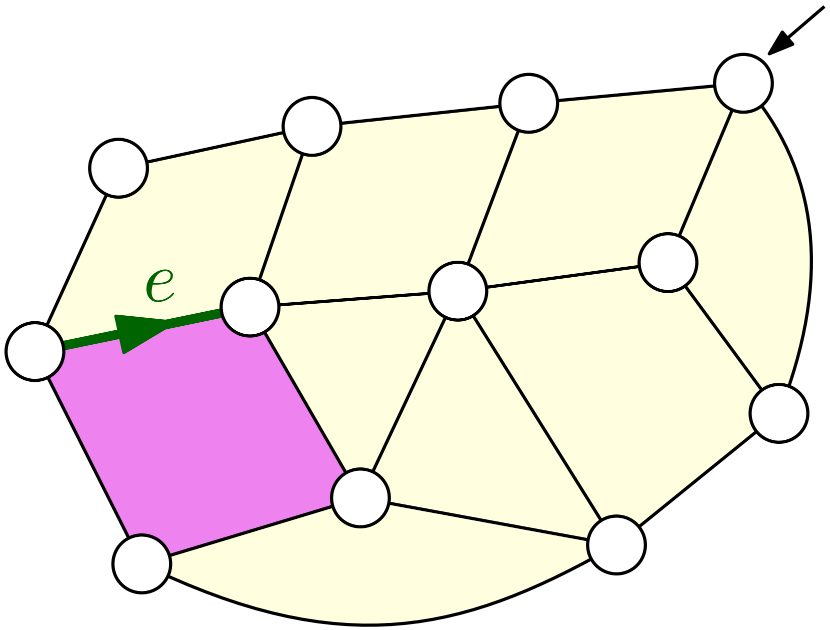
- **Combinatorial proofs**:
  - bijections to tree families
    [Cori-Vauquelin'81], [Schaeffer'97,99], [Bousquet-Mélou–Schaeffer'00], [Bouttier-Di Francesco-Guitter'02,04], [Bernardi-F'12], [Albenque-Poulalhon'15]...
  - bijections to slices & slice decompositions
    [Bouttier-Guitter'15], [Bouttier-Guitter-Miermont'22], [Bouttier-Guitter-Manet'24]
  - growth bijections using slit-slide-sew operations
    [Bettinelli'14,'20], [Bettinelli-Korkotashvili'24]

# Bi-marked maps

[Bettinelli'20]

bi-marked map = rooted map + marked oriented edge $e$ such that:

- $e$ has an inner face on its right
- end of $e$ closer to root-vertex than origin of $e$



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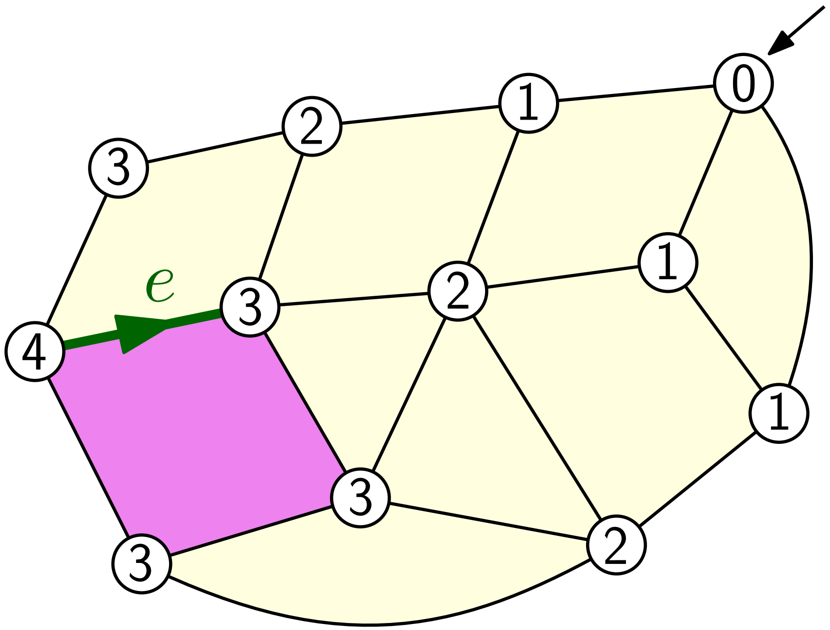

# Slit-slide-sew bijection on bi-marked maps

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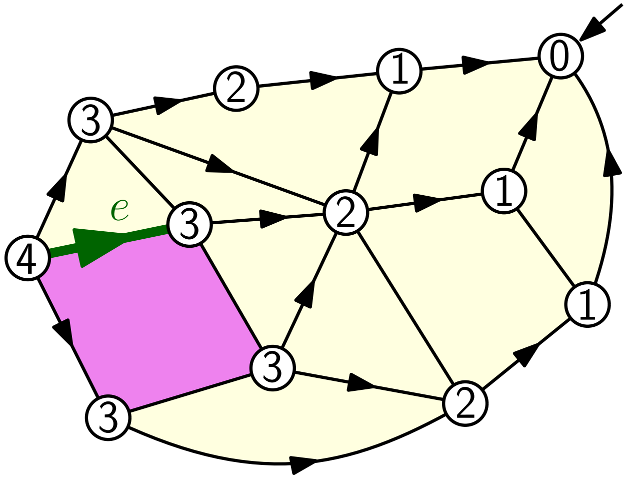

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rightmost path from $e$

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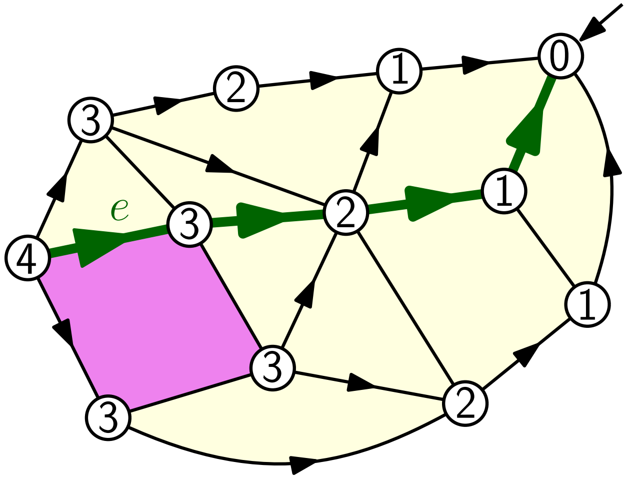

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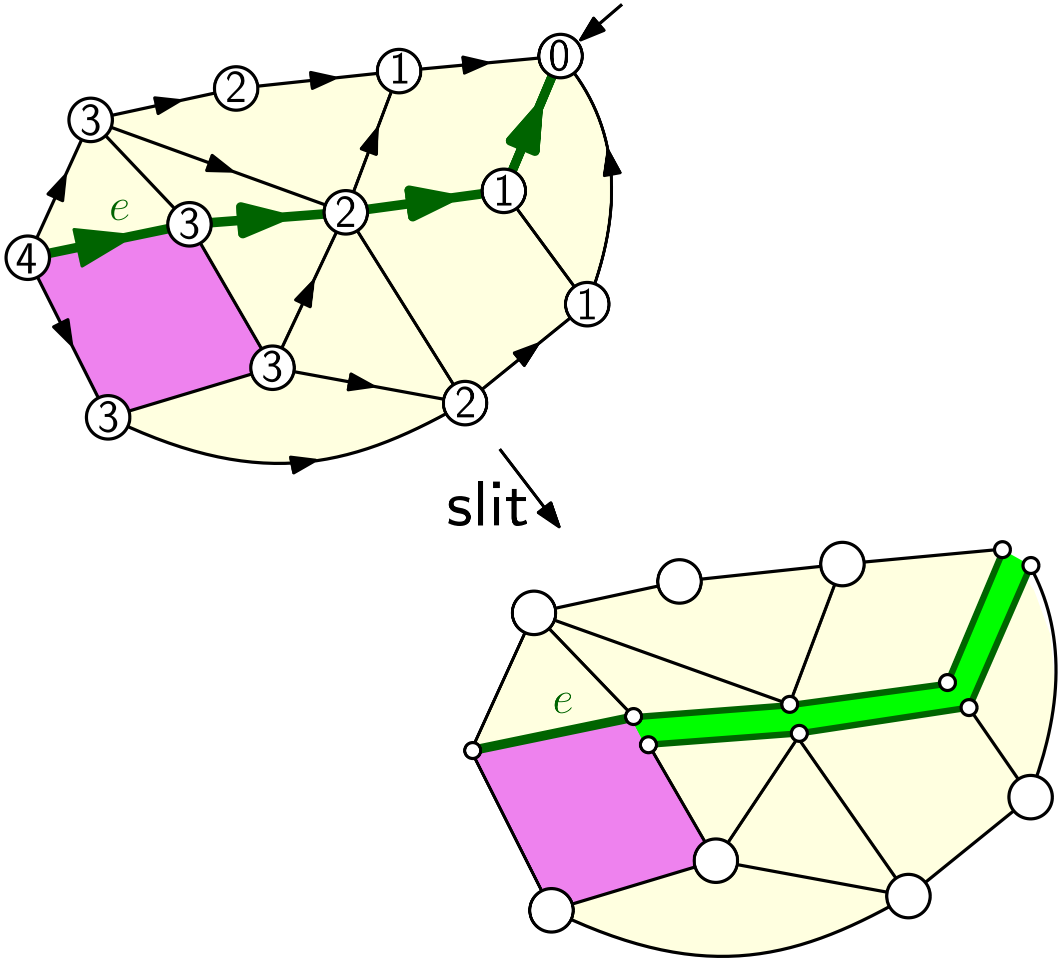

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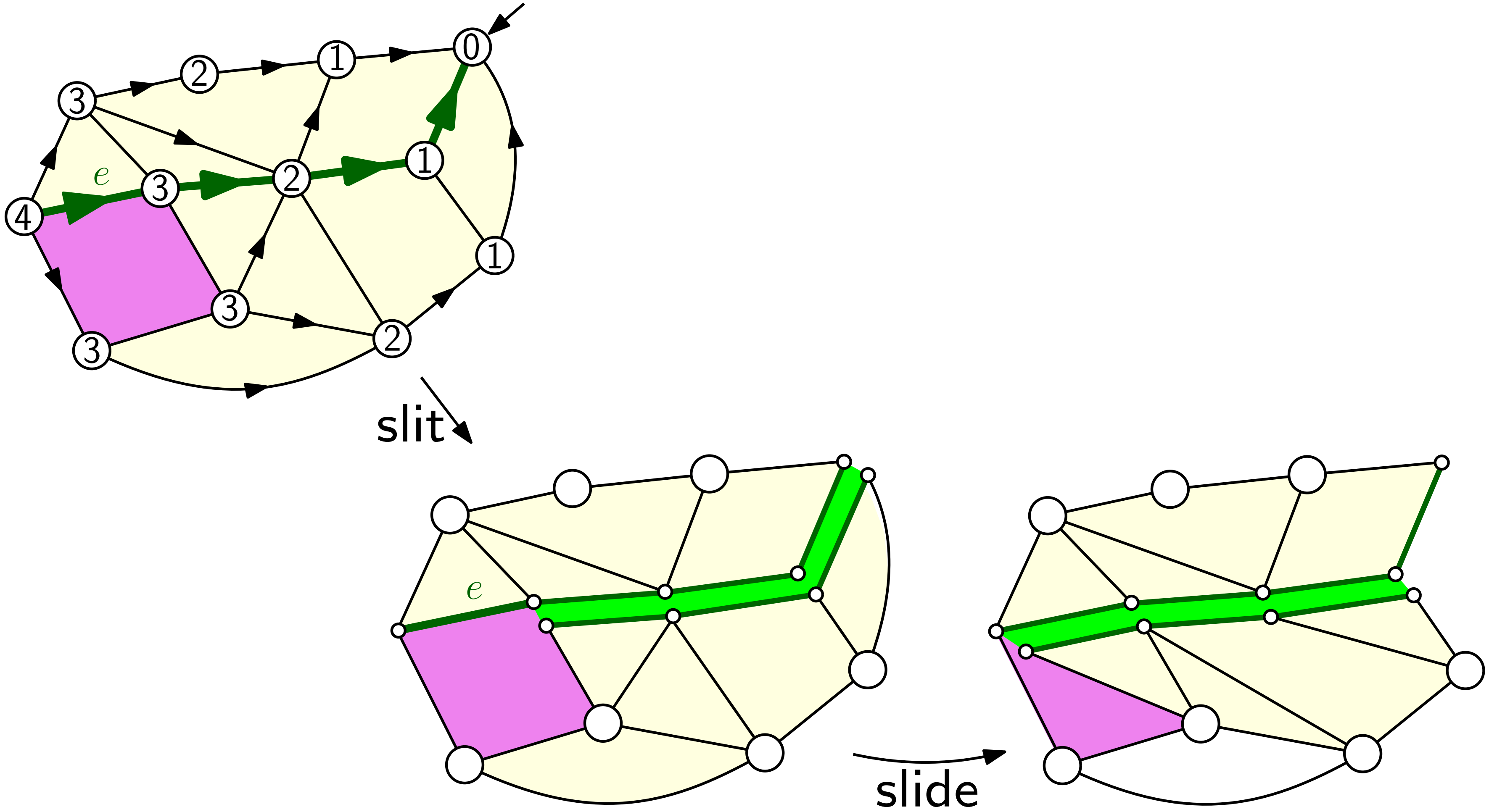

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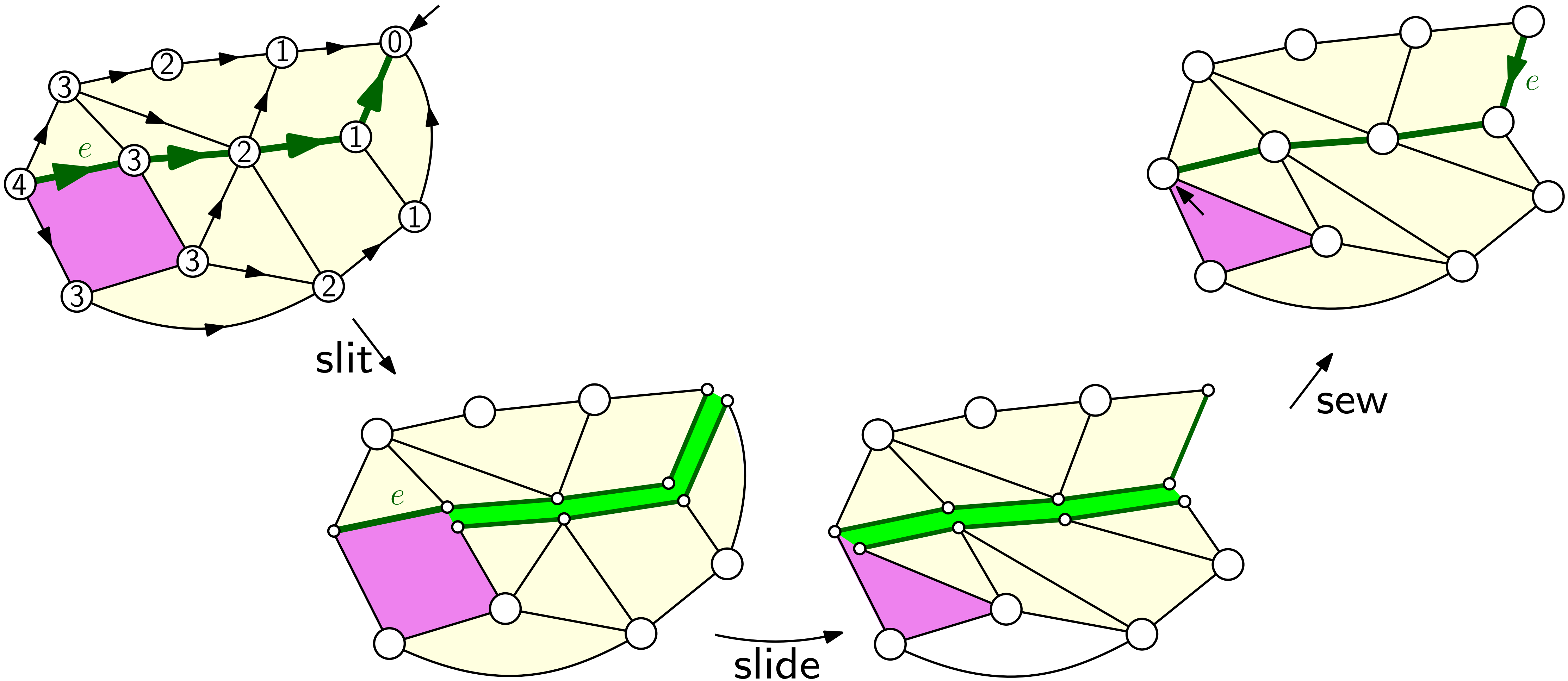

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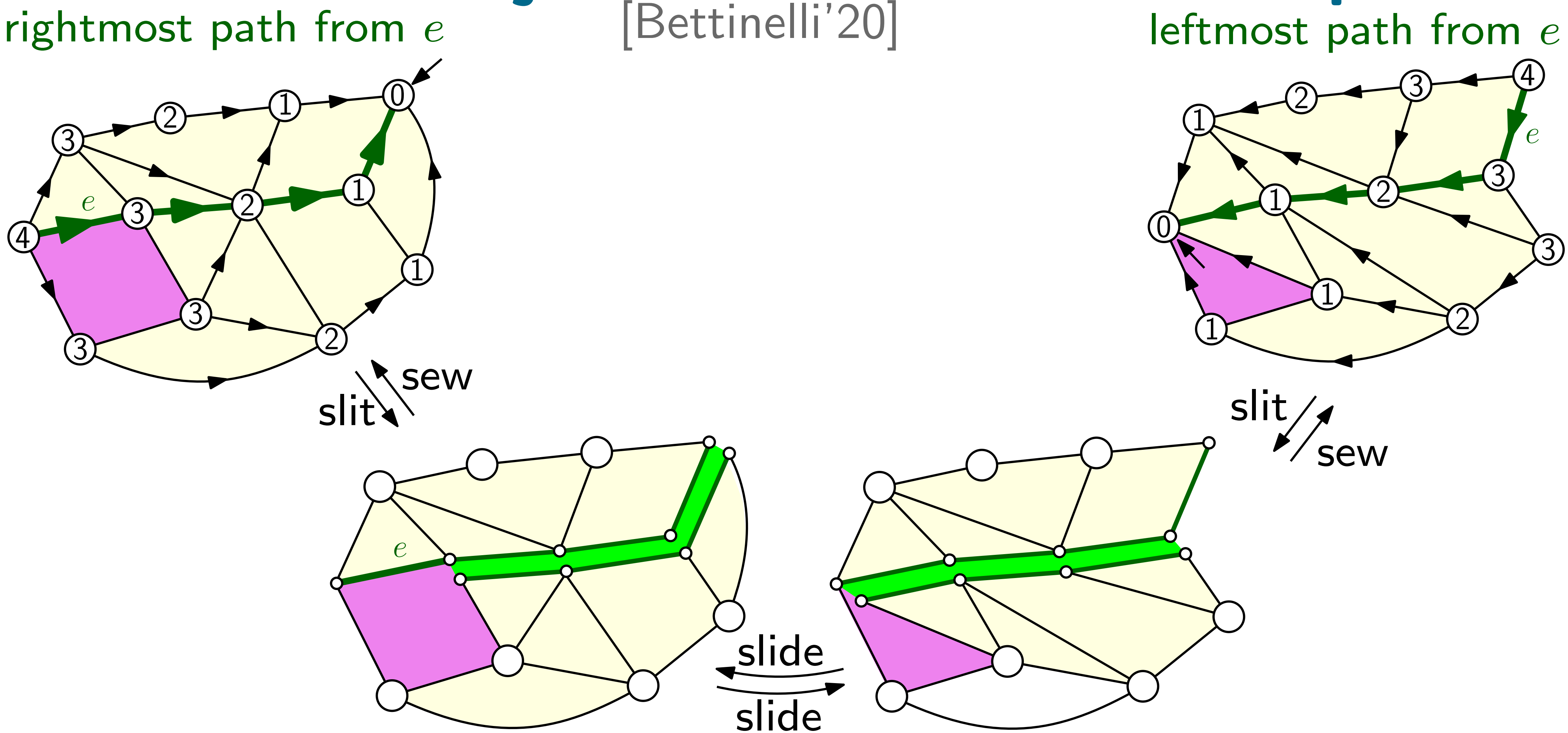

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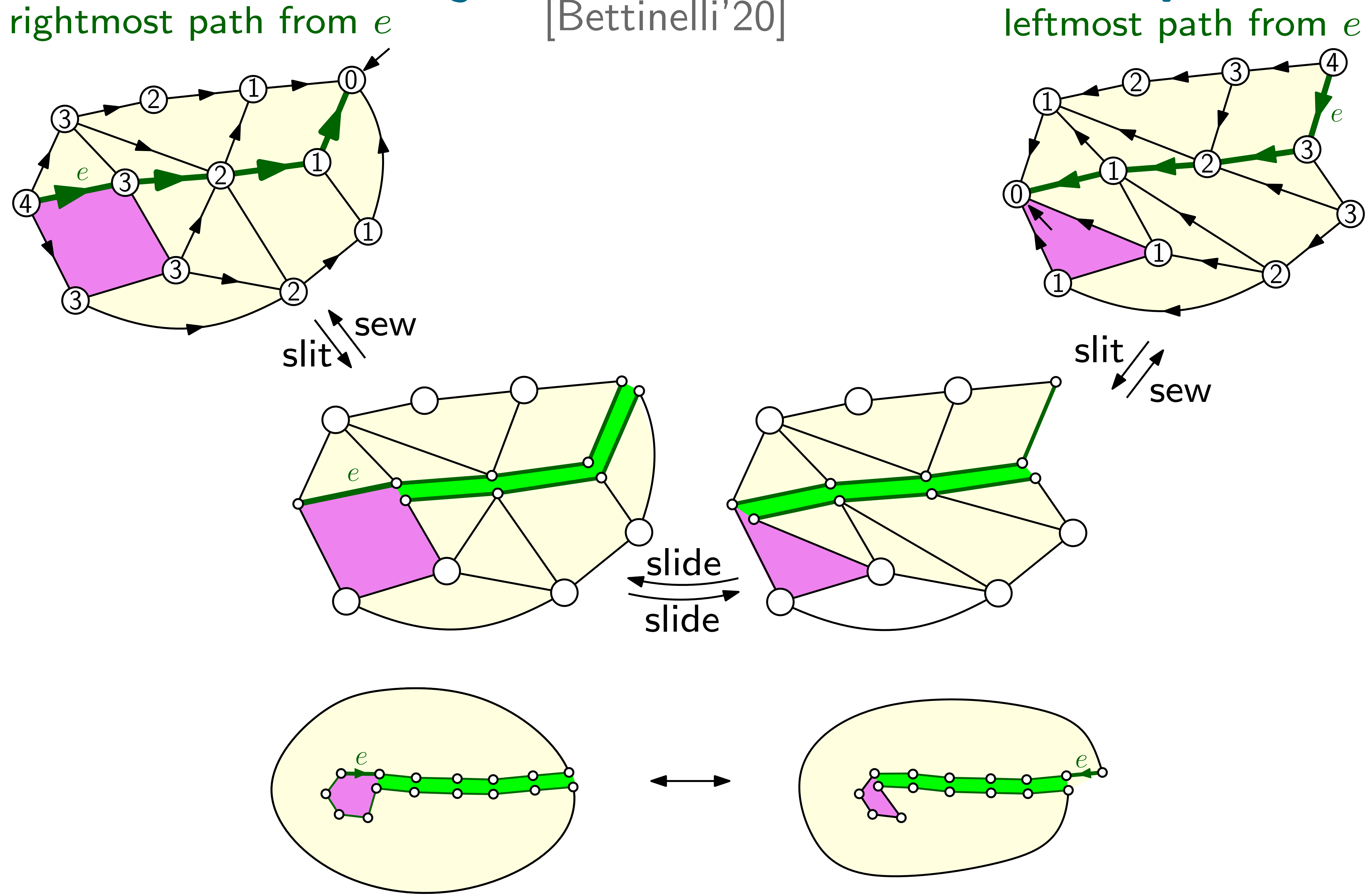
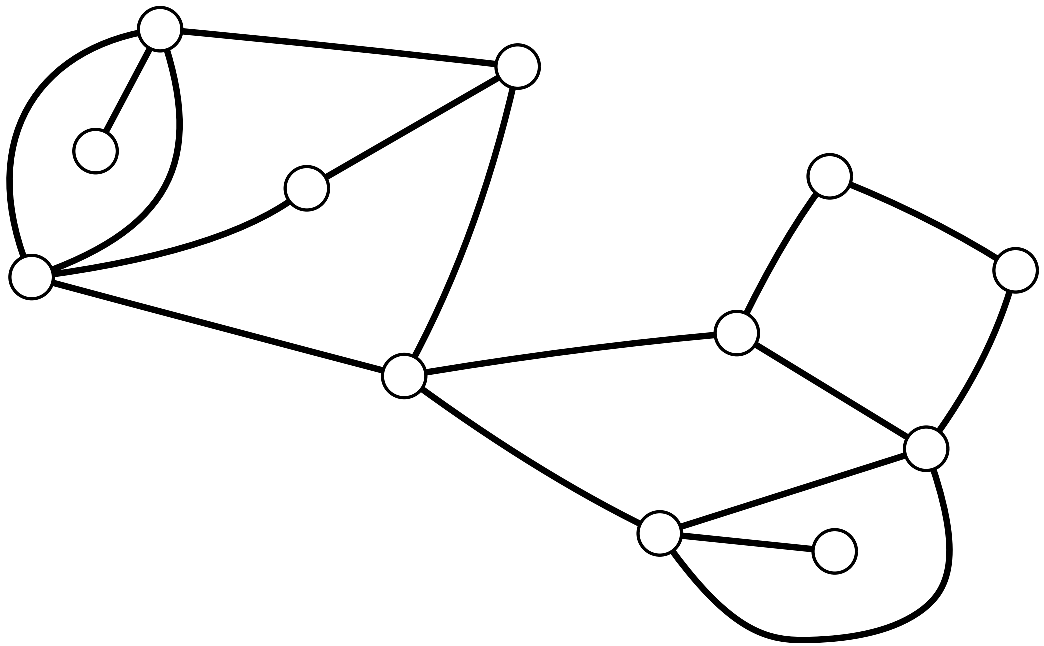


