Remixed Eulerian numbers

Solal Gaudin

Lundi 10 mars 2025

ICJ - Université Lyon 1

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- They have *size* n, and k is a parameter.
- The *remixed Eulerian numbers* $A_c(q)$ depend on some c of size n, and on a real parameter $q \geq 0$.
- A new proof of a formula from Garsia-Remmel (1984):

$$\frac{\sum_{k=0}^{n} t^k H_k(\lambda, q)}{\prod_{i=0}^{n}(1 - tq^i)} = \sum_{j \geq 0} t^j \prod_{i=1}^{n} (j + i - \lambda_{n+1-i})_q.$$

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$$A_{(0,3,0,2,0)}(q) = 2q^8+6q^7+12q^6+17q^5+17q^4+12q^3+6q^2+2q.$$

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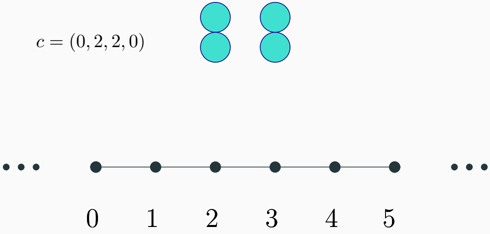
- Bilateral parking functions.

A particle model: definition



- We place n balls over $\mathbb{Z}$, forming a *configuration* c, with c_i balls over the site i.

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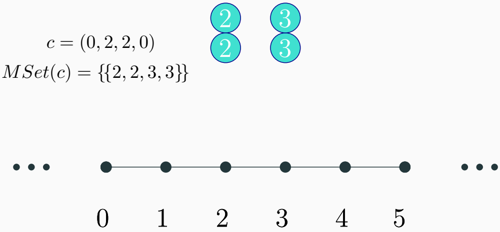

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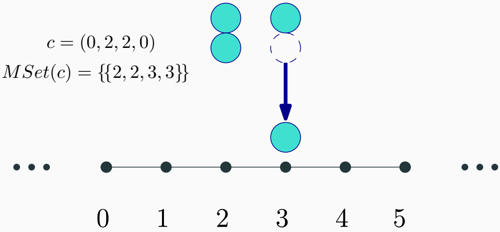

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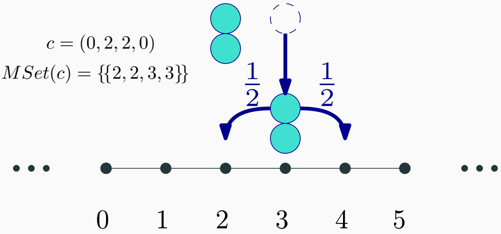

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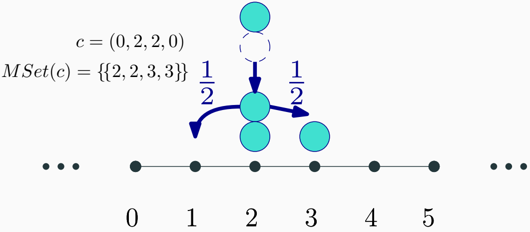

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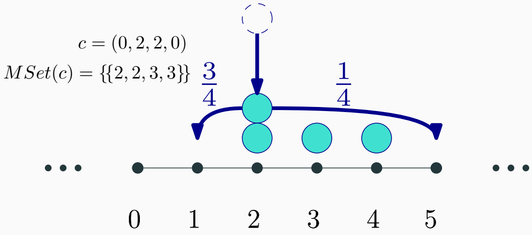

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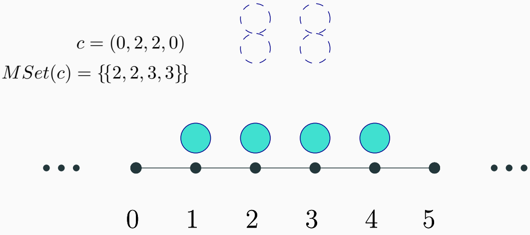
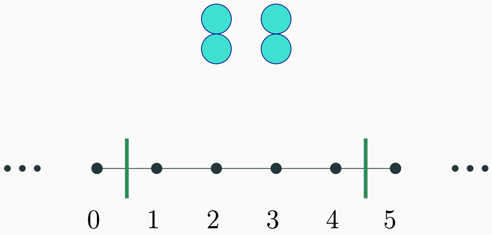


- We place n balls over $\mathbb{Z}$, forming a *configuration* c, with c_i balls over the site i.
- We then drop balls one by one.
- Balls bounce on top of one another.
- What is the final state?

A particle model: A_c



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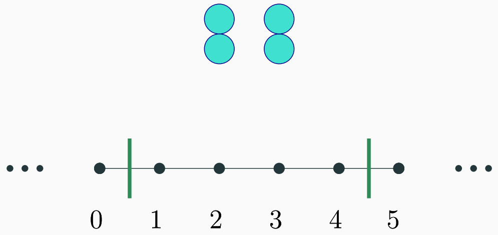

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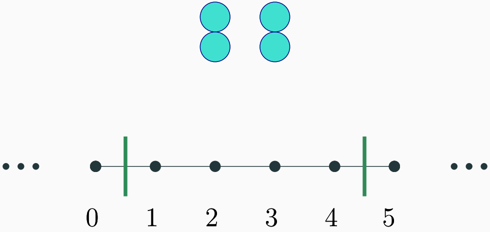