# Quadrangulations with a boundary

Let $Q_{n,j} = \#$ rooted maps with root-face degree $2j$, and $n$ inner faces all of degree $4$



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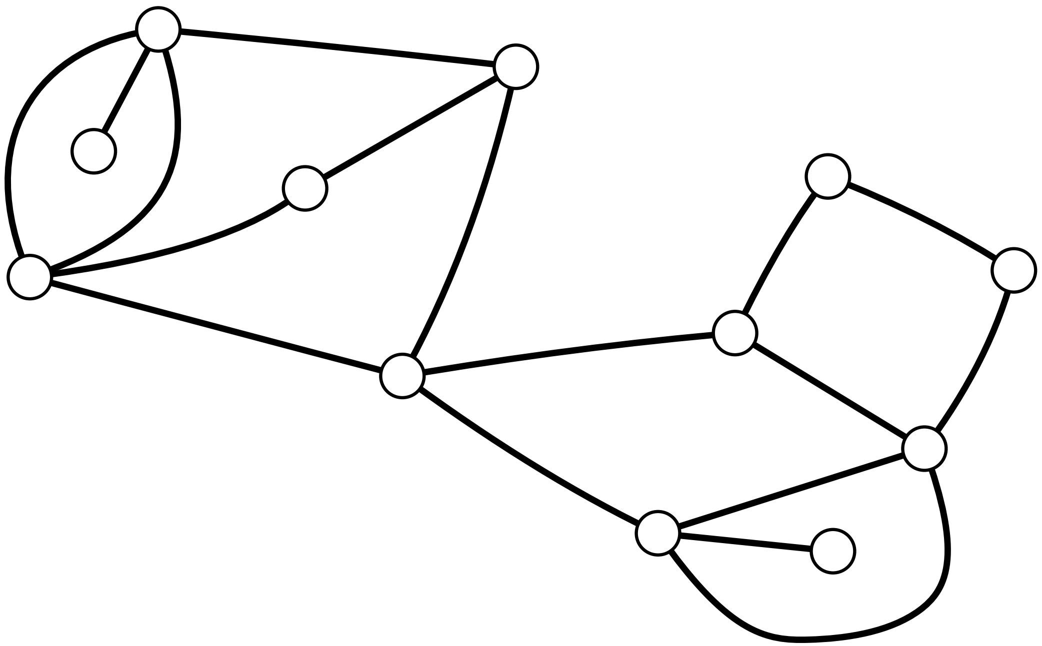

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Counting formula: [Tutte'62]

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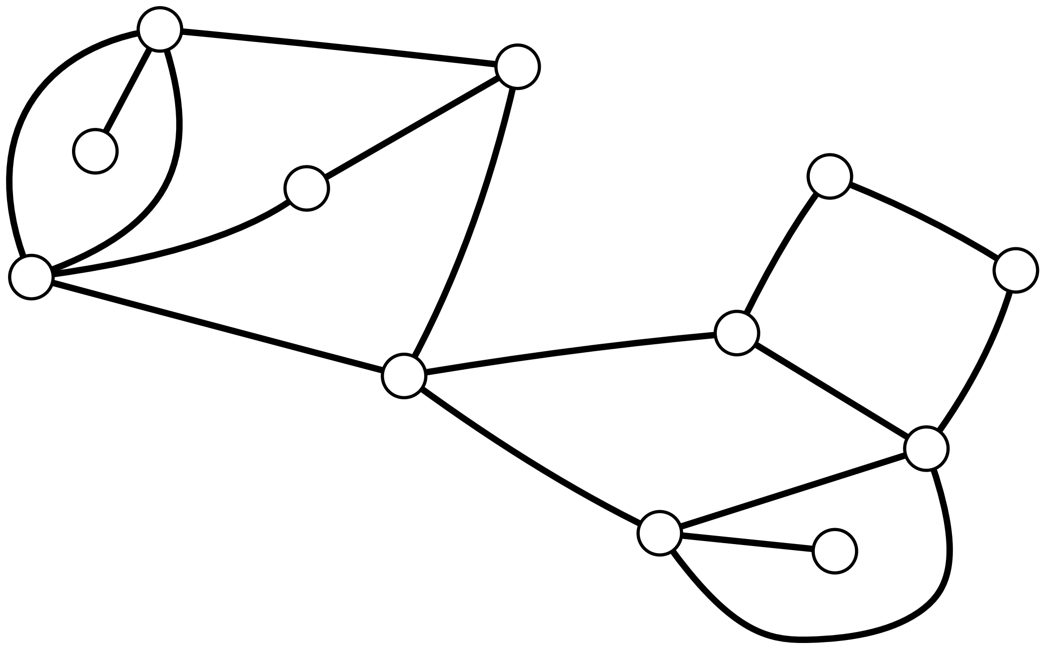

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$$\Downarrow$$

Identity:

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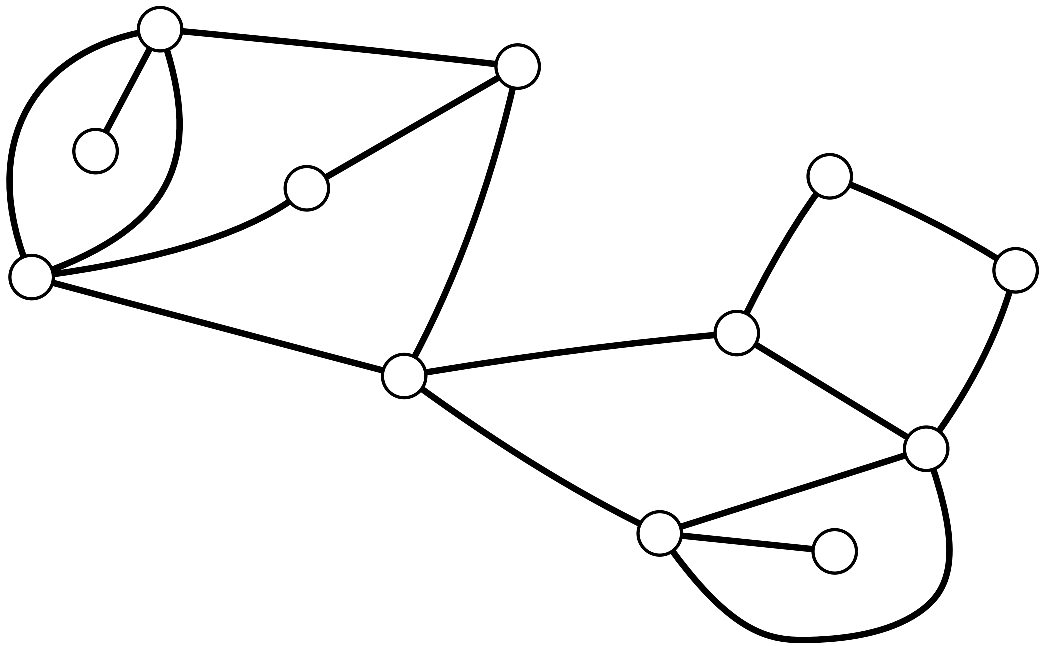

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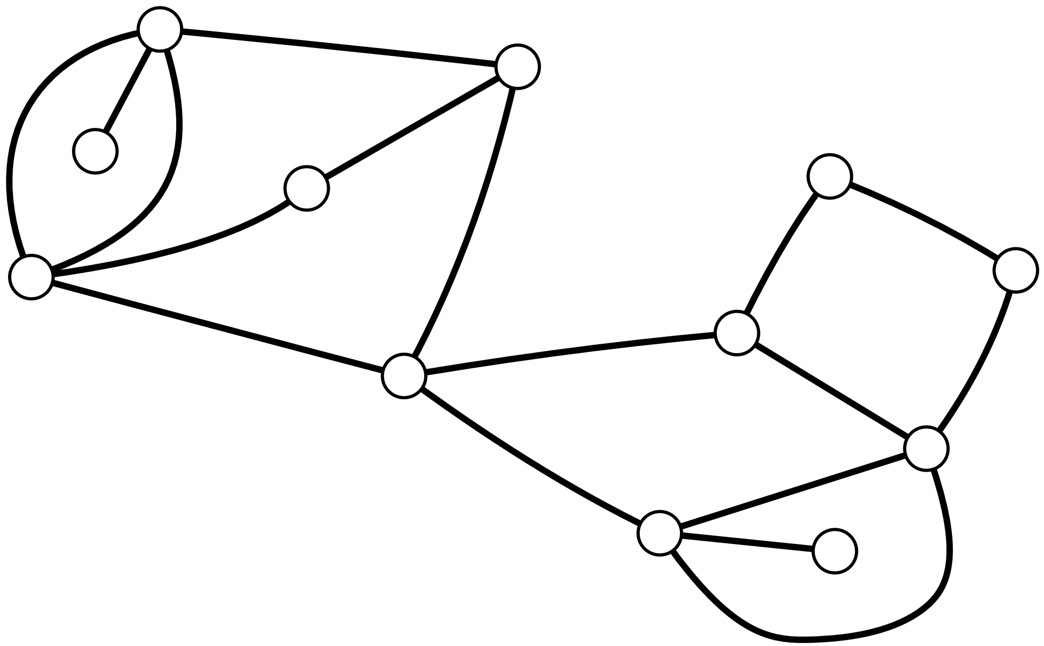

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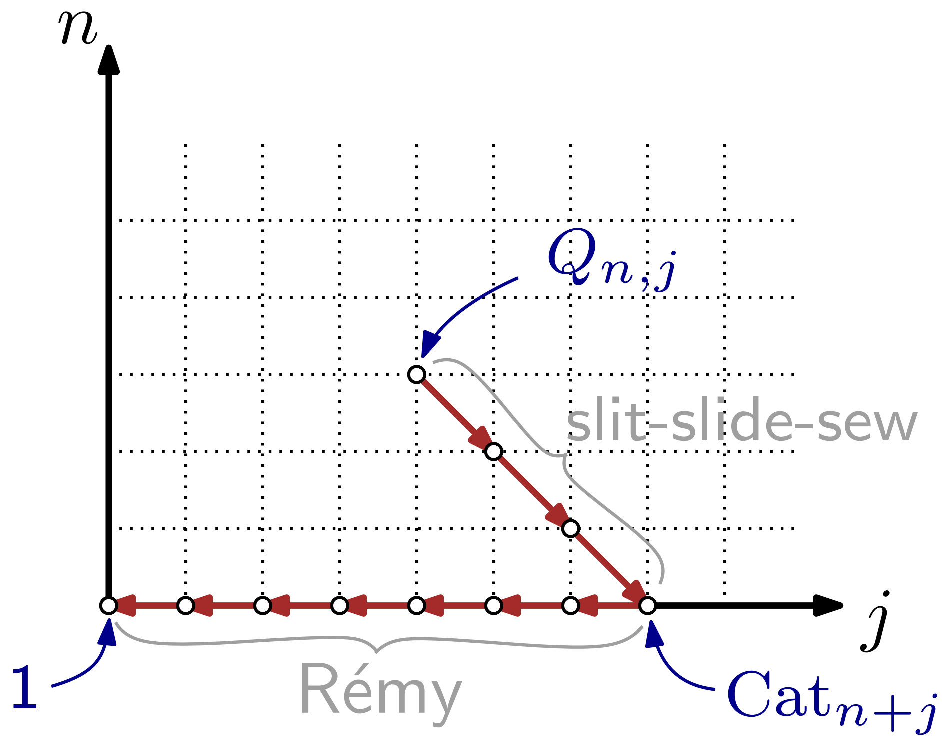
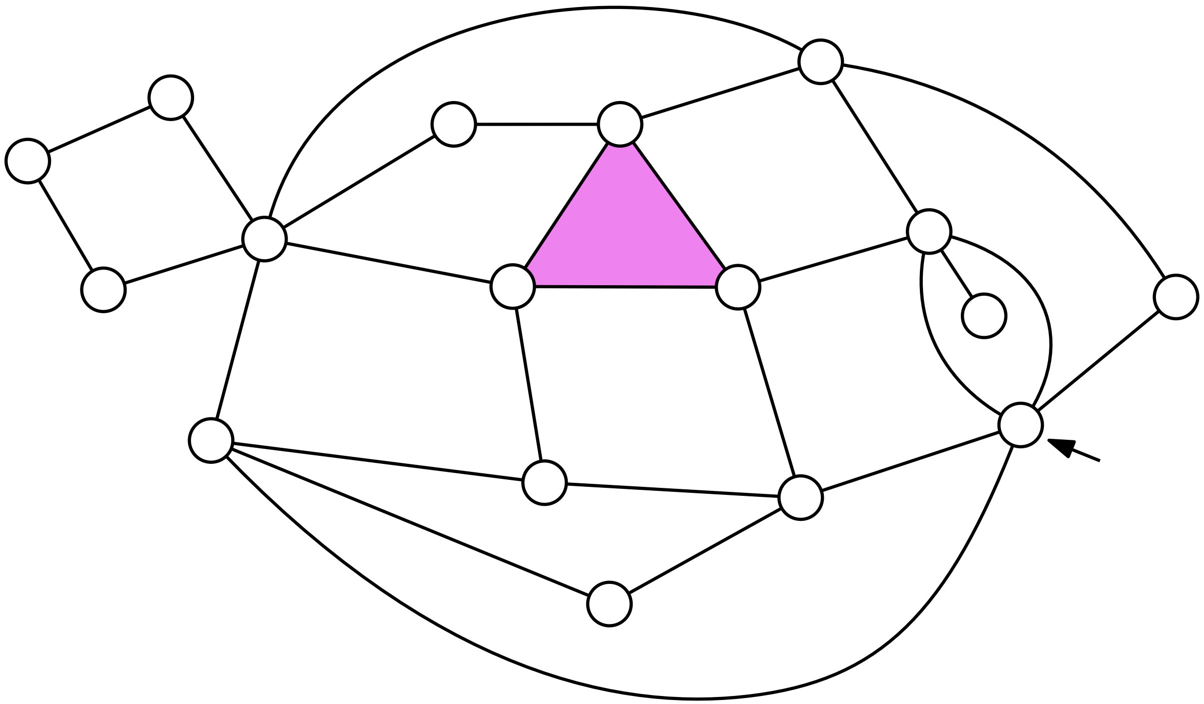


# Quasi-quadrangulations with a boundary

rooted maps with a marked inner face of degree $3$, other inner faces of degree $4$



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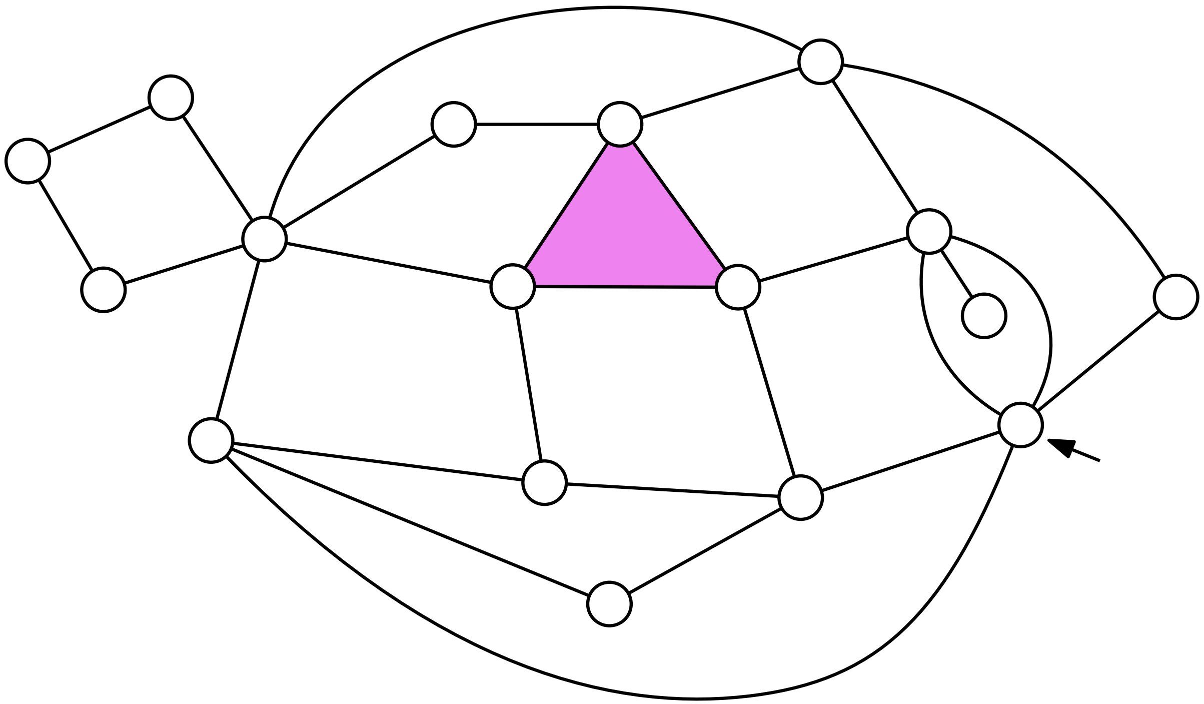

$n = 12$
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$\widetilde{Q}_{n,j} = \#$ quasi-quadrangulations with $n$ inner faces, and root-face degree $2j+1$

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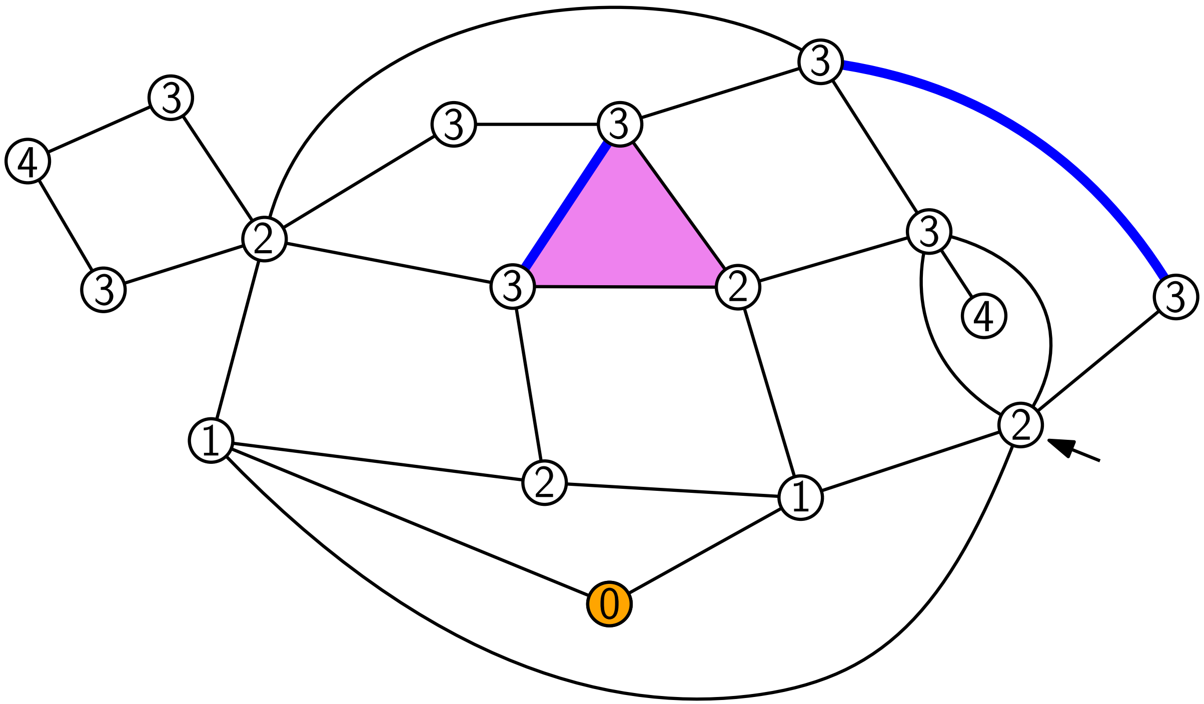


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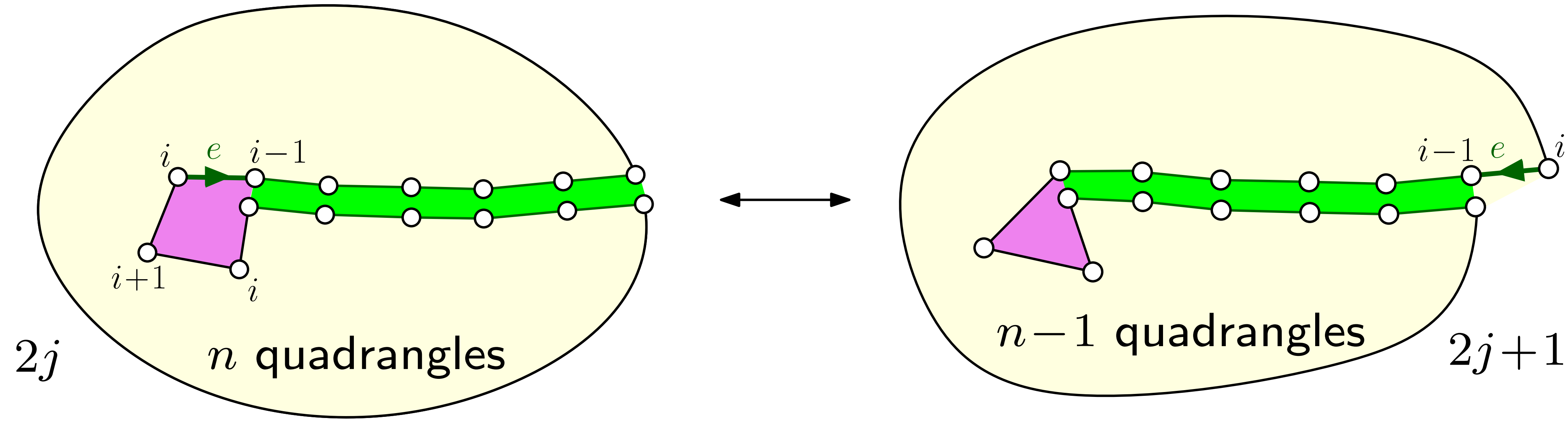
**Rk:** In distance-labeling from a given vertex, the two odd faces have a unique edge $i - i$ on their contour

# The identity via slit-slide-sew



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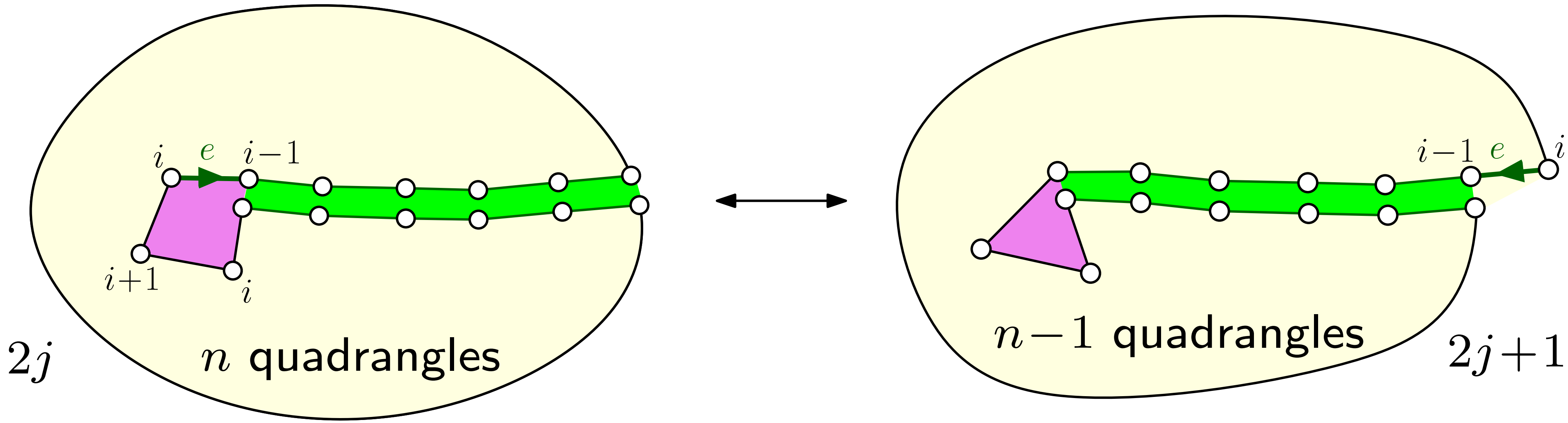


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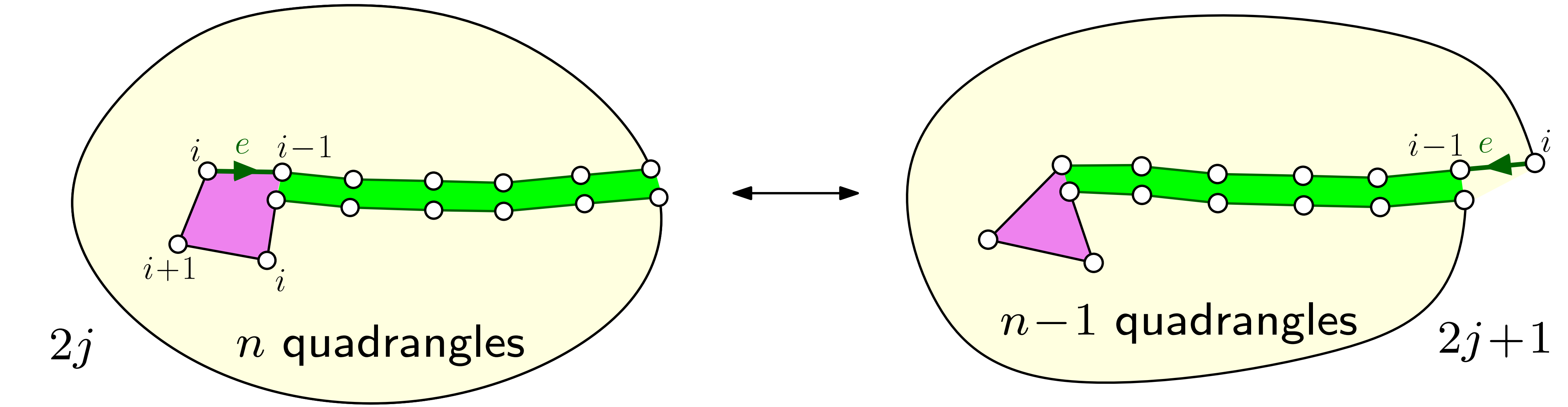


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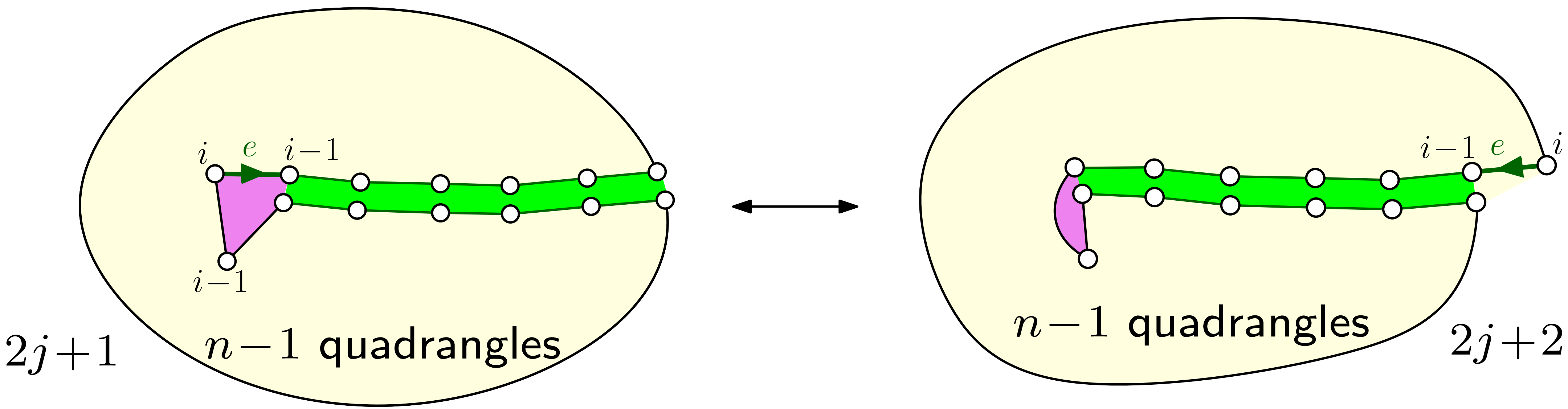
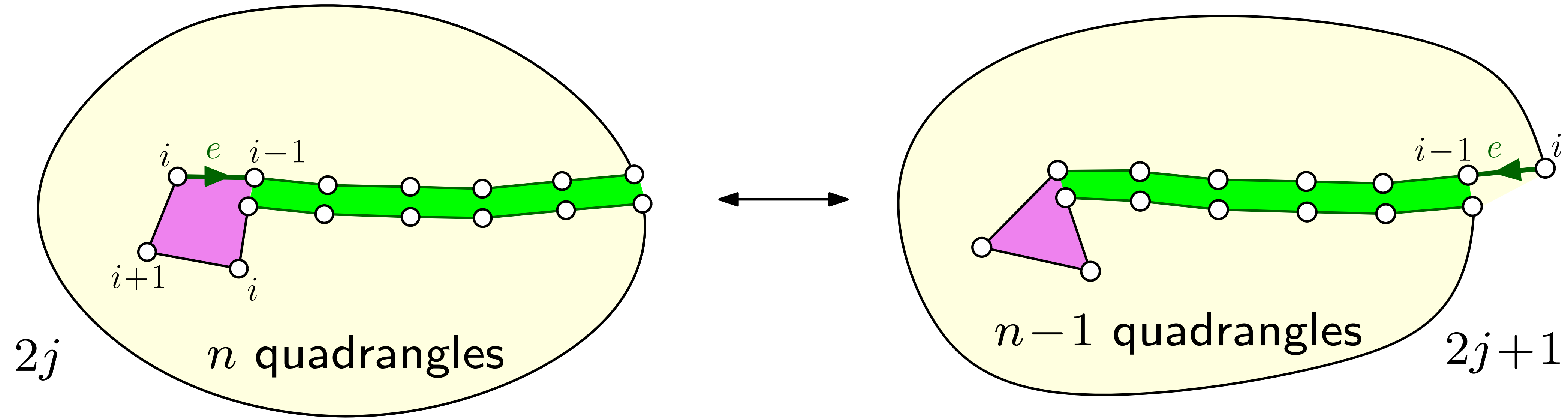


# The identity via slit-slide-sew

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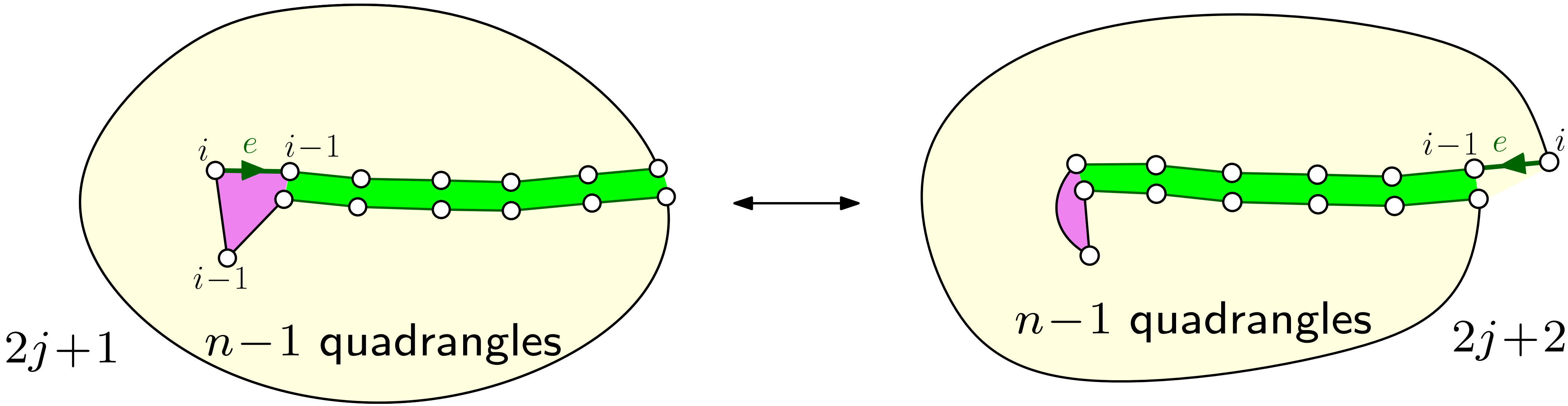
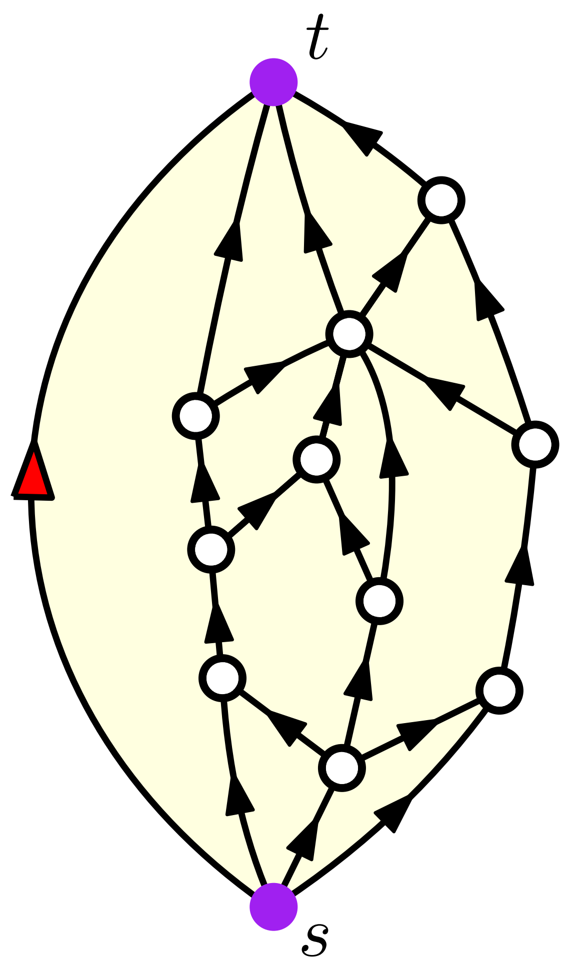



$$\widetilde{Q}_{n,j} \quad = \quad \tfrac{1}{2}\,2(2n+j-1)\,Q_{n-1,j+1}$$

# Oriented planar maps

# Plane bipolar orientations

Acyclic orientation on rooted map, where the ends of the root-edge are the unique source and unique sink



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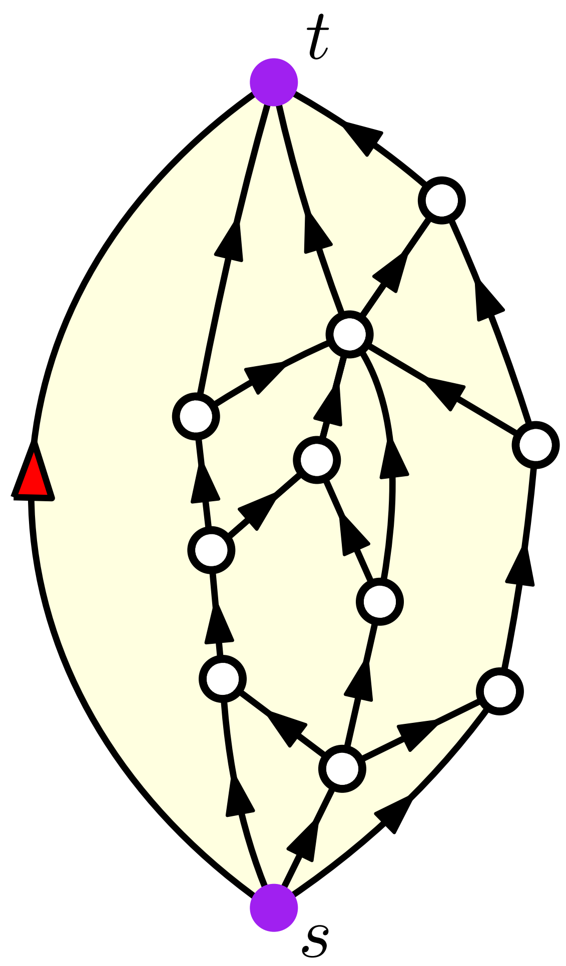


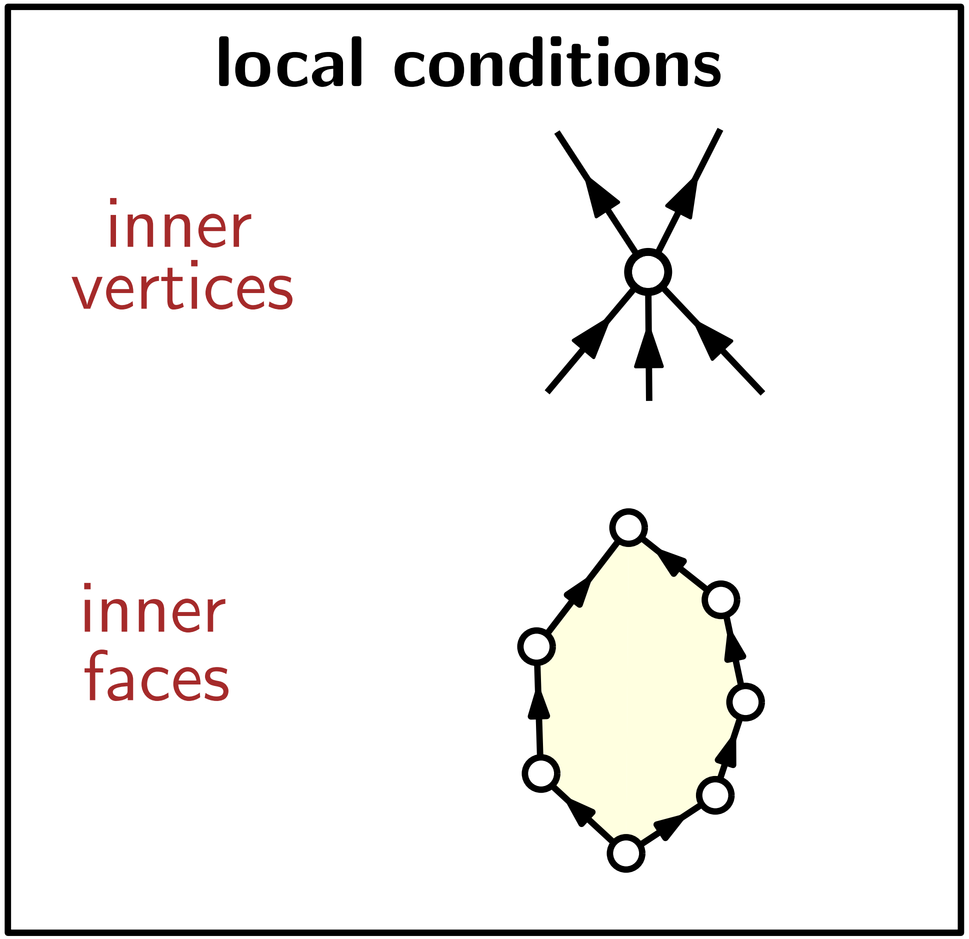

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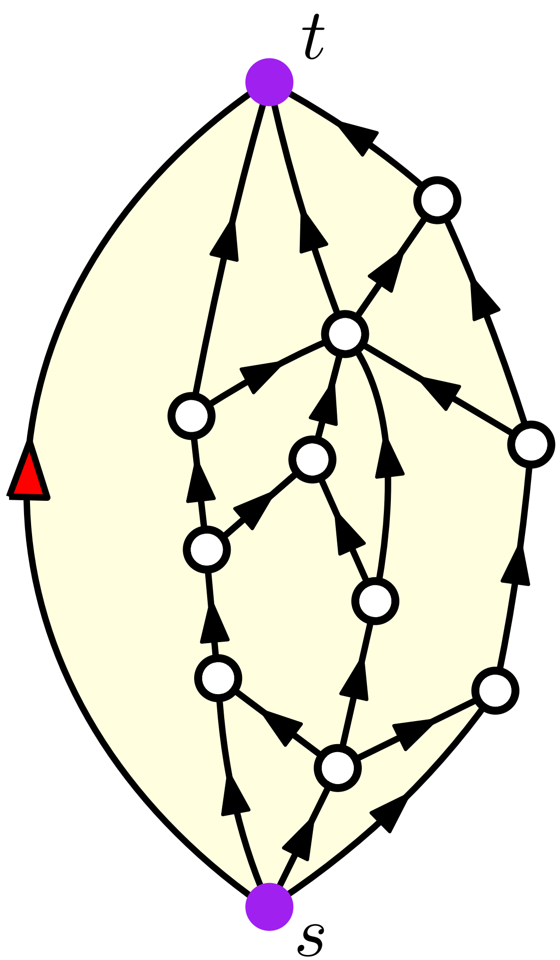
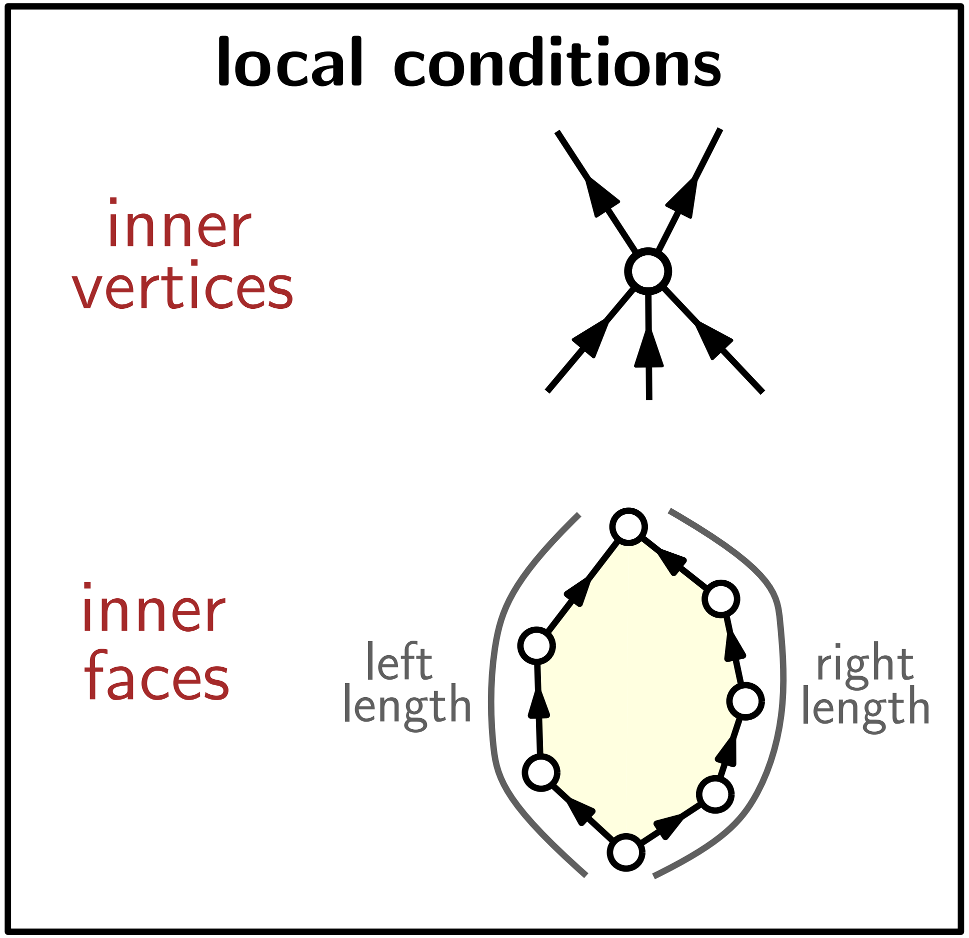
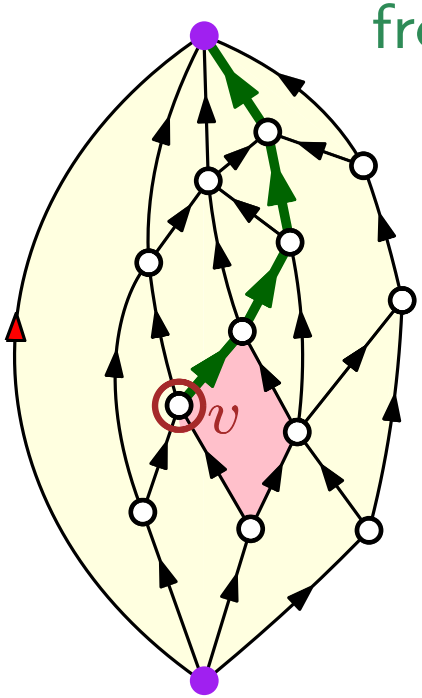


# Slit-slide-sew bijection for bipolar orientations

bipolar orientation with
marked inner vertex $v$

rightmost path
from $v$




$n$ inner vertices
$j$ outer vertices

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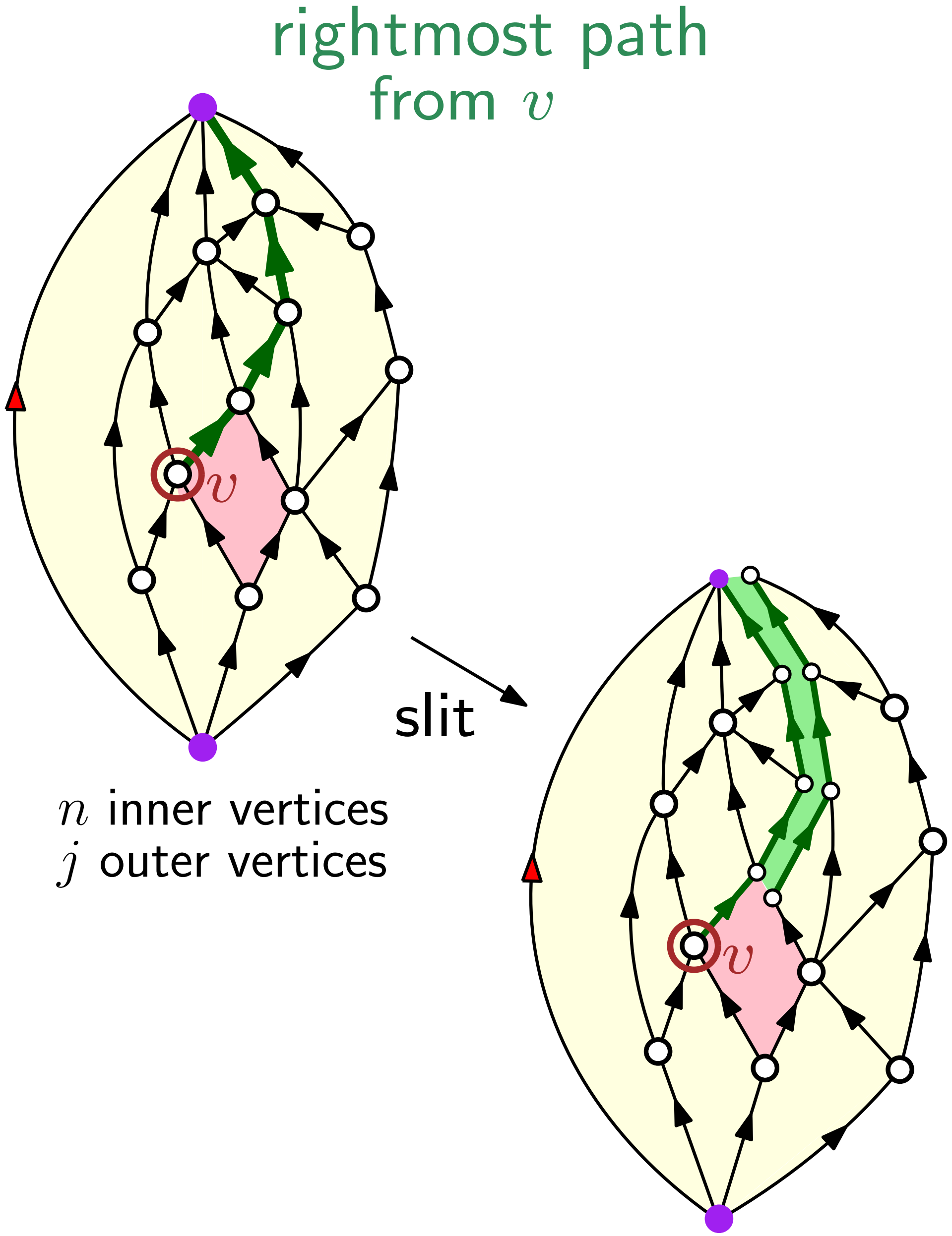

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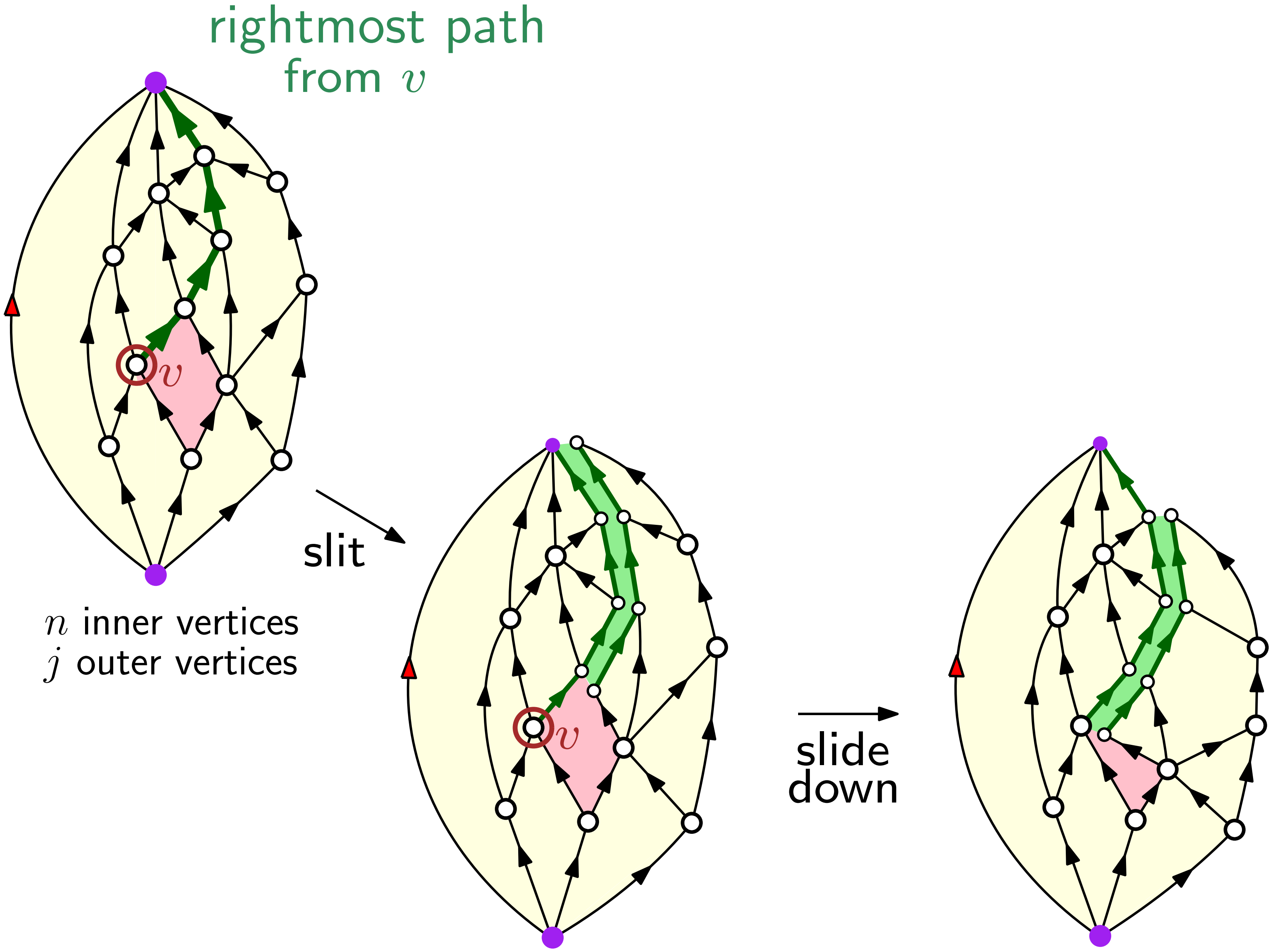

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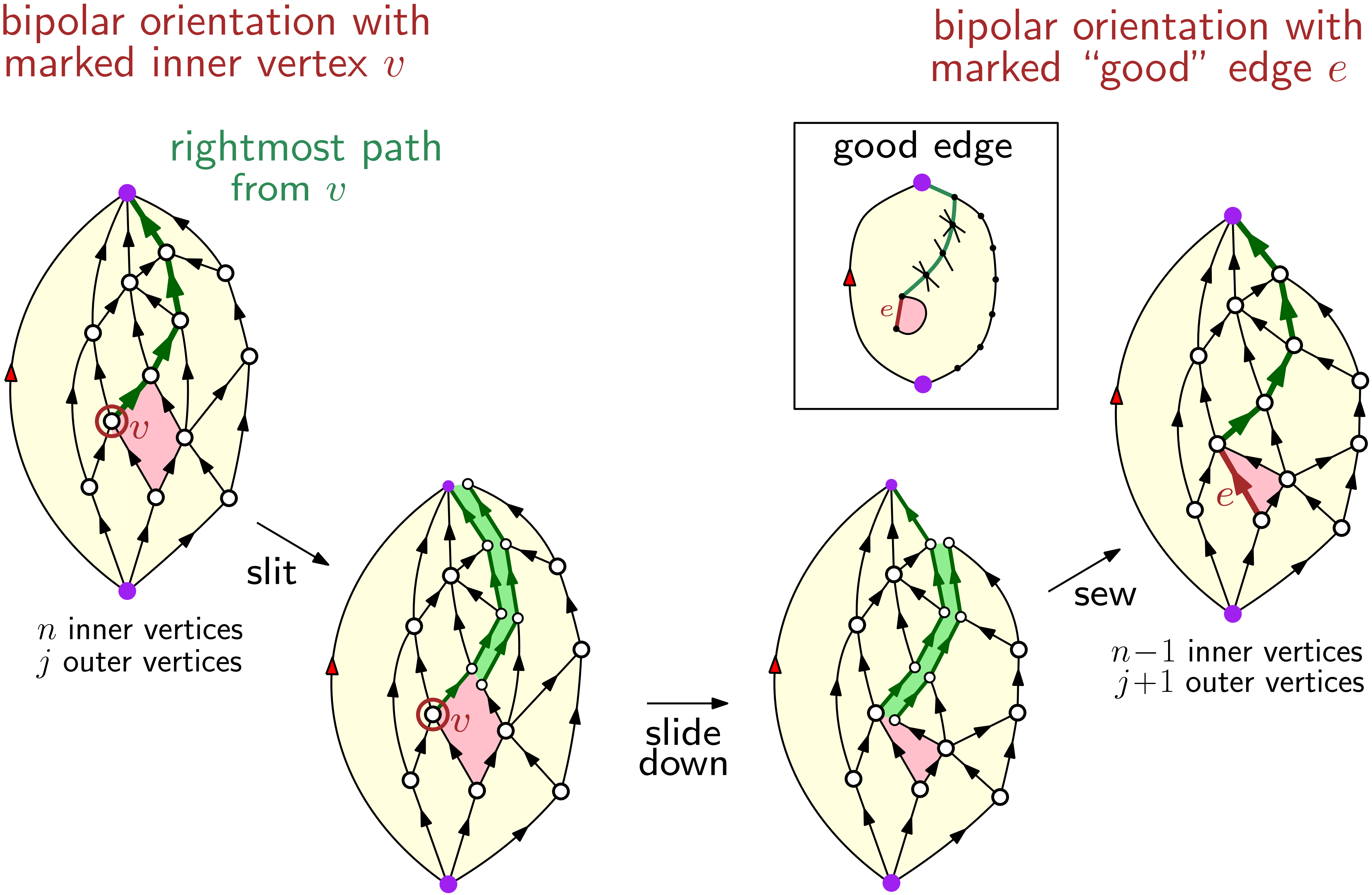
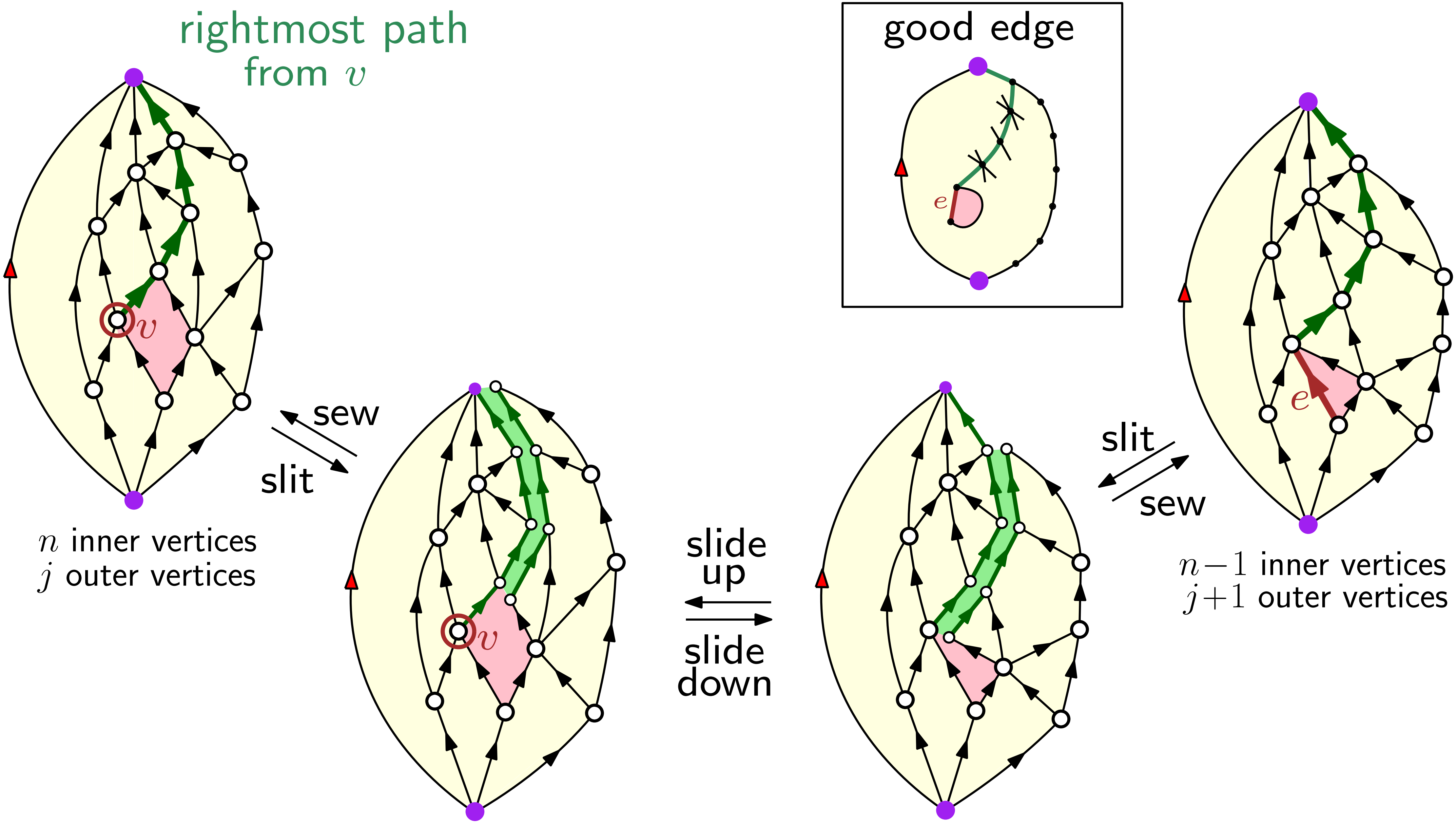


# Slit-slide-sew bijection for bipolar orientations

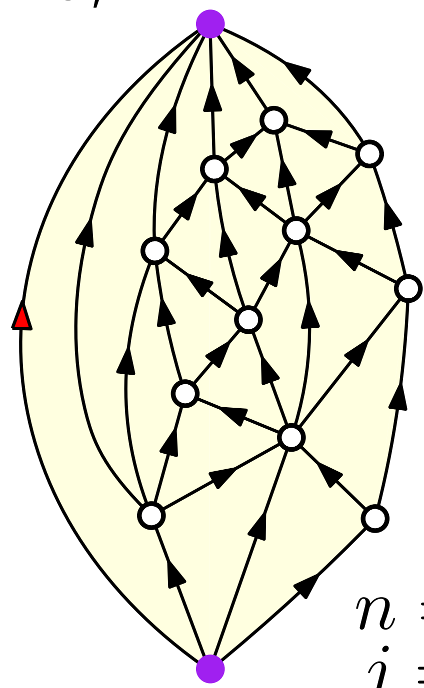
bipolar orientation with marked inner vertex $v$

bipolar orientation with marked "good" edge $e$



# Inner-triangulated bipolar orientations

$T_{n,j} :=$ # bipolar orientations with inner faces of degree $3$,
$n$ inner vertices and $j$ outer vertices



$n = 8$
$j = 5$

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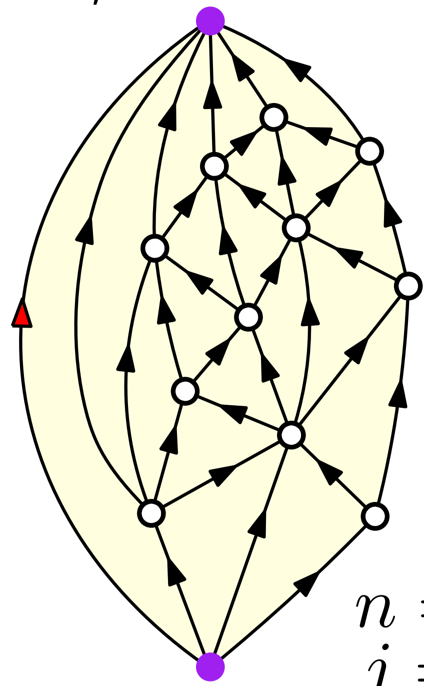
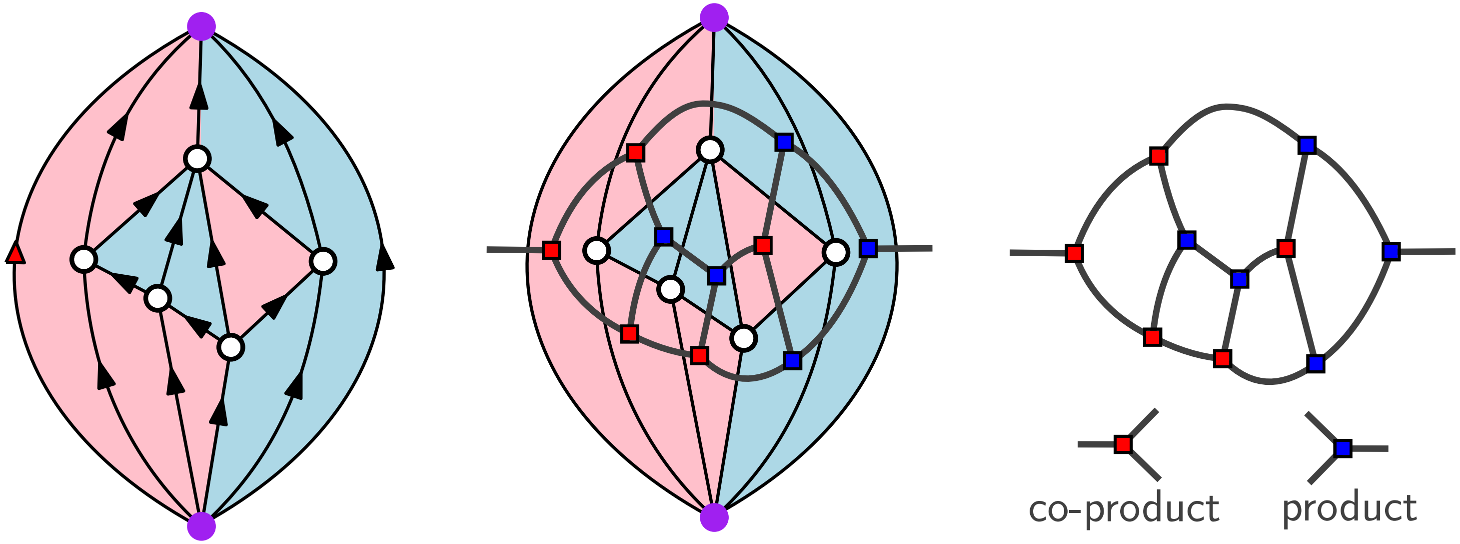



**Rk:** By duality, $T_{n,2} =$ # prographs with $2n$ operators



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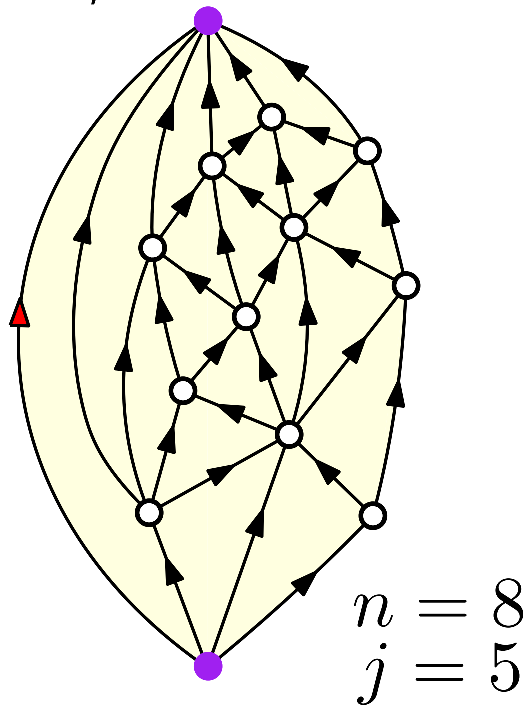


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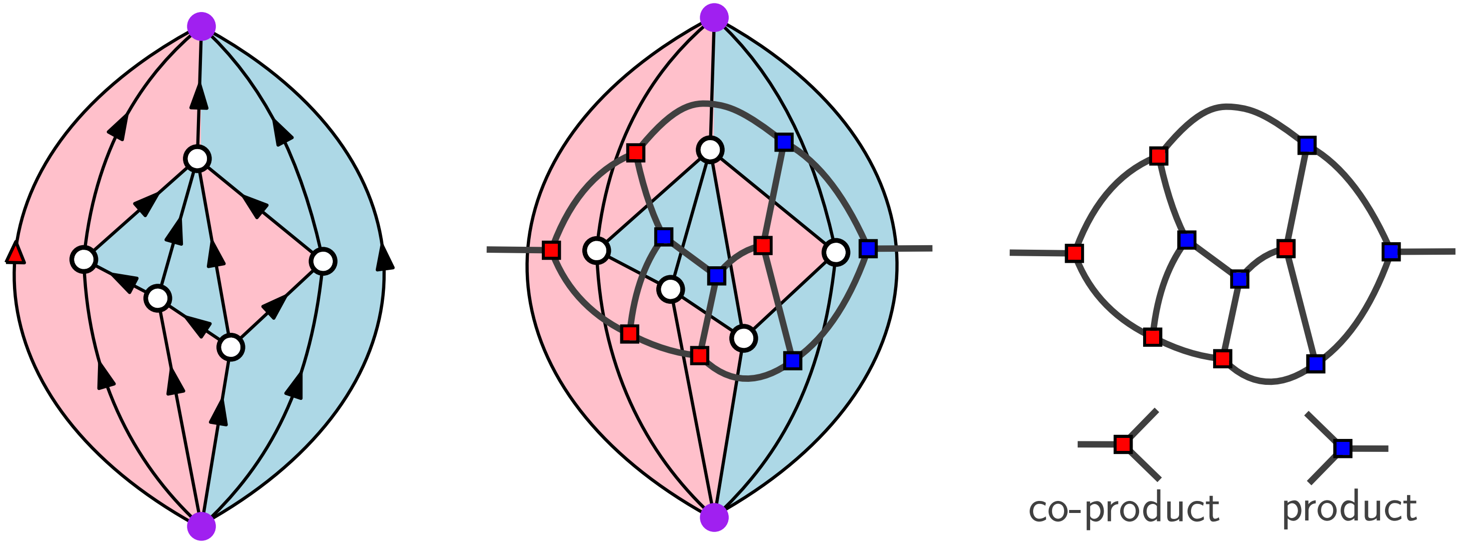


Counting formula: [Tutte'73, Bousquet-Mélou'11]

$$T_{n,j} = j(j-1)\frac{(3n+2j-4)!}{n!(n+j-1)!(n+j)!}$$

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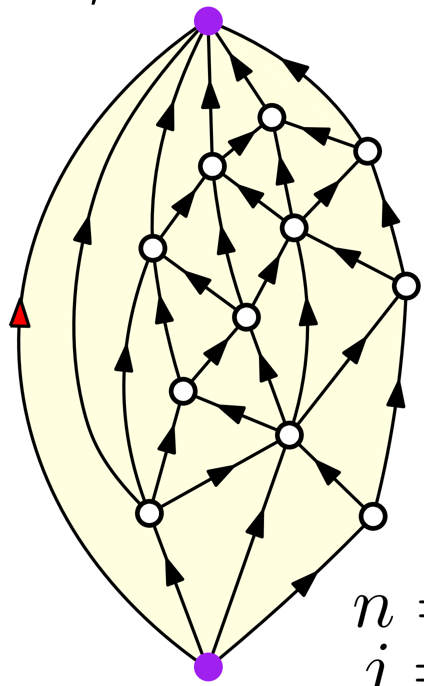

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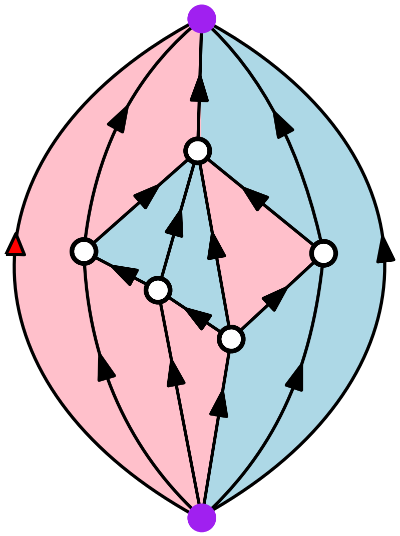

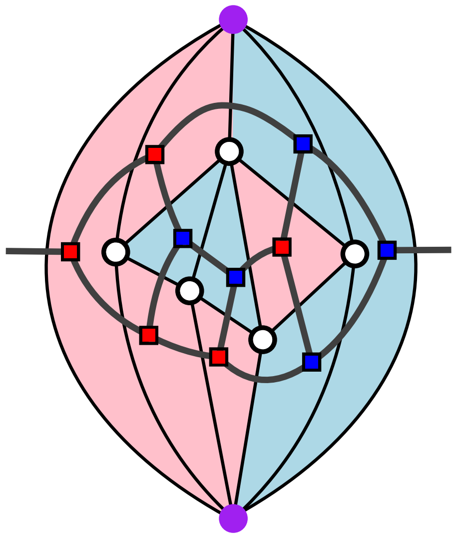

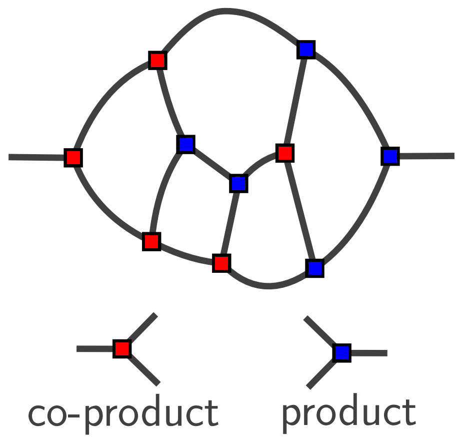


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$\Downarrow$

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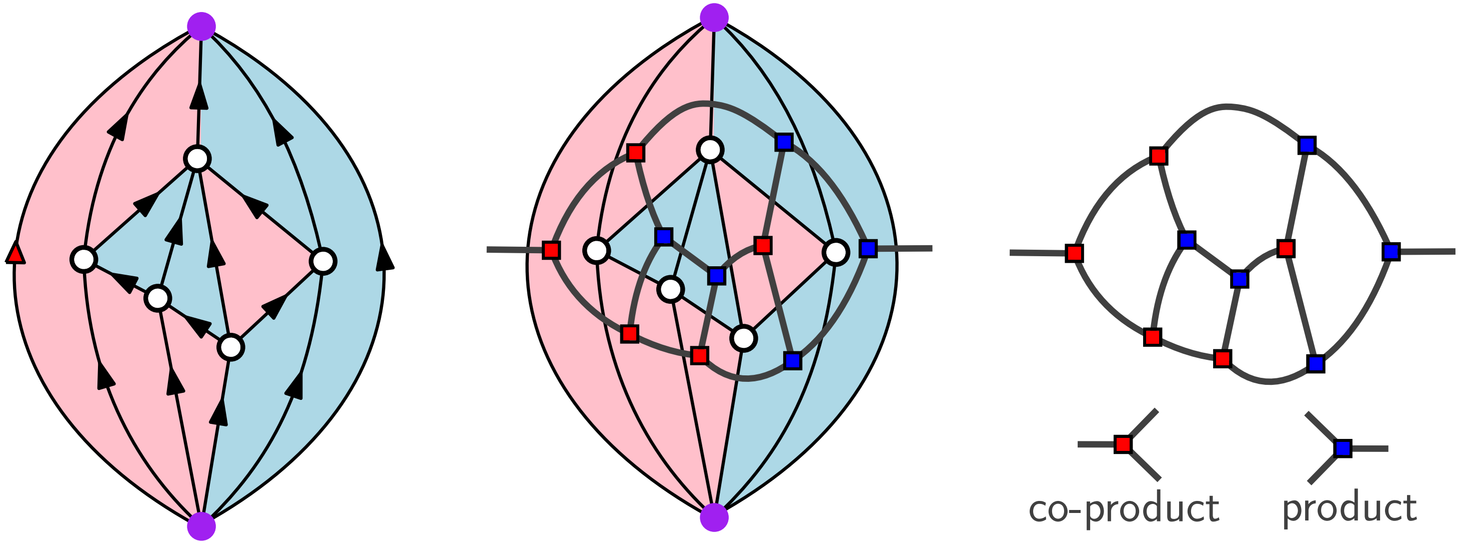


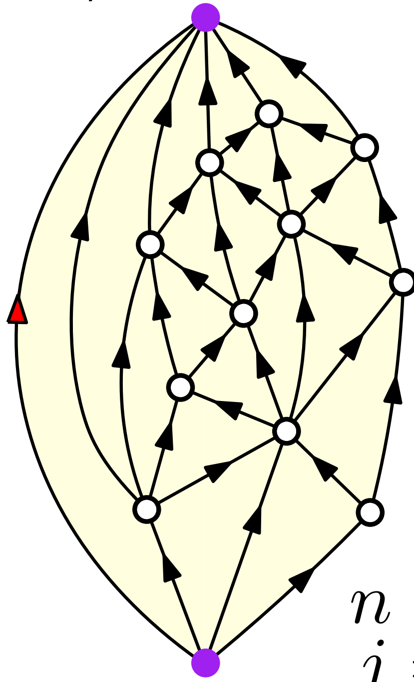

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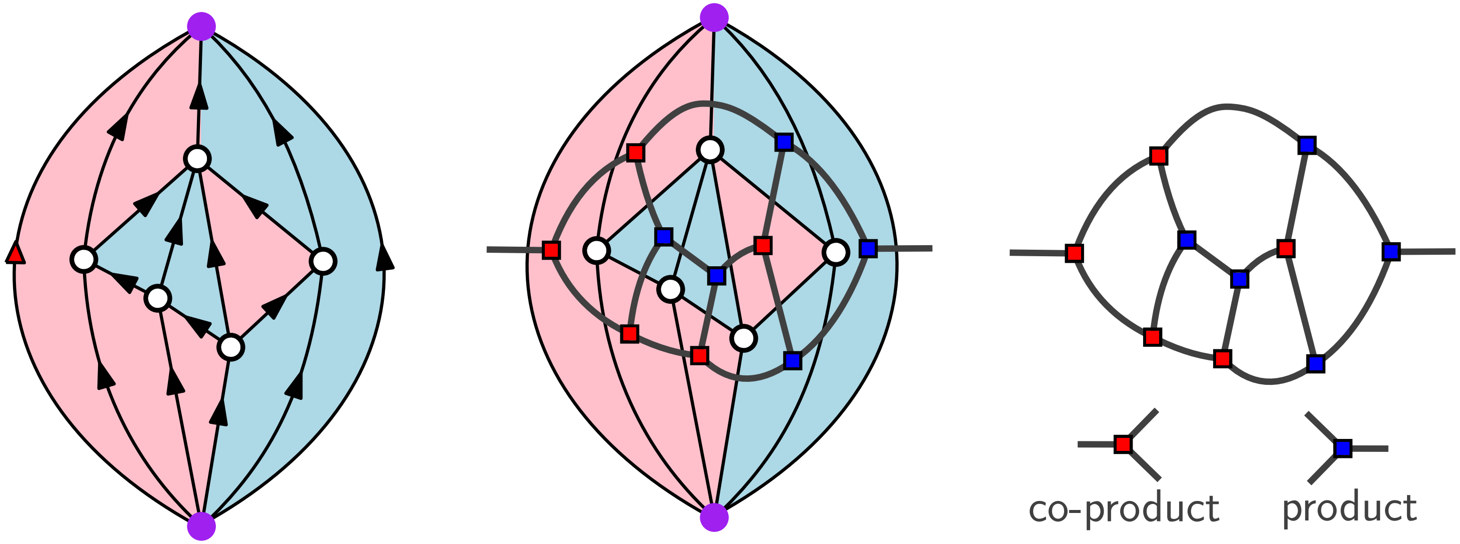


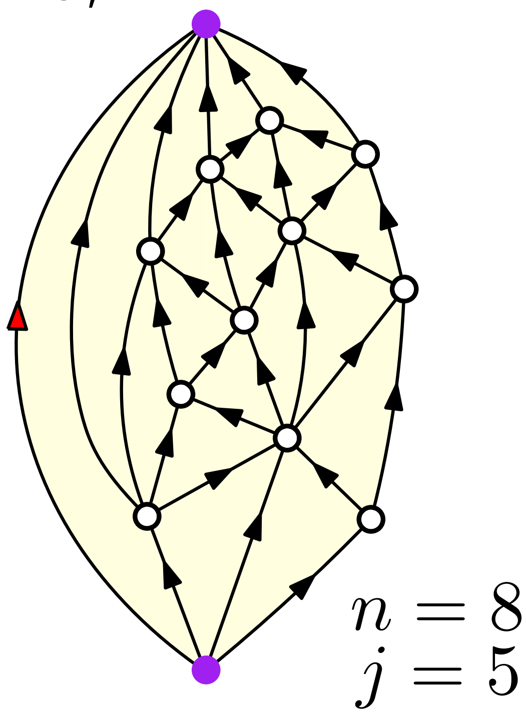


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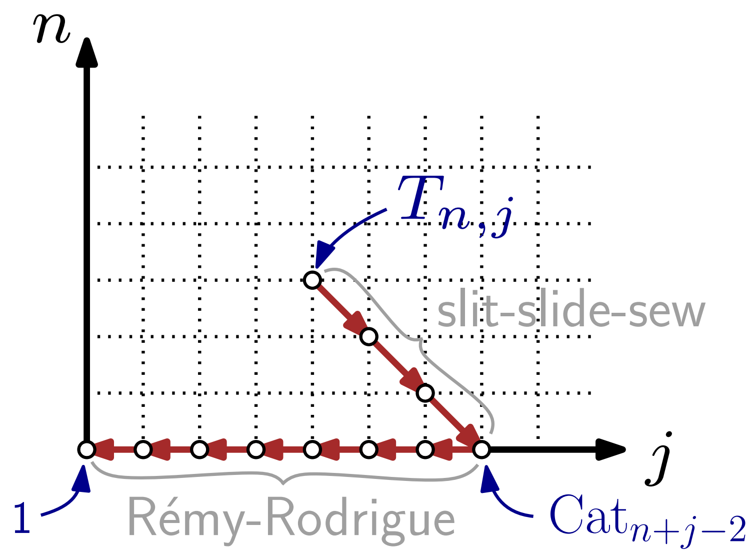

# Marked bipolar orientations

$$n\,T_{n,j} = \frac{j-1}{j+1}(3n+2j-4)\,T_{n-1,j+1}$$

$\mathcal{T}_{n,j} = \{$ inner-triangulated bipolar orientations with $n$ inner vertices, $j$ outer vertices $\}$

$\mathcal{T}^{\bullet}_{n,j}$ those with a marked inner vertex $\overline{\mathcal{T}}_{n,j}$ those with a marked edge

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**Rk:** Identity also reads $$|\mathcal{T}^{\bullet}_{n,j}| = \frac{j-1}{j+1}\,|\overline{\mathcal{T}}_{n-1,j+1}|$$

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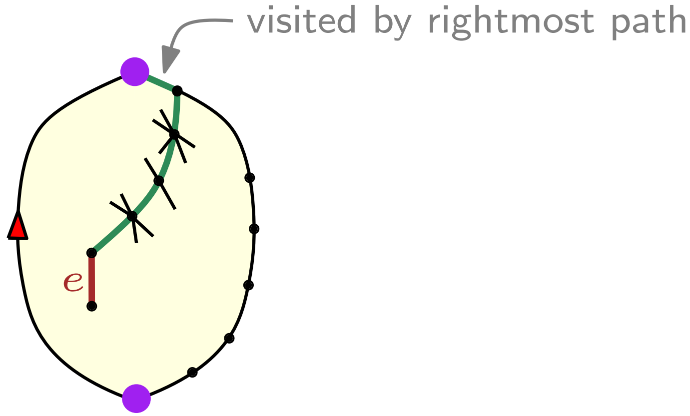
**Def:** An edge $e$ is **boundary-reaching** if




$\overline{\mathcal{T}}^{\partial}_{n,j} \subset \overline{\mathcal{T}}_{n,j}$ subfamily where the marked edge is boundary-reaching

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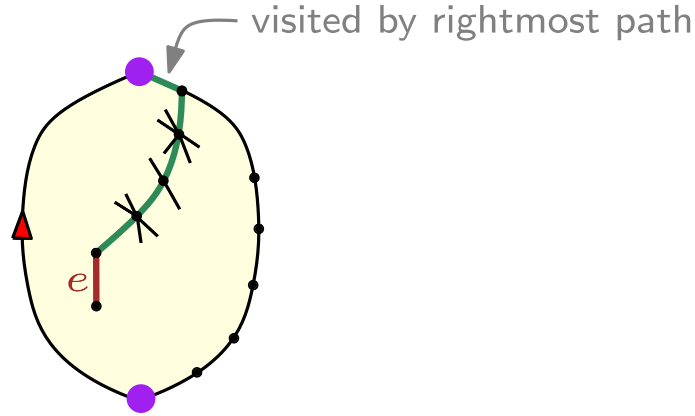
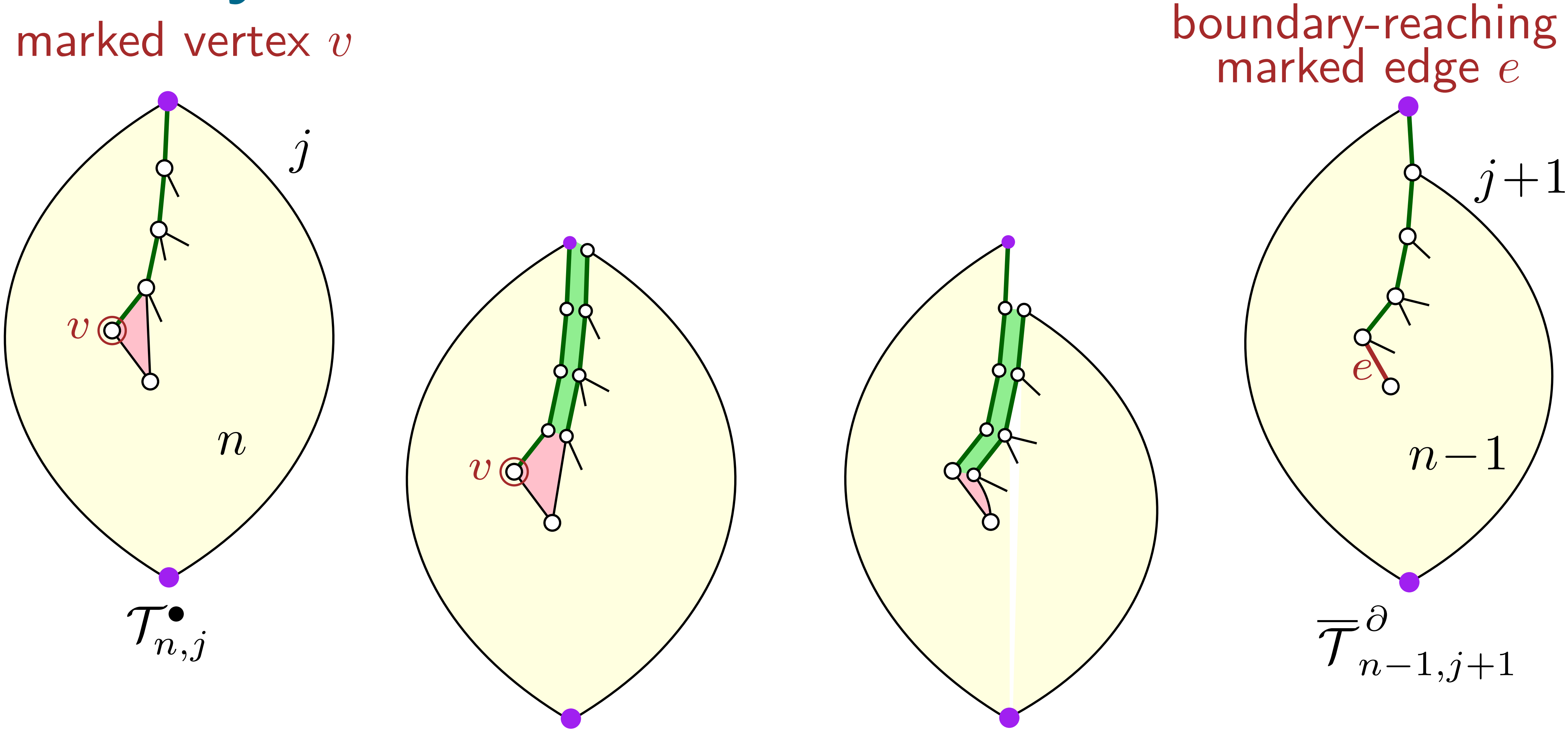



$\overline{\mathcal{T}}^{\partial}_{n,j} \subset \overline{\mathcal{T}}_{n,j}$ subfamily where the marked edge is boundary-reaching

$$\pi_{n,j} := \frac{|\overline{\mathcal{T}}^{\partial}_{n,j}|}{|\overline{\mathcal{T}}_{n,j}|}$$ (boundary-reaching probability)

# Identity via slit-slide-sew



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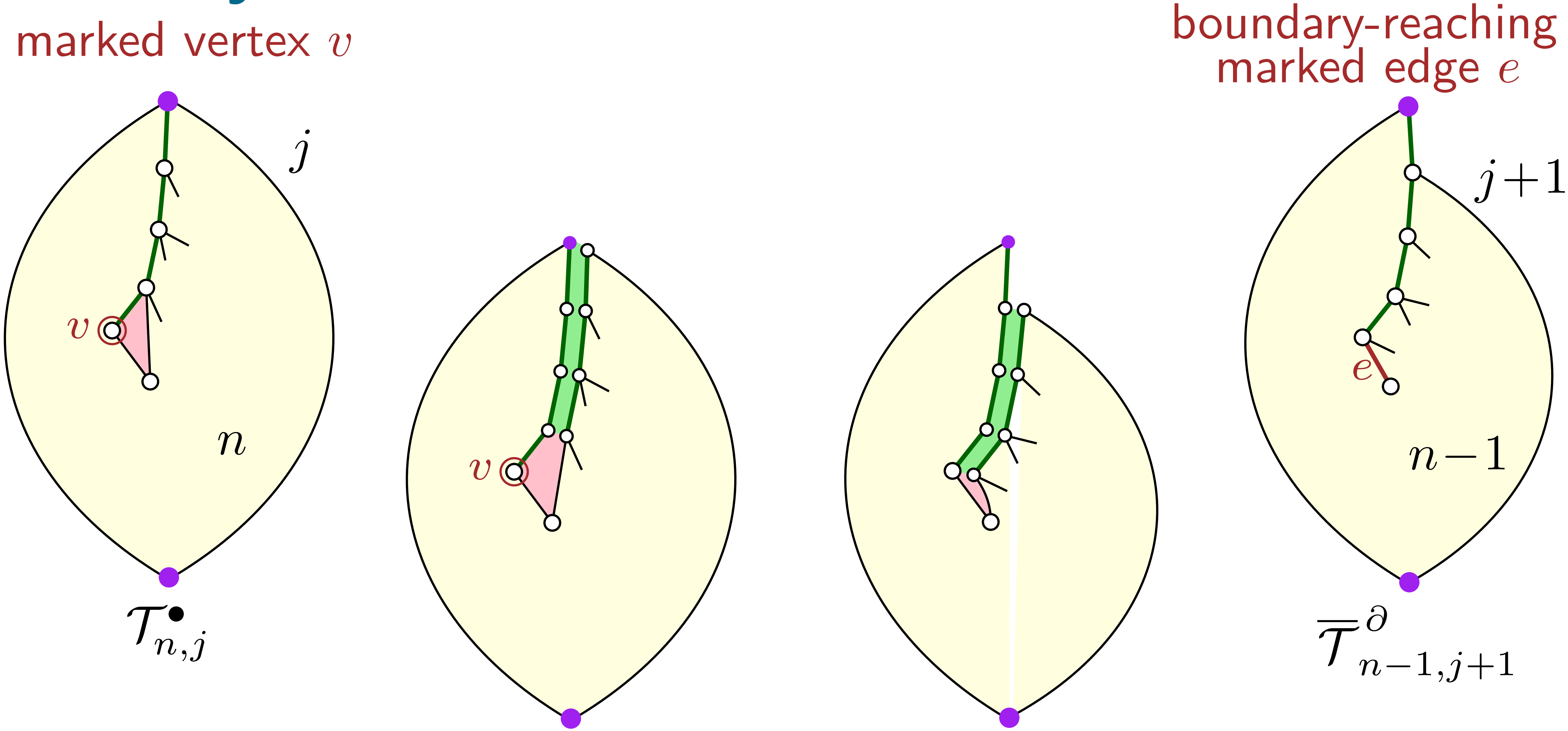


$$\Rightarrow \quad |\mathcal{T}^\bullet_{n,j}| = |\overline{\mathcal{T}}^{\partial}_{n-1,j+1}| \quad = \quad \pi_{n-1,j+1} |\overline{\mathcal{T}}_{n-1,j+1}|$$

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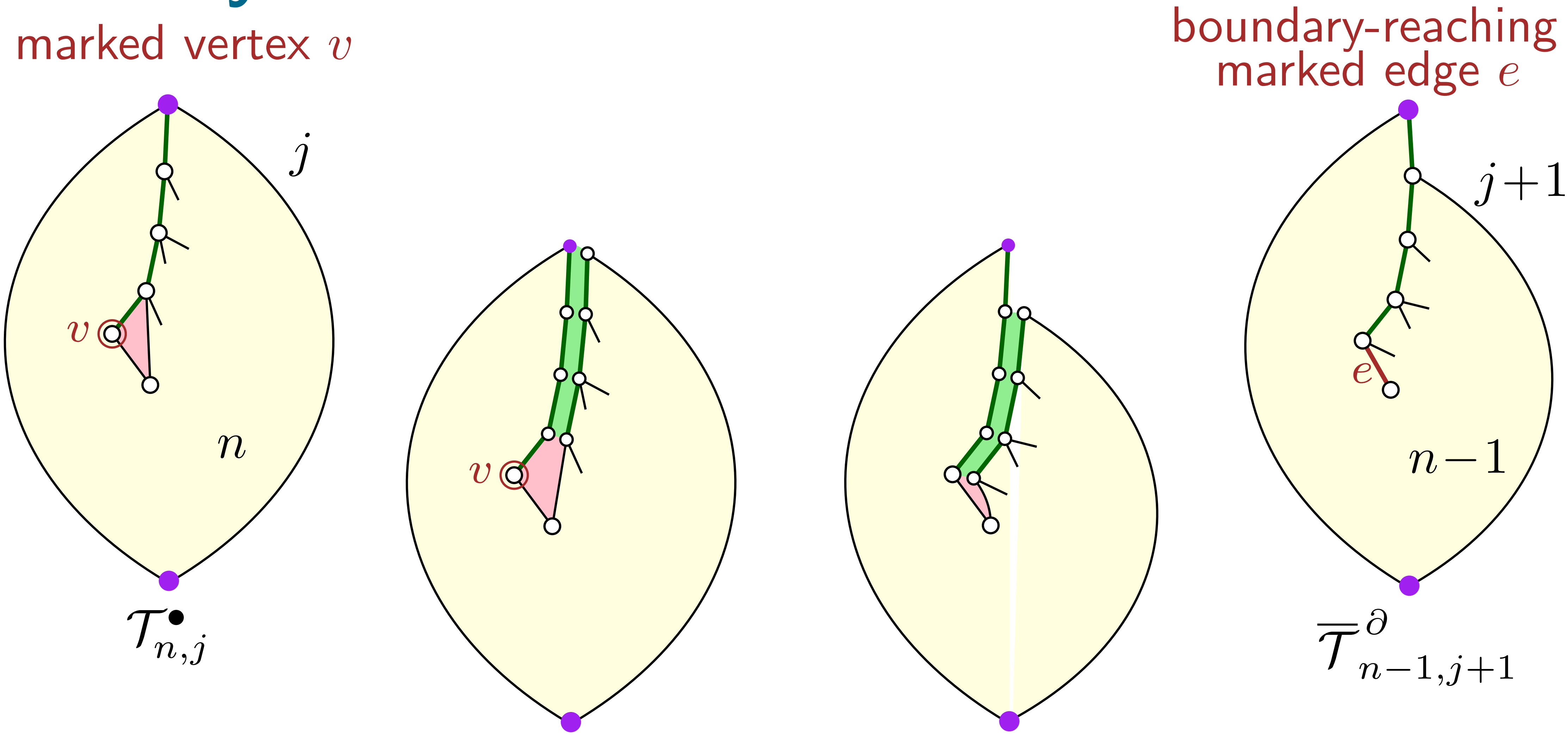


$$\Rightarrow \quad |\mathcal{T}^{\bullet}_{n,j}| = |\overline{\mathcal{T}}^{\partial}_{n-1,j+1}| \;=\; \pi_{n-1,j+1}|\overline{\mathcal{T}}_{n-1,j+1}|$$

whereas Identity $\Leftrightarrow \quad |\mathcal{T}^{\bullet}_{n,j}| = \frac{j-1}{j+1} \; |\overline{\mathcal{T}}_{n-1,j+1}|$

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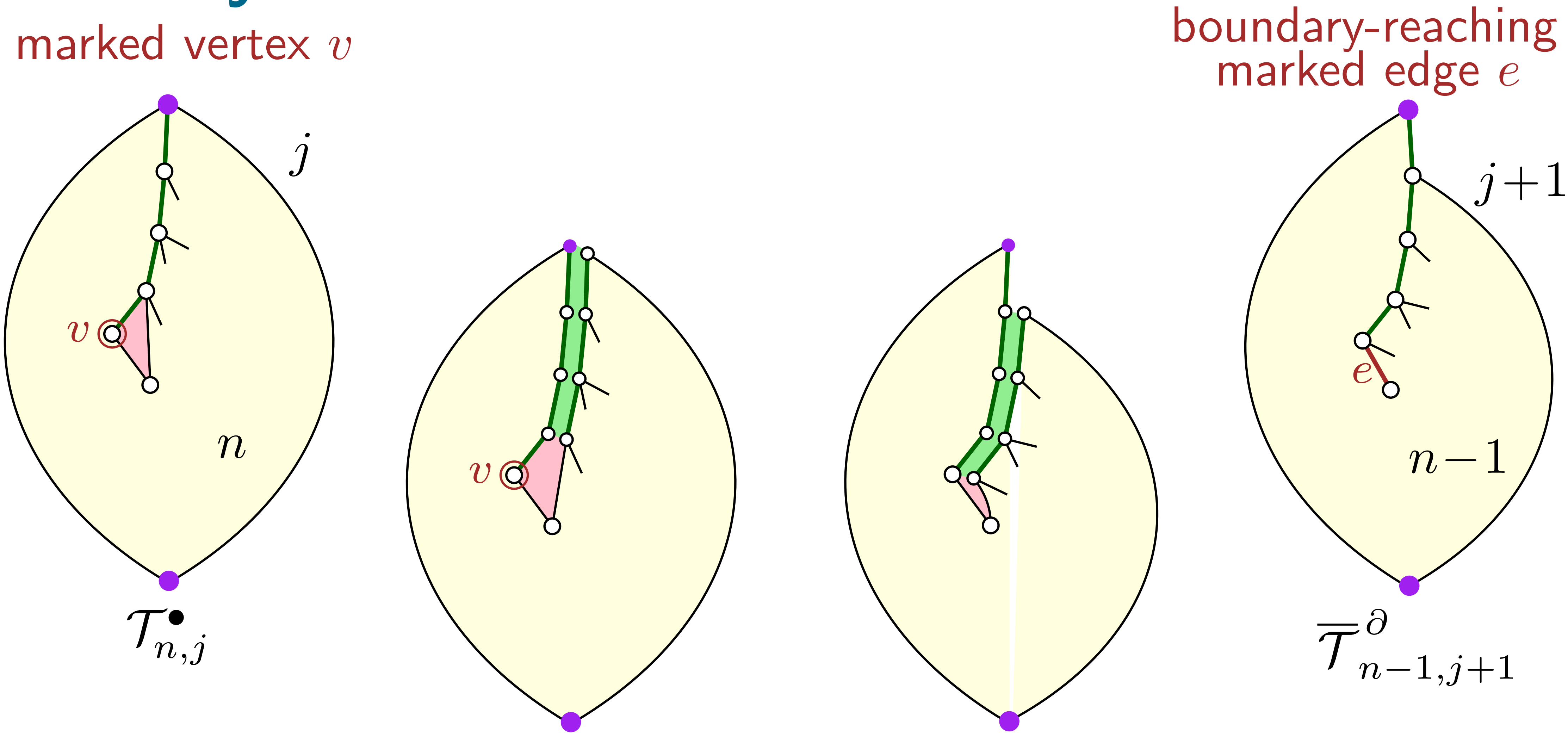
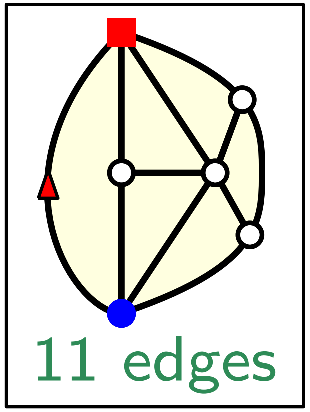



$$\Rightarrow \quad |\mathcal{T}^\bullet_{n,j}| = |\overline{\mathcal{T}}^\partial_{n-1,j+1}| \quad = \quad \pi_{n-1,j+1}|\overline{\mathcal{T}}_{n-1,j+1}|$$

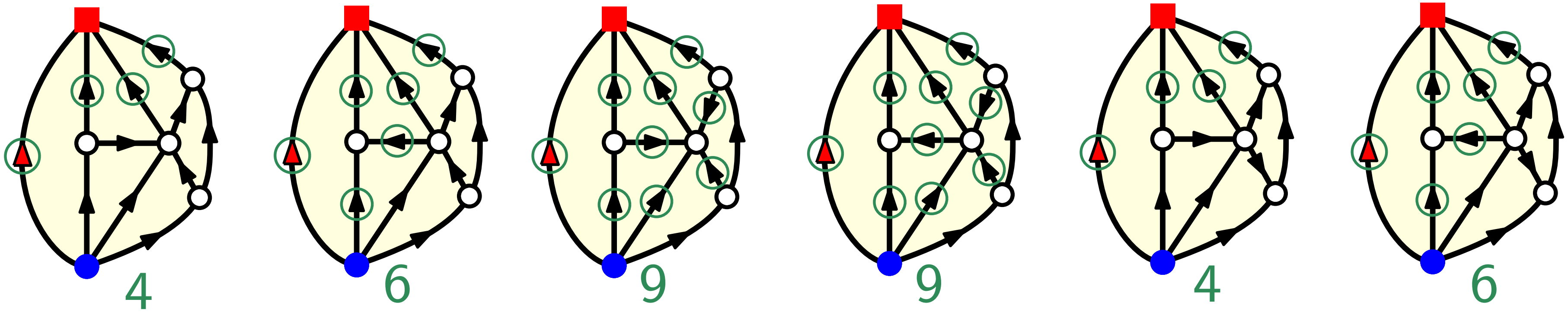
whereas Identity $\Leftrightarrow$ $|\mathcal{T}^\bullet_{n,j}| = \frac{j-1}{j+1} \, |\overline{\mathcal{T}}_{n-1,j+1}|$

it remains to prove that $\boxed{\pi_{n,j} = 1 - \frac{2}{j}}$

# Boundary-reaching ratio on a fixed map?

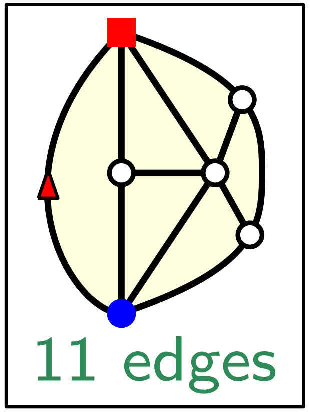



edges not boundary-reaching are surrounded



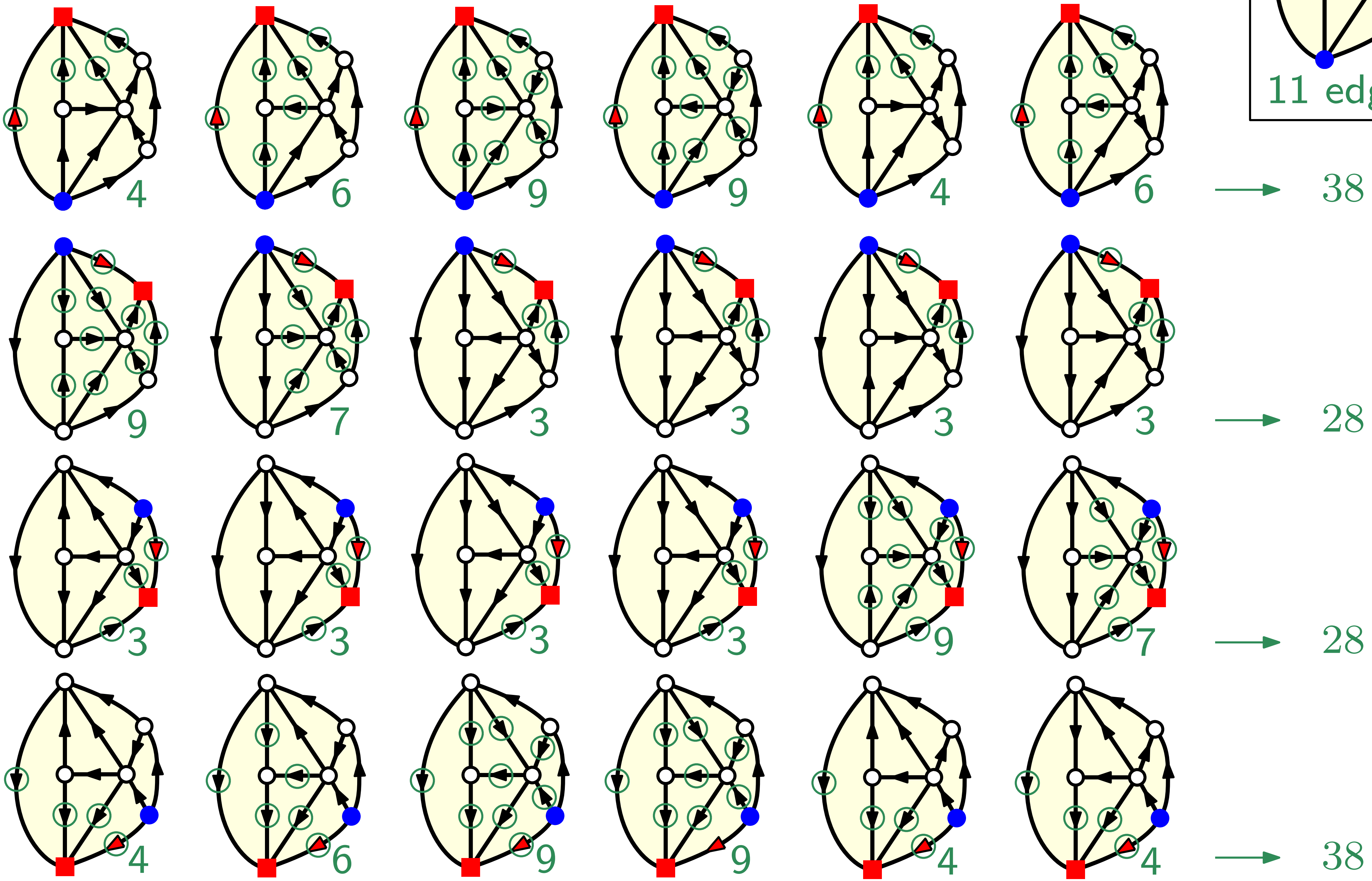

$\longrightarrow \quad 38 \neq 33$

# Boundary-reaching ratio on a fixed map?




edges not boundary-reaching are surrounded

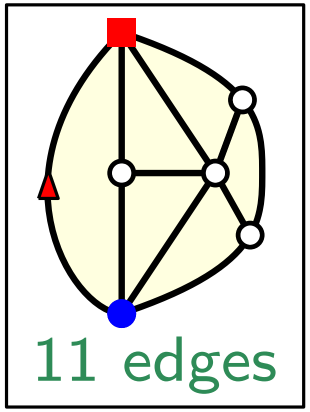
include bipolar orientations rooted at other outer edges

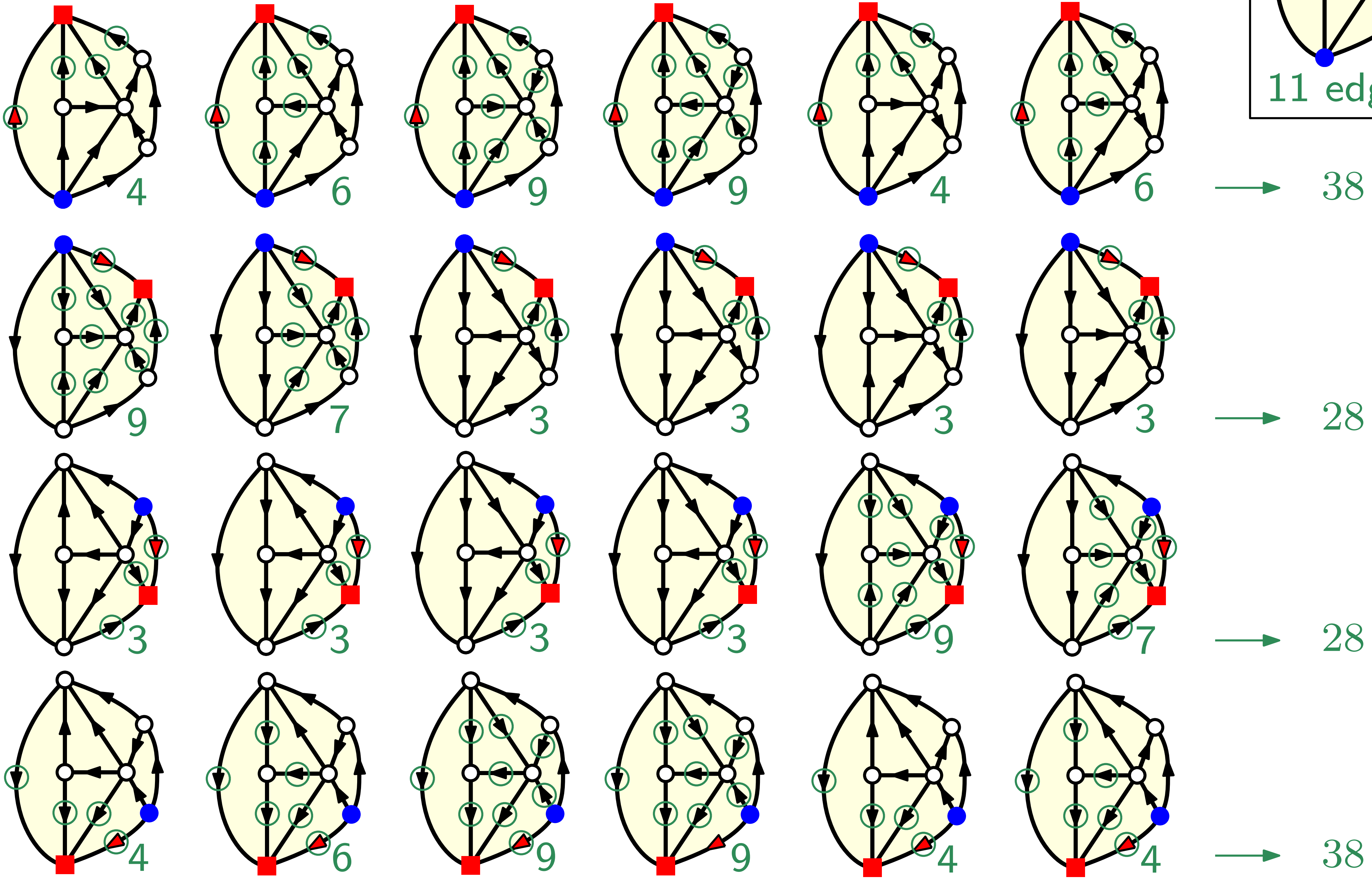


# Boundary-reaching ratio on a fixed map?

edges not boundary-reaching are surrounded

each edge is surrounded $12$ times out of $24$ !

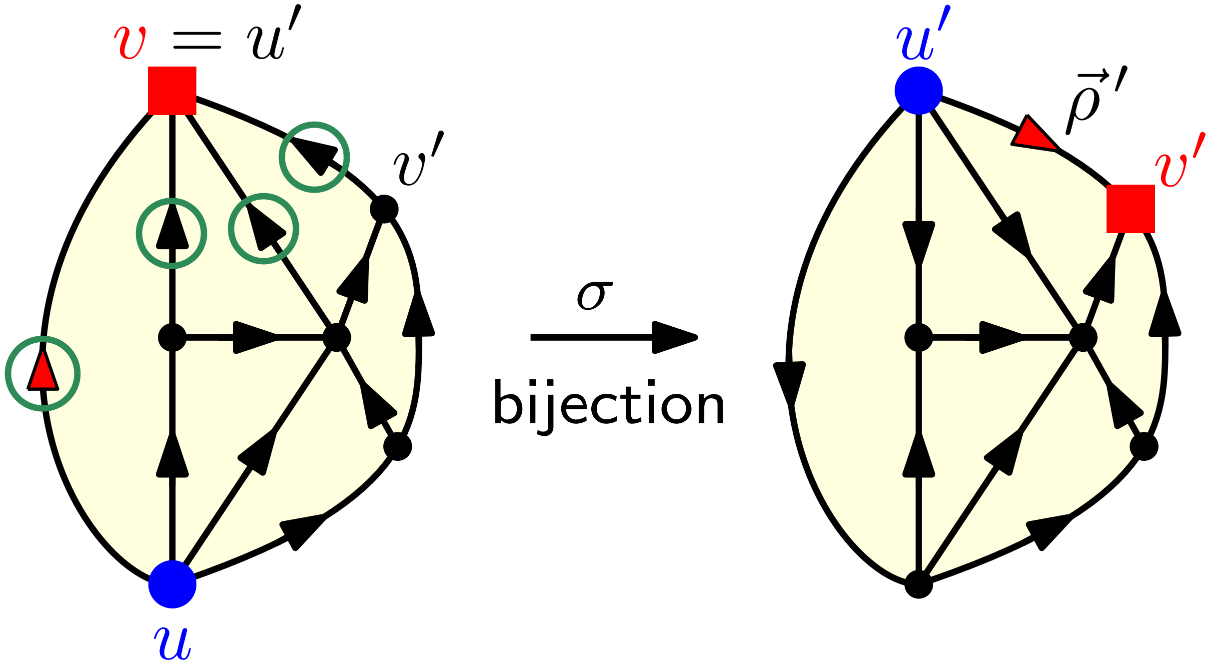


# Rerooting operator

[de Fraysseix et al'95]

Number of bipolar orientations does not depend on choice of root-edge




edges that can reach $v'$ are unchanged, other ones are returned

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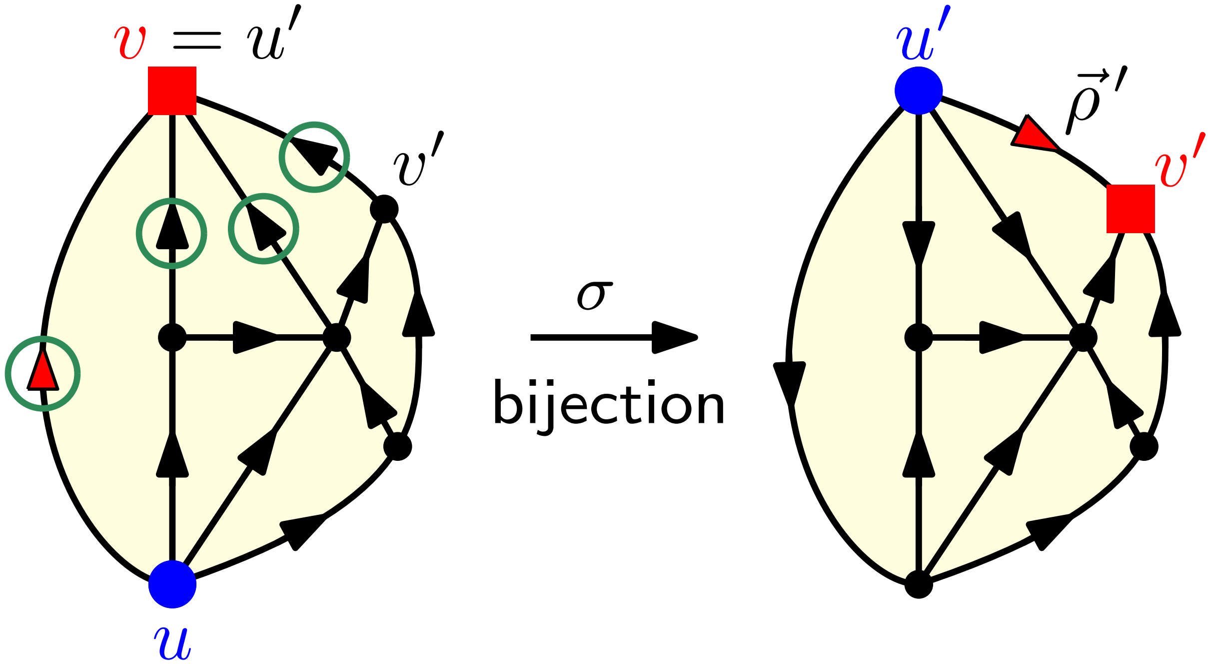
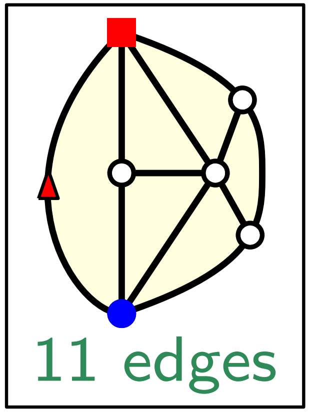
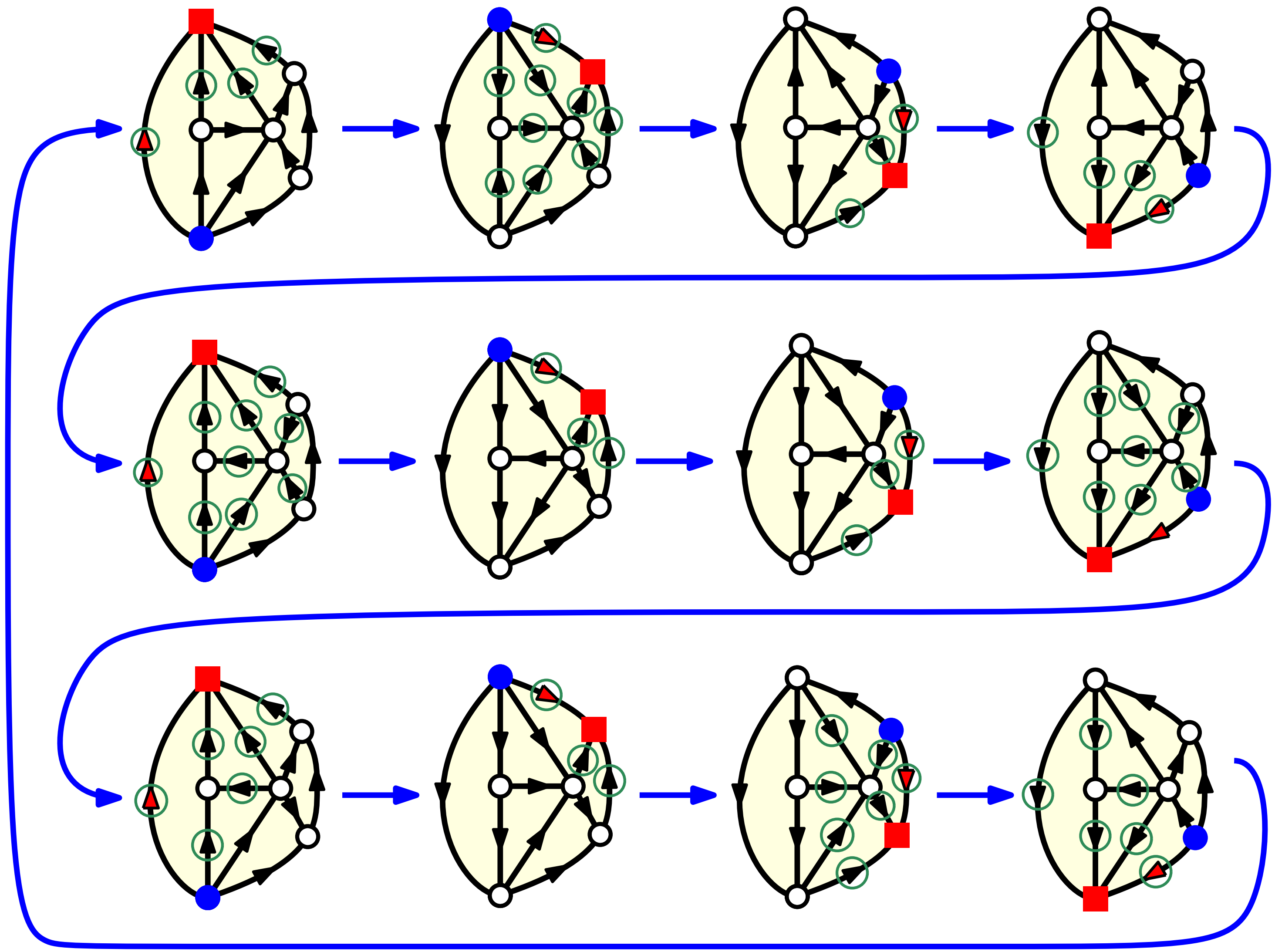



edges that $\underbrace{\text{can reach } v'}_{\text{boundary-reaching}}$ are unchanged, other ones are returned

# Orbit property

along a $\sigma$-orbit, any edge is reversed $2/j$ of the times



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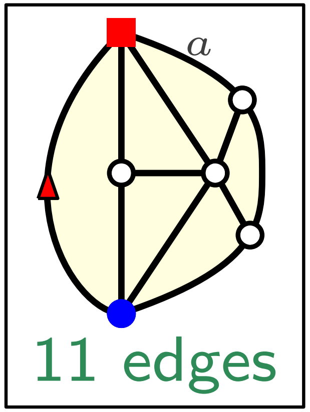


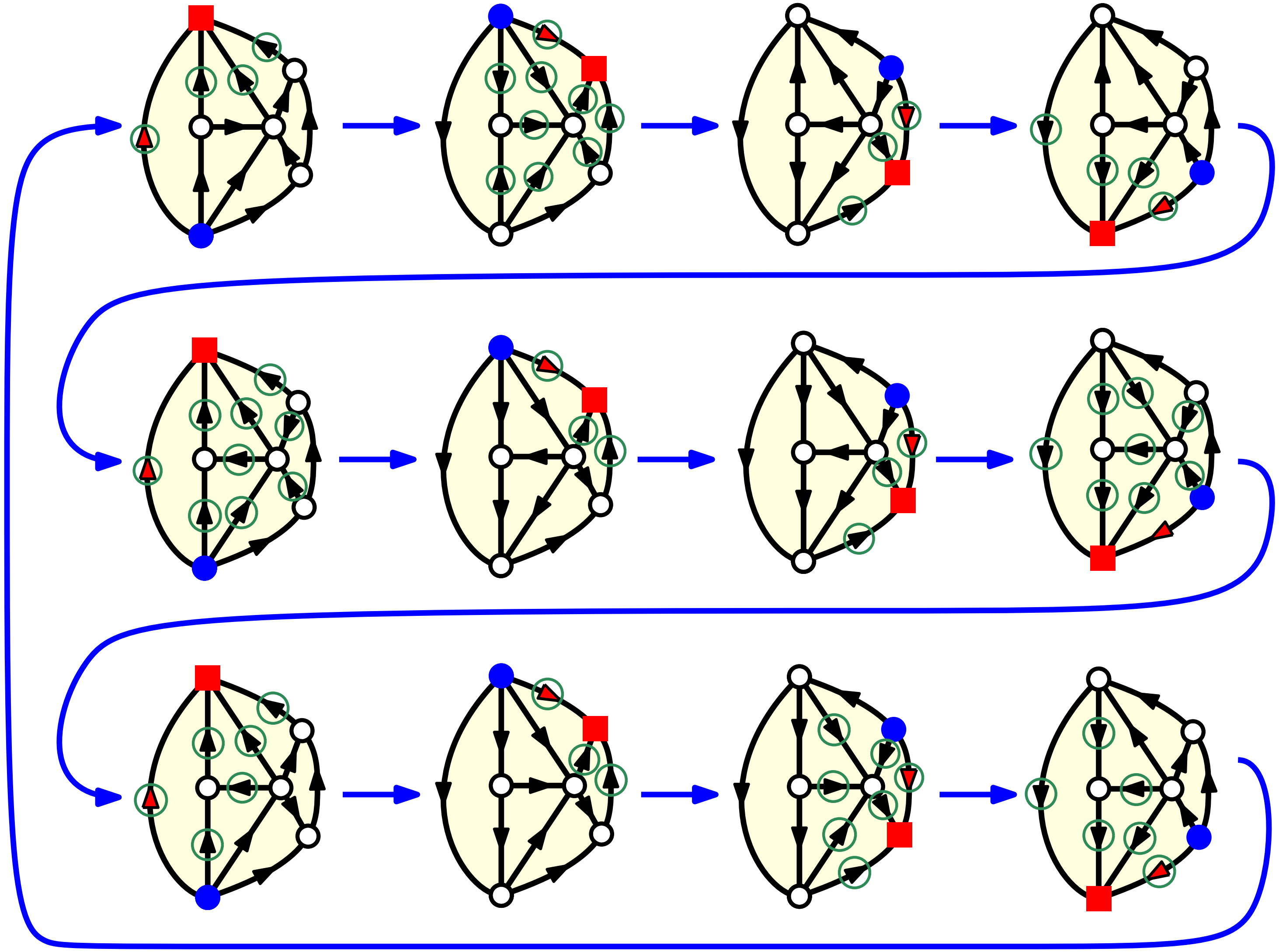

$a : 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0$

# Orbit property

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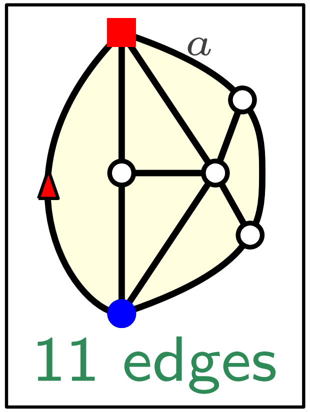
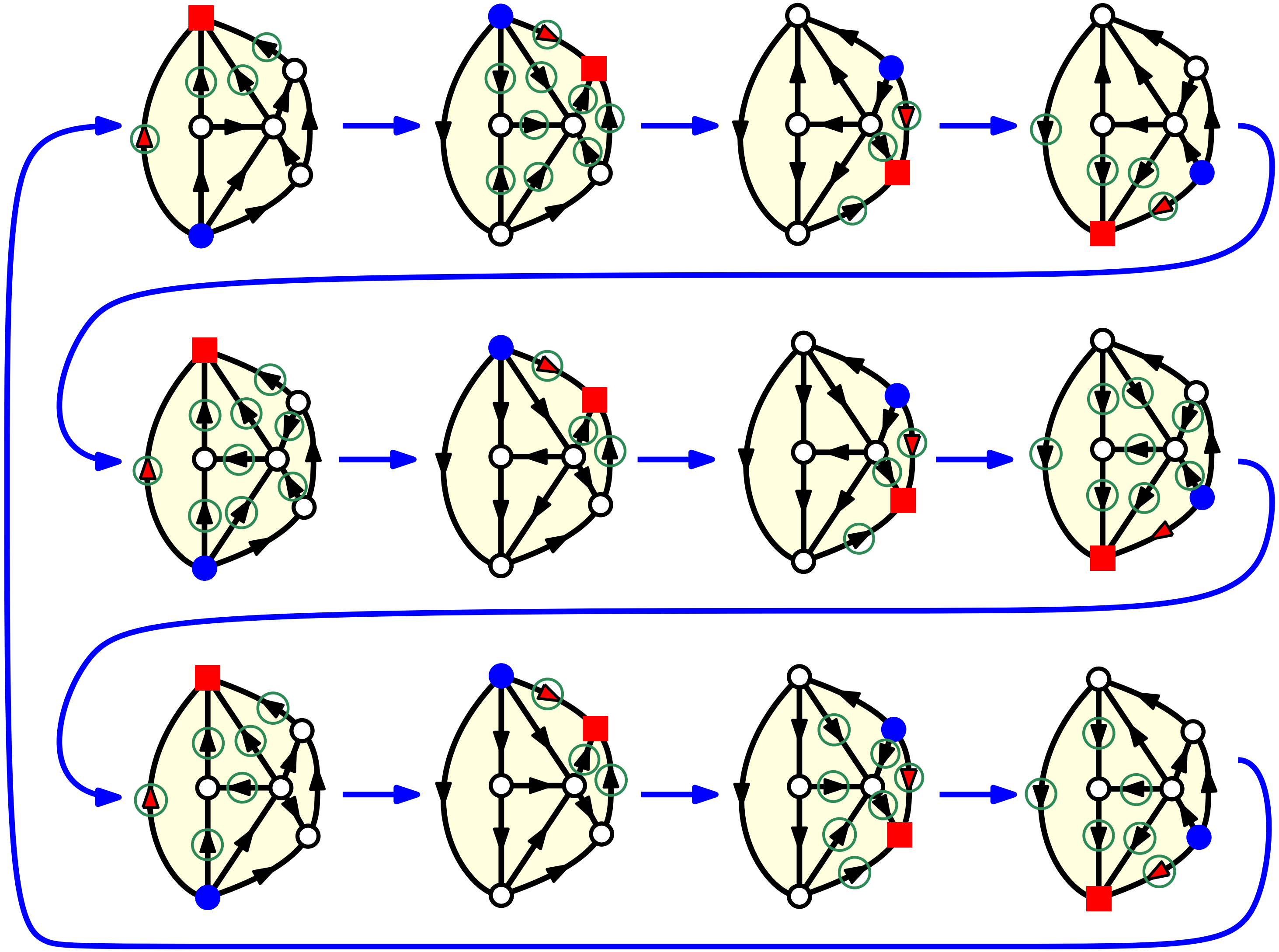
- true for outer edges $1^2 0^{j-2} 1^2 0^{j-2} \dots$



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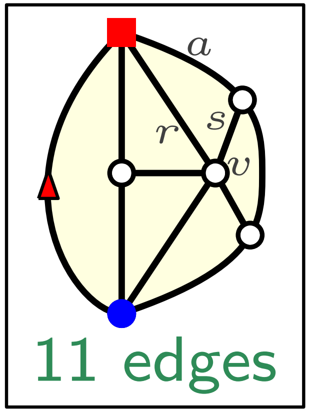
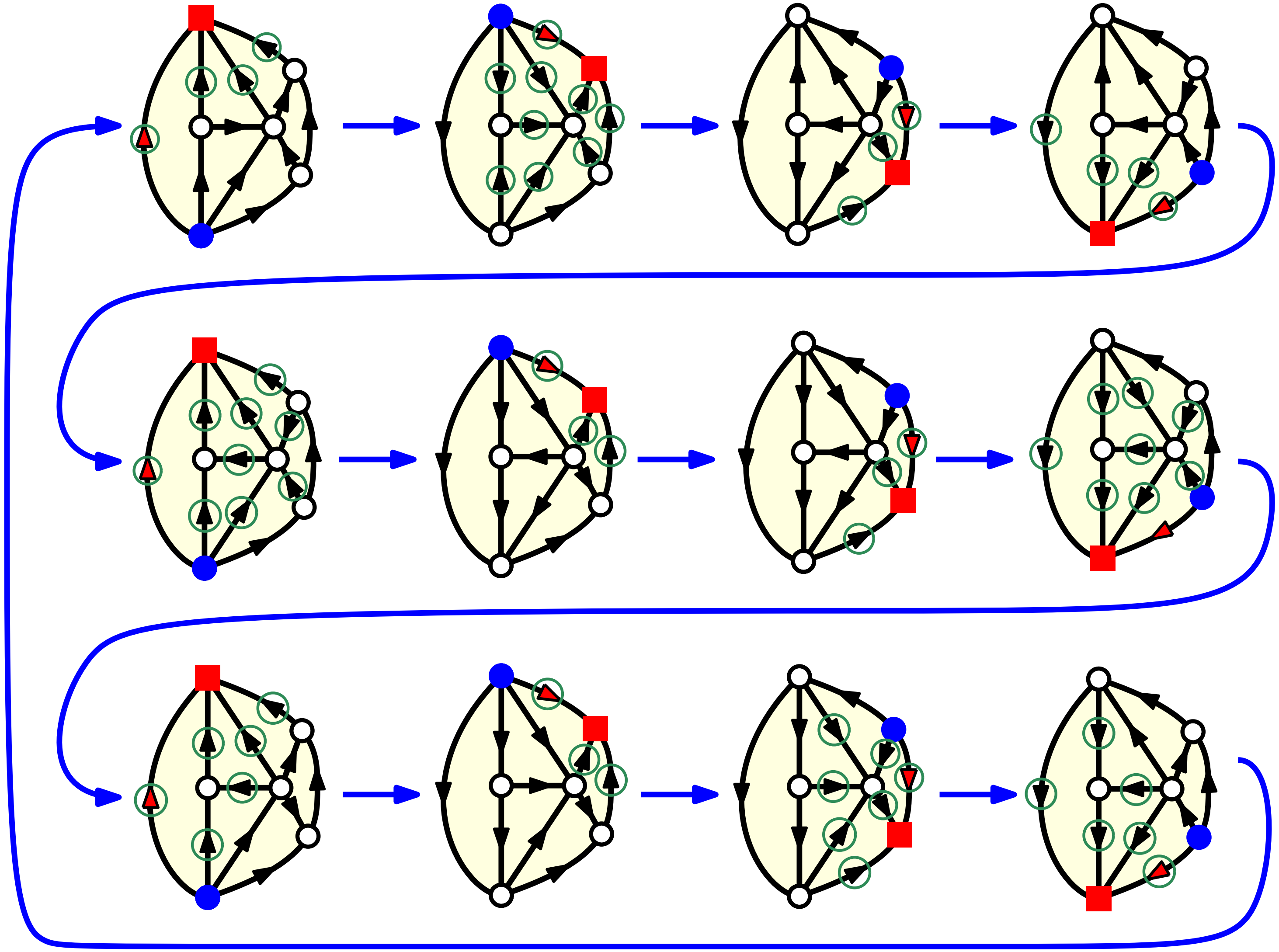


$a : 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0$

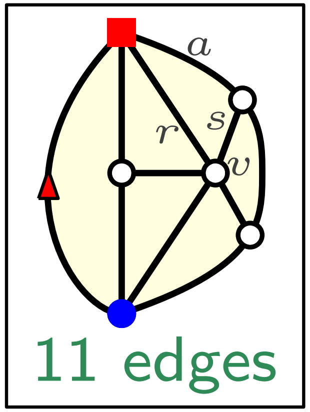
$r : 1\,1\,0\,0\,1\,0\,0\,1\,1\,0\,1\,0$

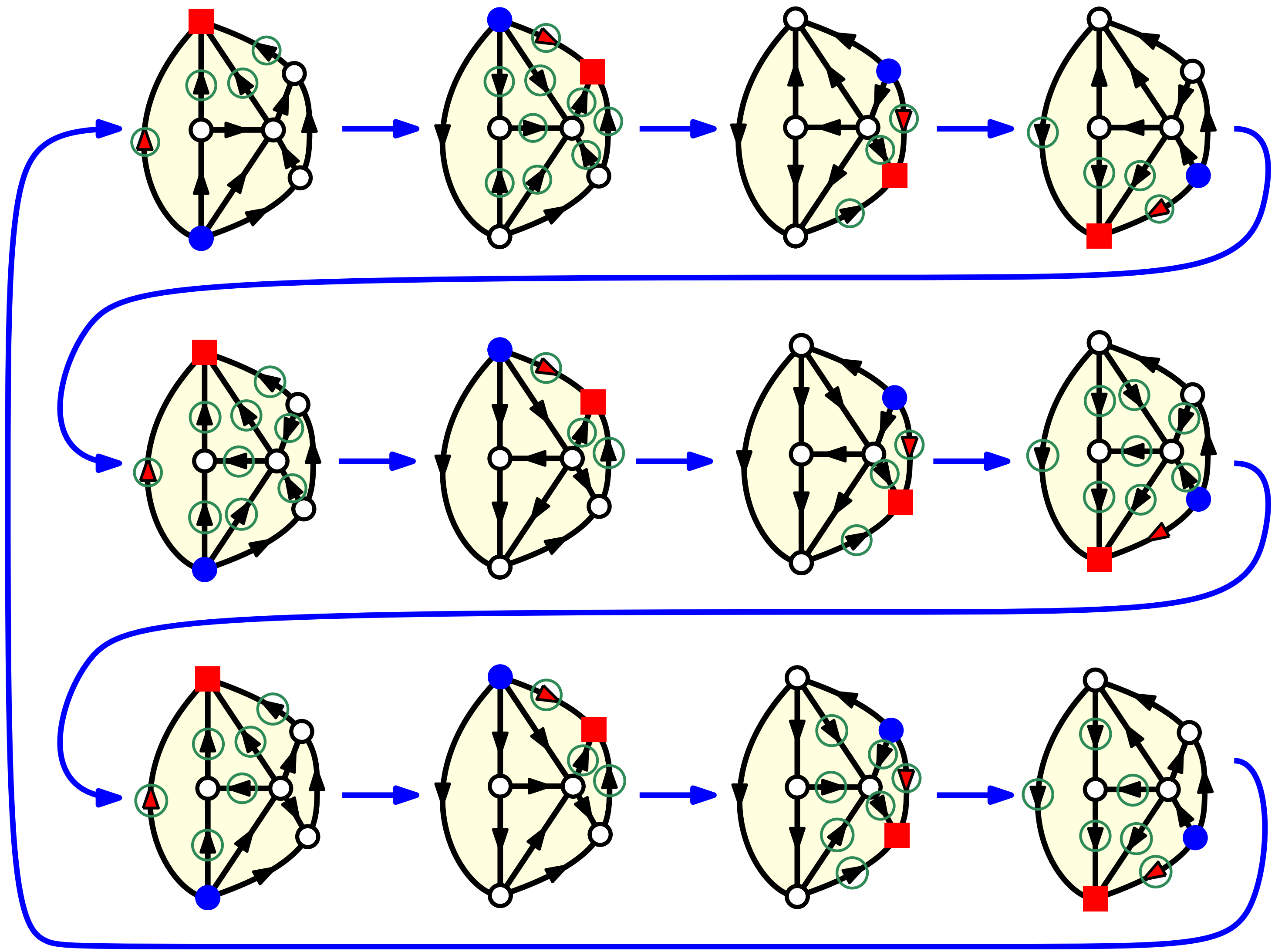
$s : 0\,1\,0\,0\,1\,1\,0\,1\,0\,1\,1\,0$

# Orbit property

along a $\sigma$-orbit, any edge is reversed $2/j$ of the times

- true for outer edges $1^2 0^{j-2} 1^2 0^{j-2} \dots$



$a : 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0$

alternation property

$r : 1\,1\,0\,0\,1\,0\,0\,1\,1\,0\,1\,0$

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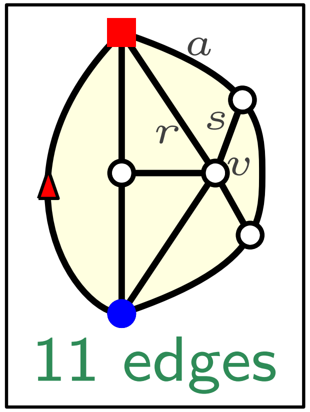


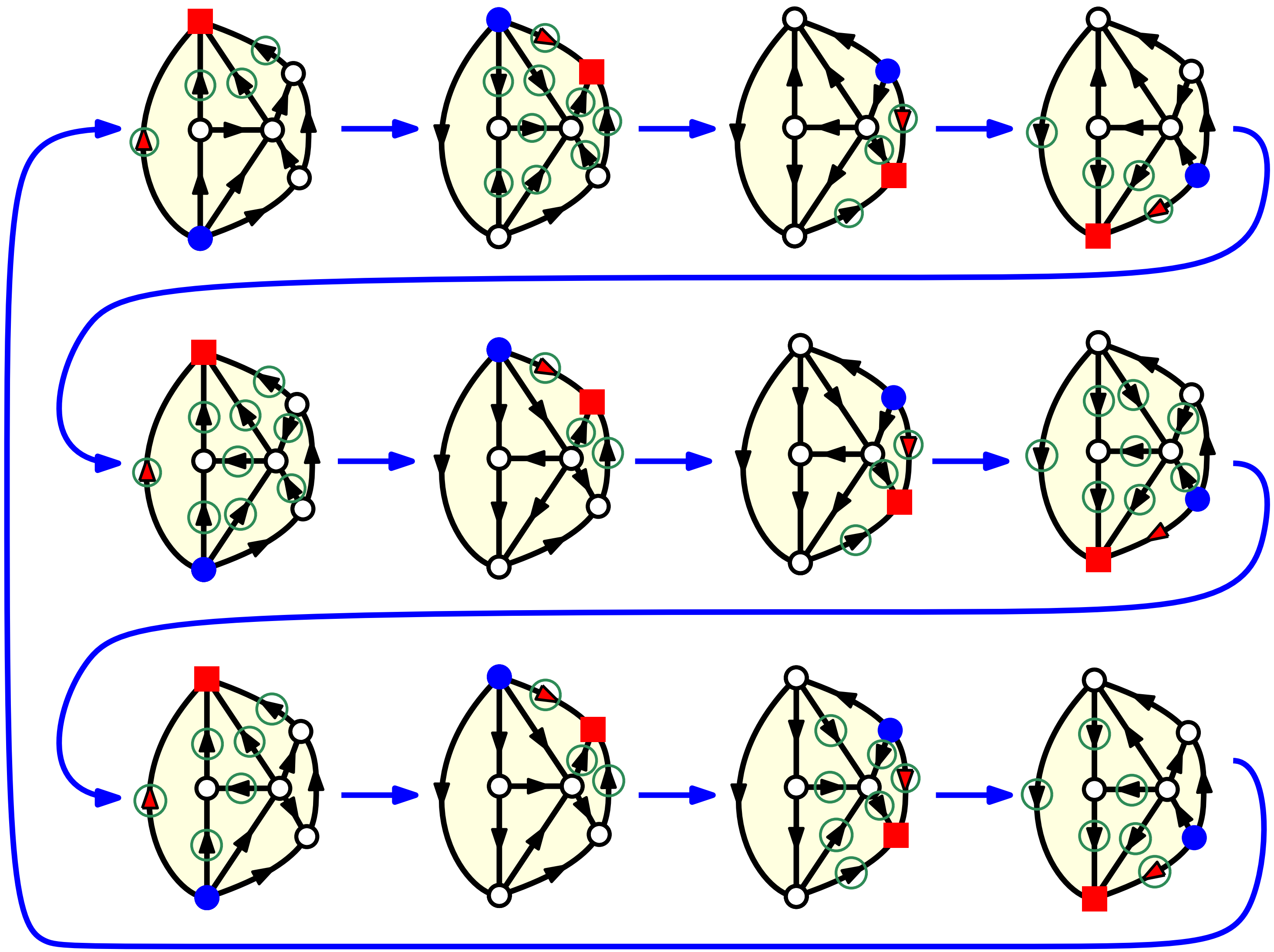

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disagree at $v$ — agree at $v$

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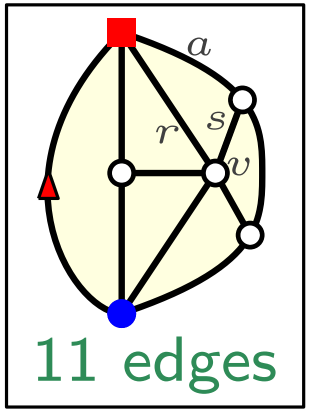


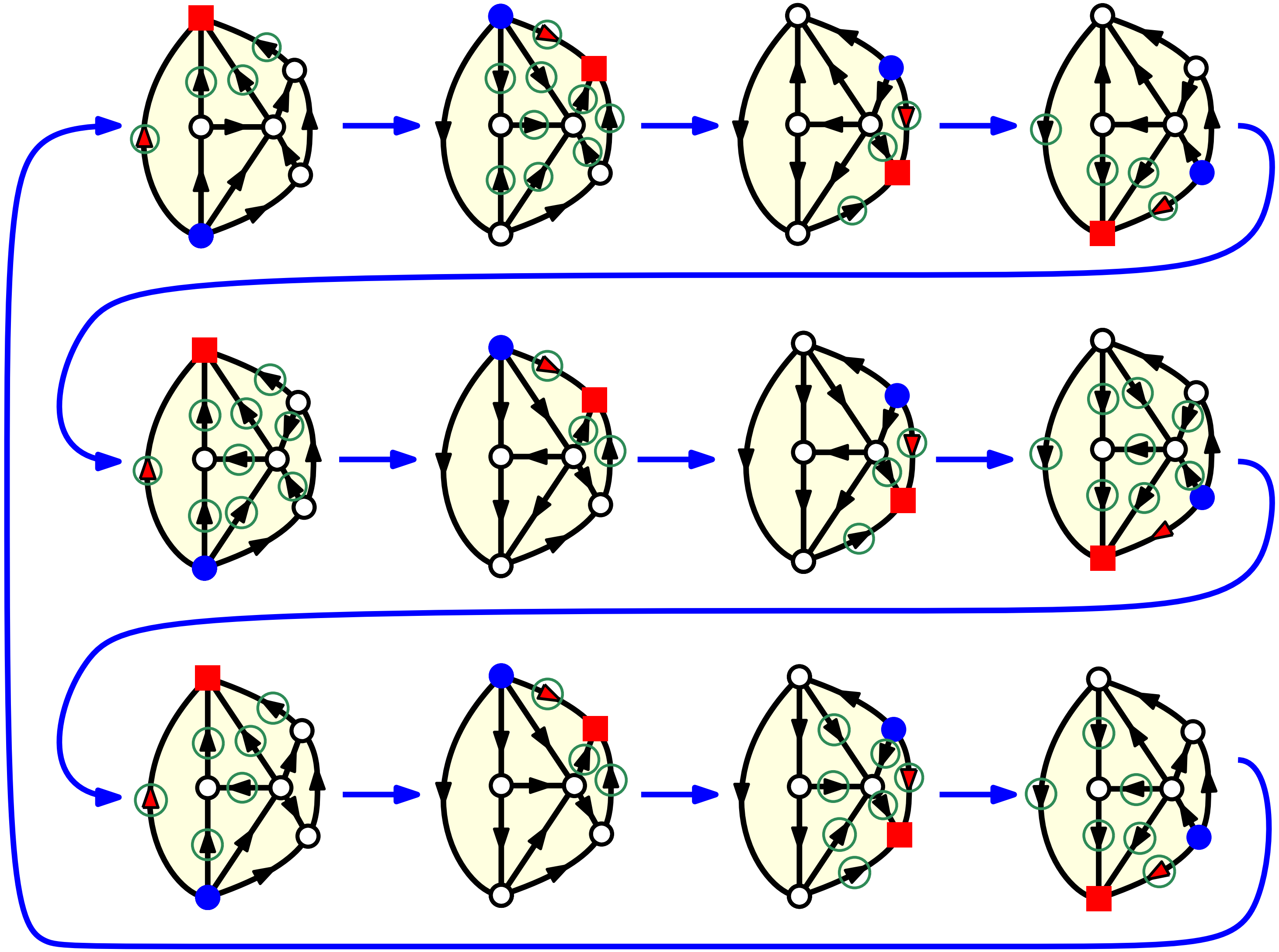

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alternation property

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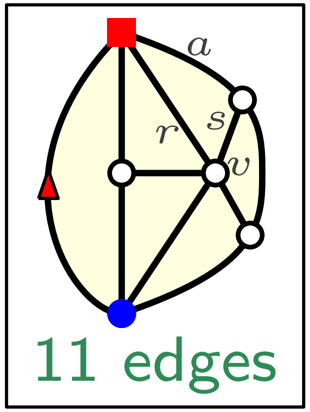


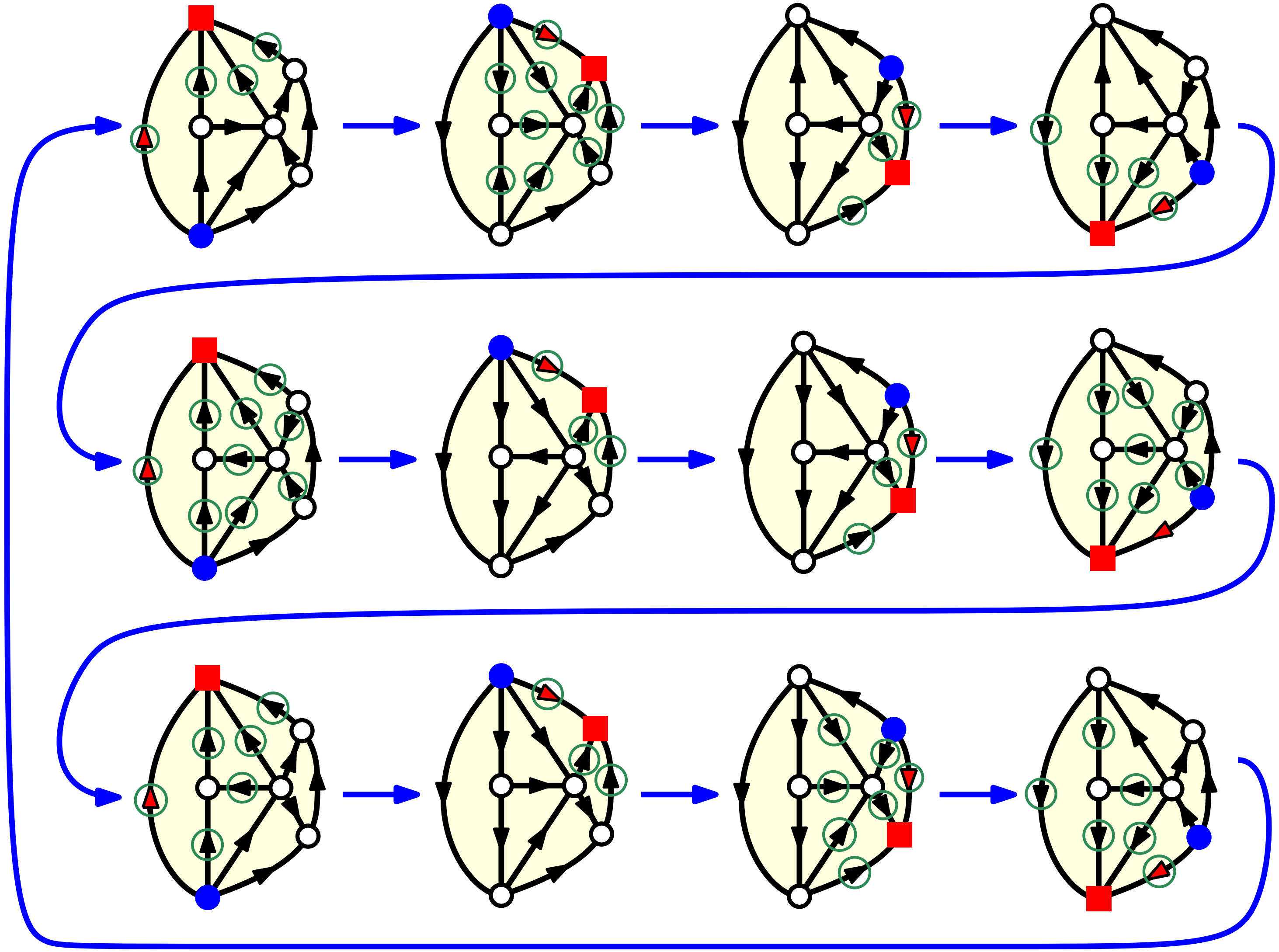

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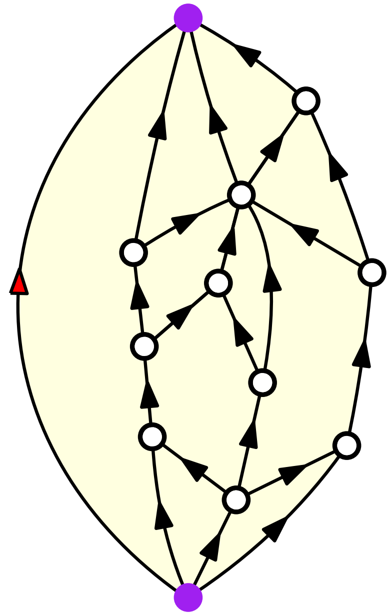
disagree at $v$

agree at $v$

$\Rightarrow$ $\pi_{n,j} = 1 - \frac{2}{j}$ by summation over edges and orbits

# Counting by vertices and faces

$B_{n,f,j} = \#$ bipolar orientations with $f$ inner faces, $n$ inner vertices, $j$ outer vertices

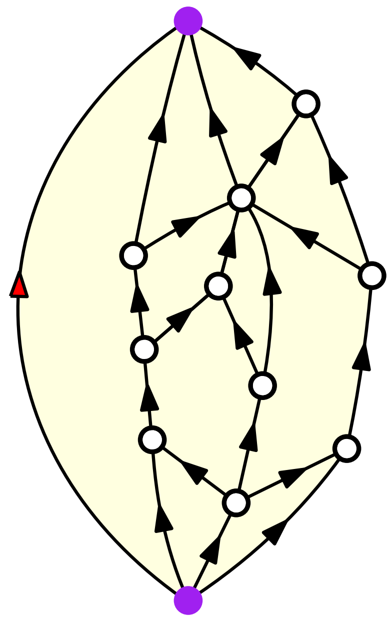


$f = 10$

$n = 7$

$j = 5$

# Counting by vertices and faces

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Similarly (slit-slide-sew & orbit property), we obtain identity

$$n\, B_{n,f,j} = \frac{j-1}{j+1}(n+f-2)\, B_{n-1,f,j+1}$$

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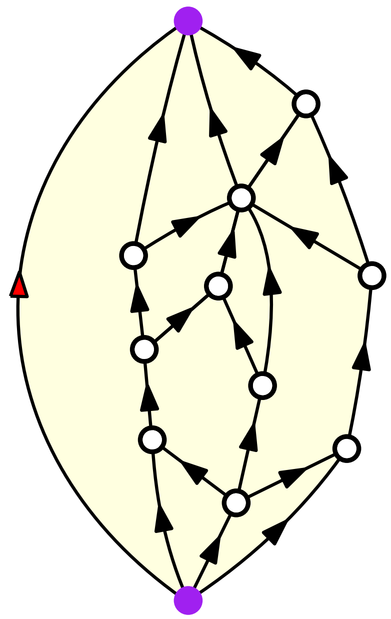


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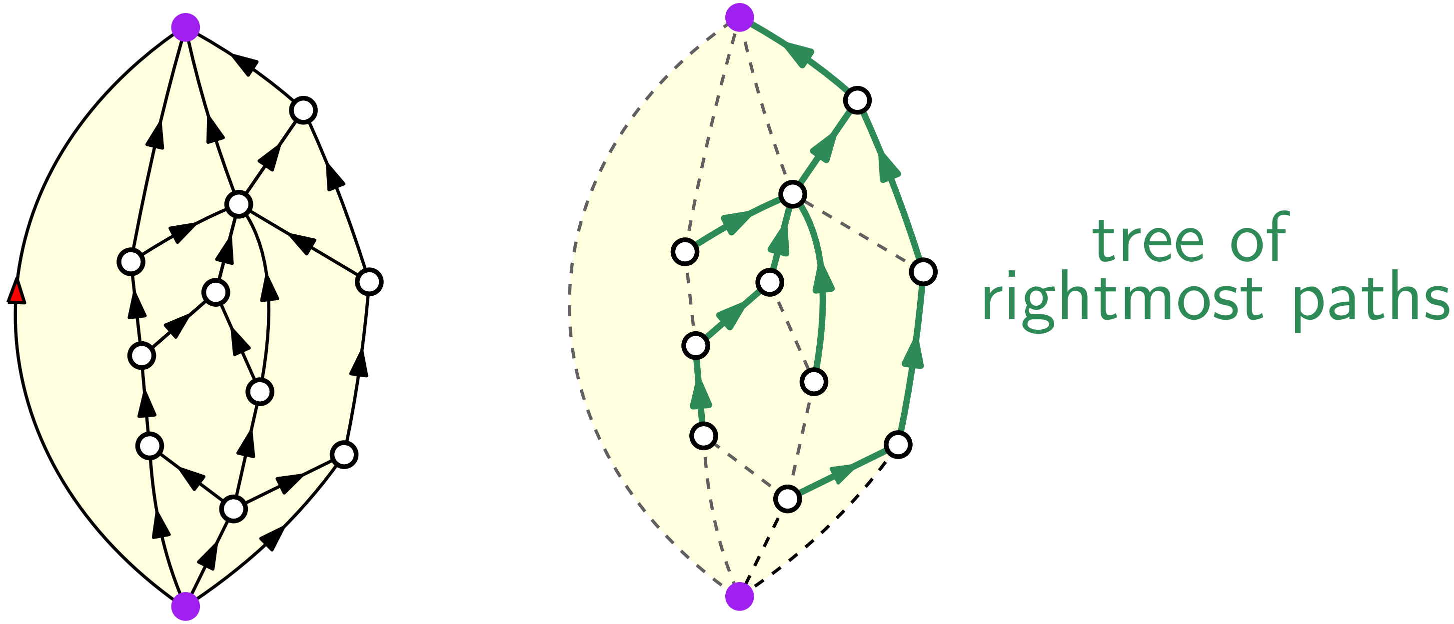
$\Downarrow$ using $B_{0,a,b} = \mathrm{Nar}_{a,b-1}$

$$B_{n,f,j} = j(j-1)\frac{(n+f-2)!(n+f+j-2)!(n+f+j-3)!}{n!(n+j)!(n+j-1)!(n+j-2)!}$$

[Baxter'01, Bousquet-Mélou'11]

# Bijective encoding by tableaux



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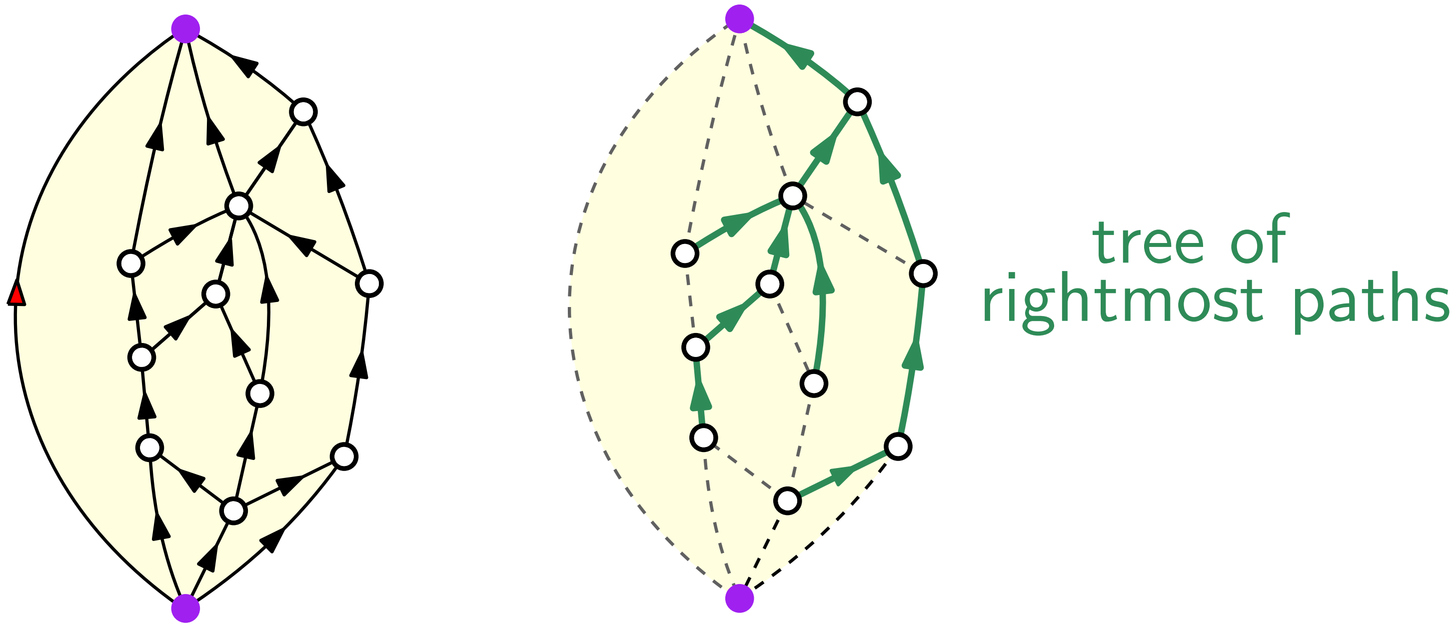
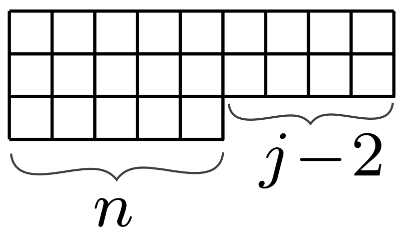



walk around the tree of rightmost paths

⇓

$T_{n,j} = \#$ SYT on

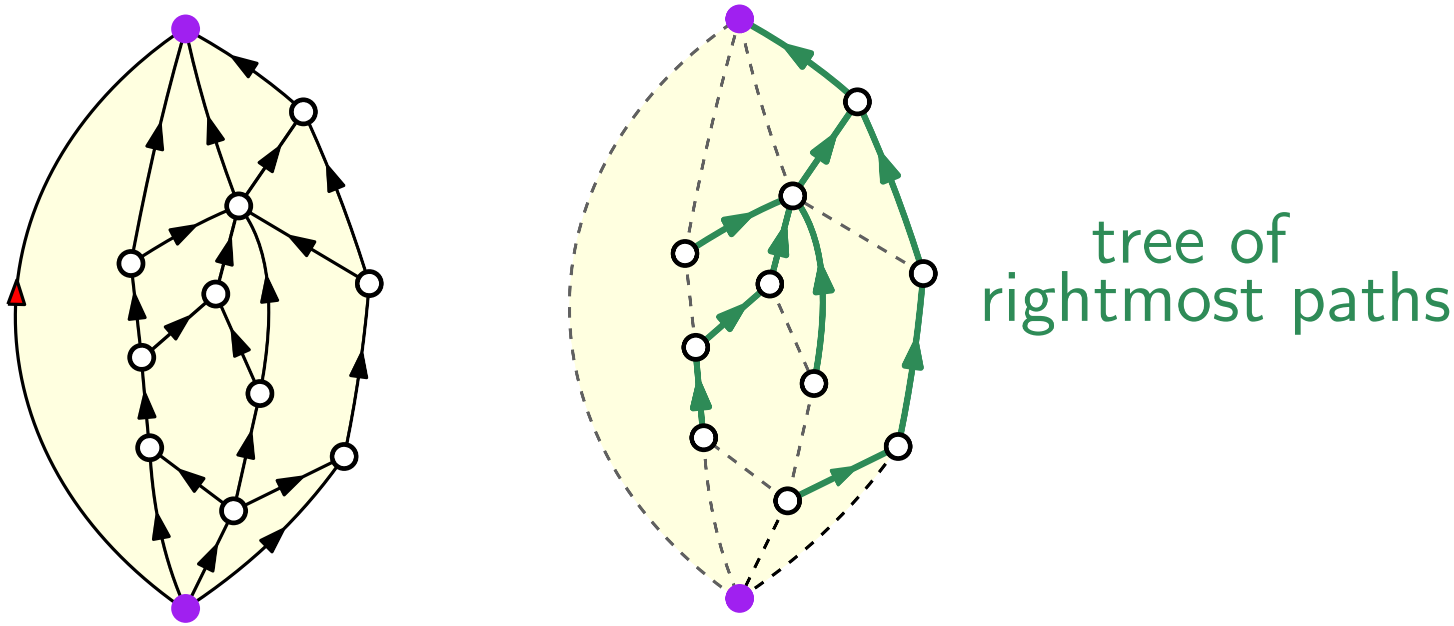
$B_{n,f,j} = \#$ SSYT$_{\leq f+1}$ on




[Borie'17, Kenyon et al'15]

[Albenque-Poulalhon'15]

# Bijective encoding by tableaux




walk around the tree of rightmost paths

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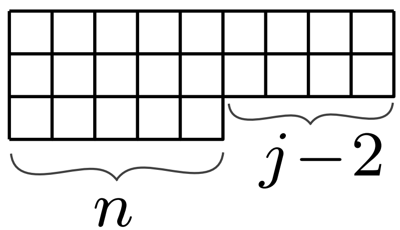


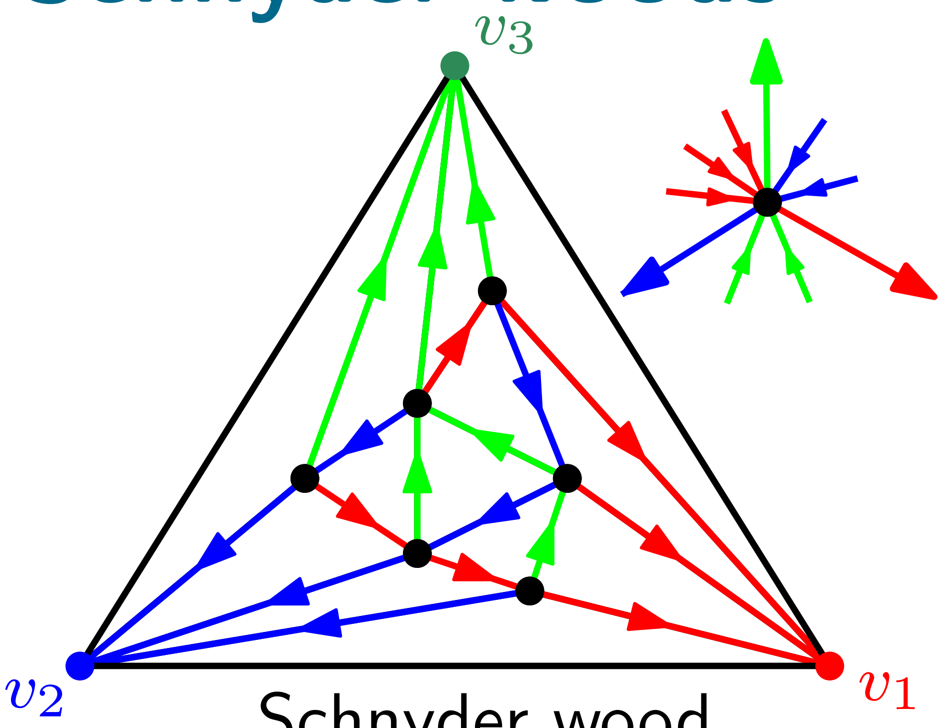

[Borie'17, Kenyon et al'15]

[Albenque-Poulalhon'15]

hook-length formula $\Rightarrow$ formula for $T_{n,j}$

hook-content formula $\Rightarrow$ formula for $B_{n,f,j}$

# Schnyder woods




Schnyder wood

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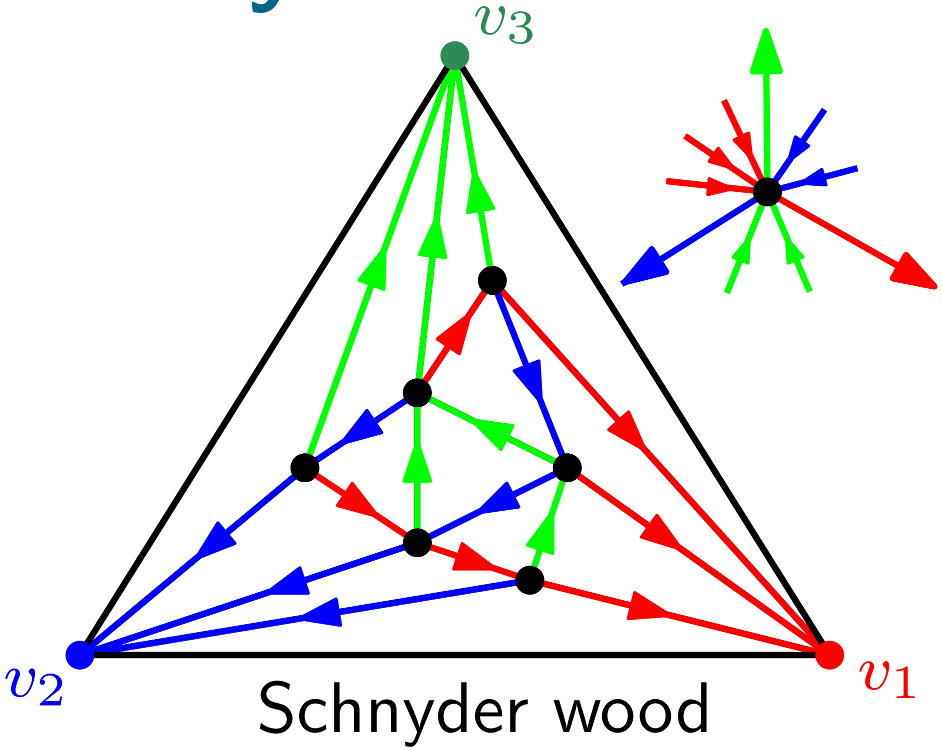
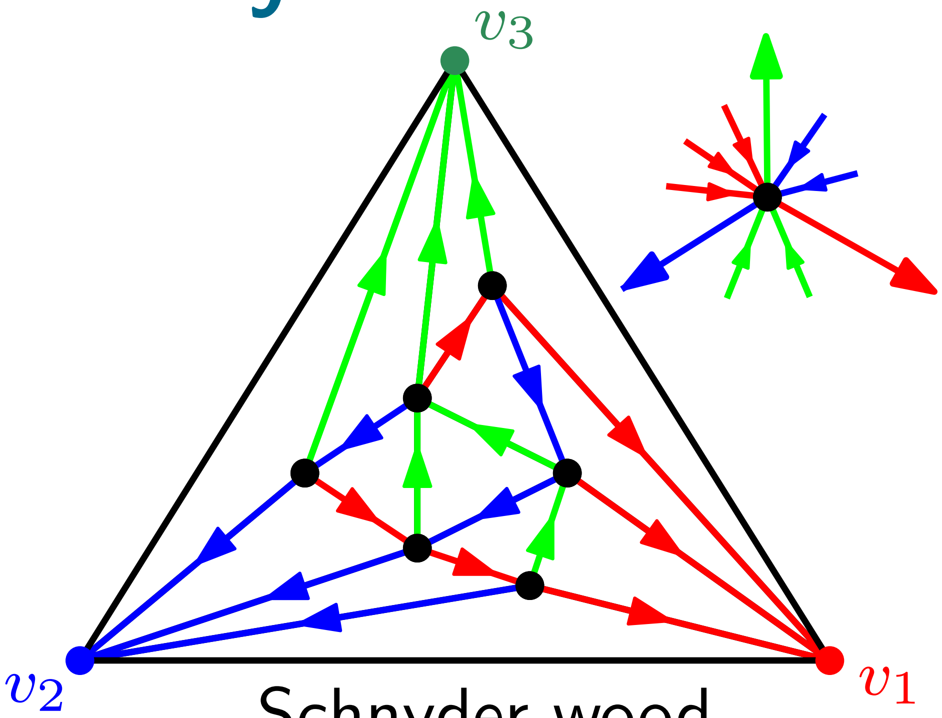



$S_{n,j} = \#$ Schnyder woods with $n+j+1$ vertices and $\deg(v_1) = j$

$$S_{n,j} = j(j-1)(j-2)\frac{(2n+2j-4)!(2n+j-3)!}{n!(n+j)!(n+j-1)!(n+j-2)!}$$

[Bernardi-Bonichon'09]

# Schnyder woods

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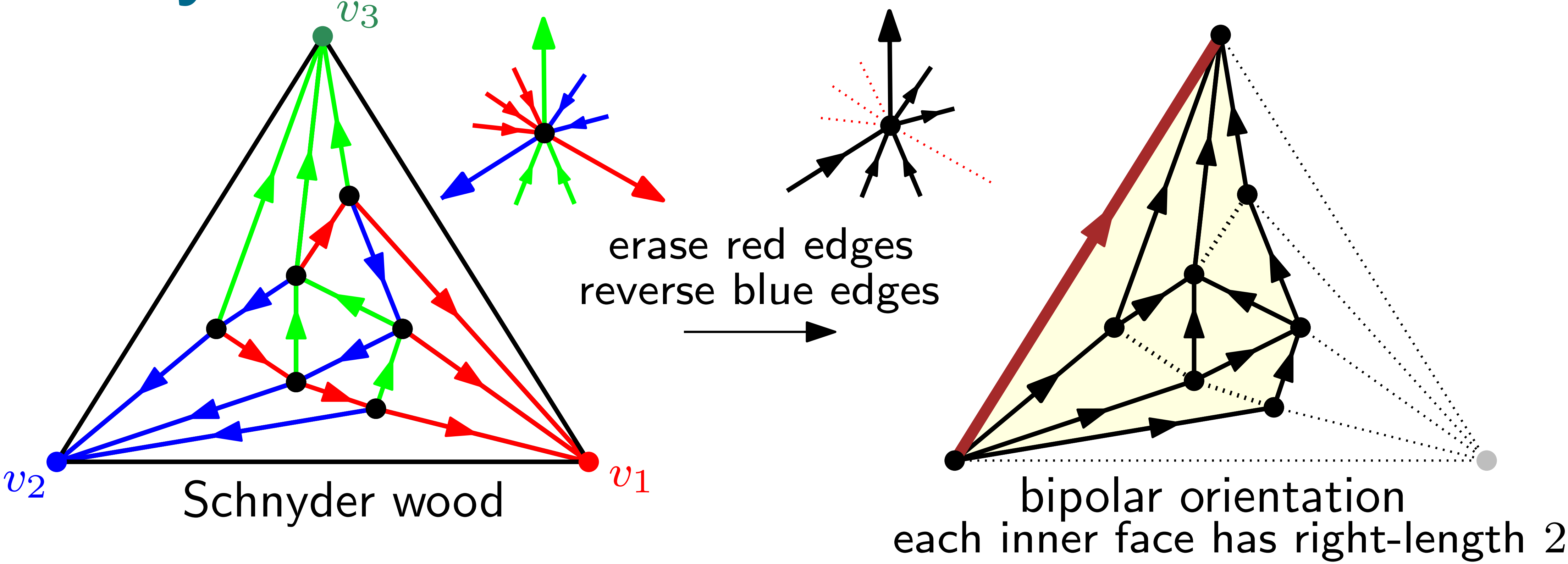
[Bernardi-Bonichon'09]

$\Downarrow \Uparrow$ using $S_{0,k} = \mathrm{Cat}_{k-2}$

Identity $$n\, S_{n,j} = \frac{j-2}{j+1}(2n+j-3)\, S_{n-1,j+1}$$

# Schnyder woods

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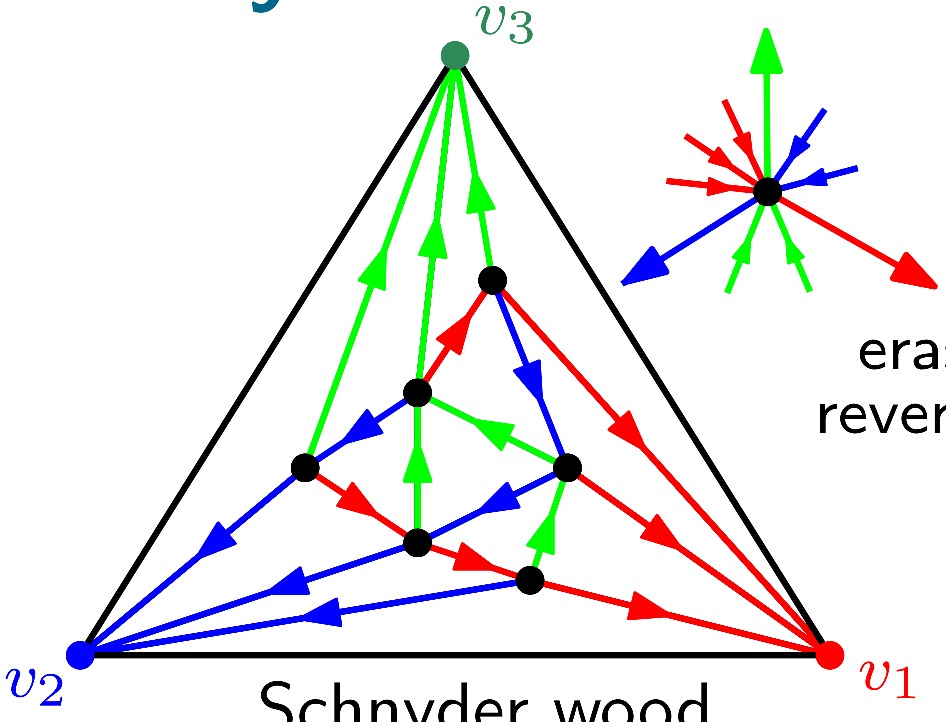
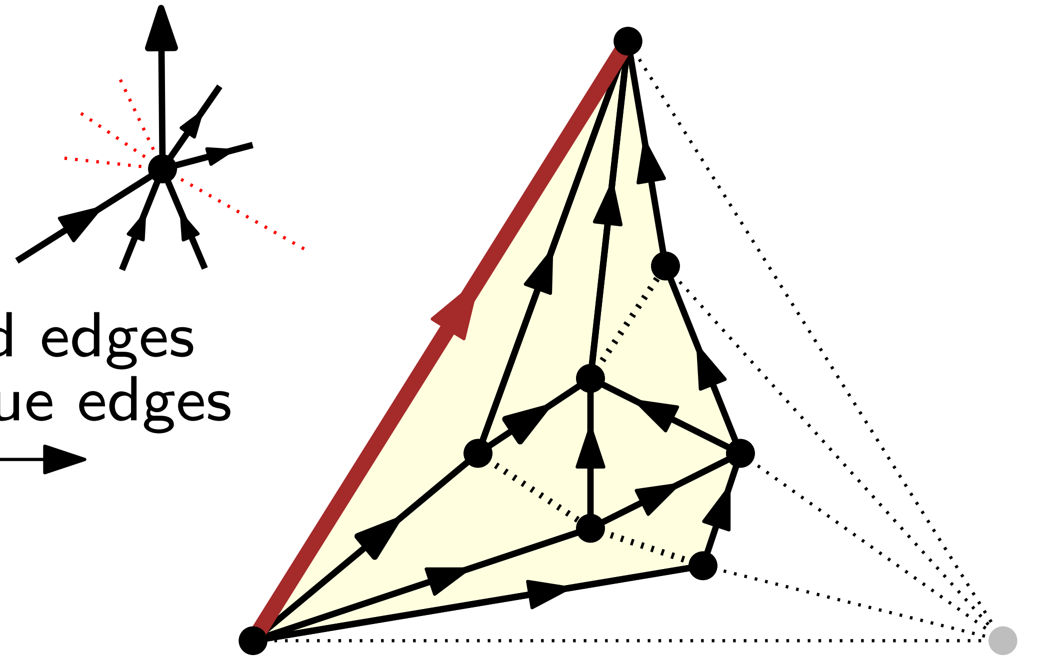



erase red edges
reverse blue edges

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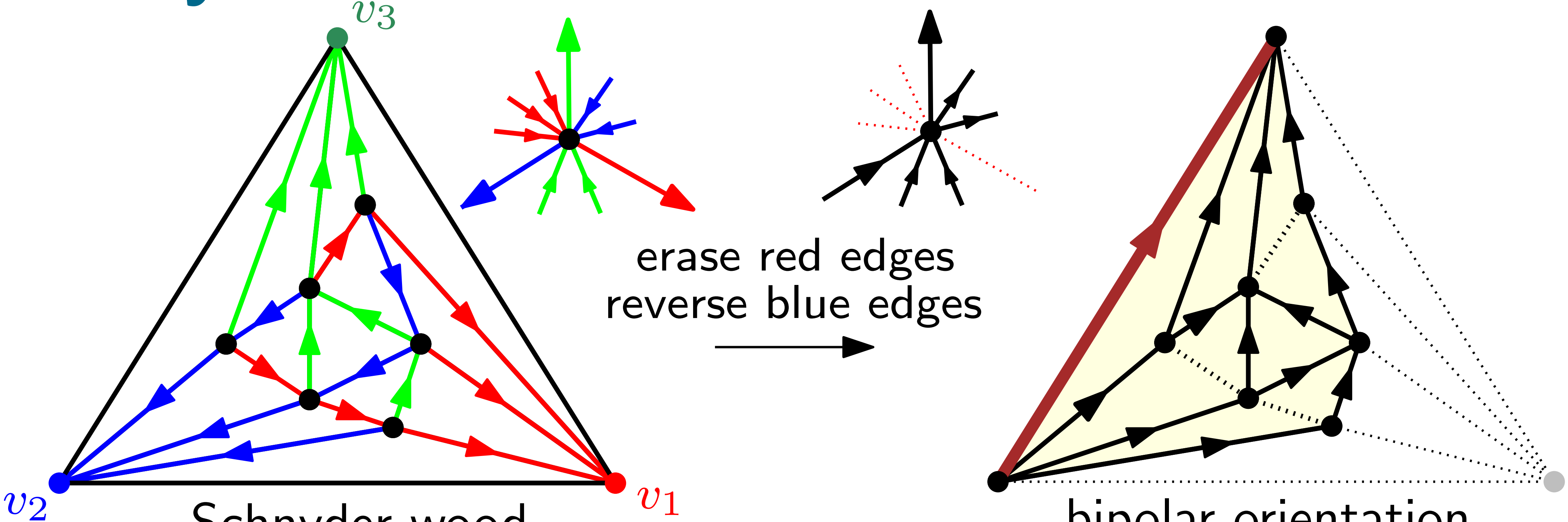


Schnyder wood

bipolar orientation
each inner face has right-length 2
(stable under slit-slide-sew bijection!)

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$$S_{n,j} = j(j-1)(j-2)\frac{(2n+2j-4)!(2n+j-3)!}{n!(n+j)!(n+j-1)!(n+j-2)!}$$

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$$\pi_{n,j} = 1 - \frac{3}{j}$$

rerooting operator
for Schnyder woods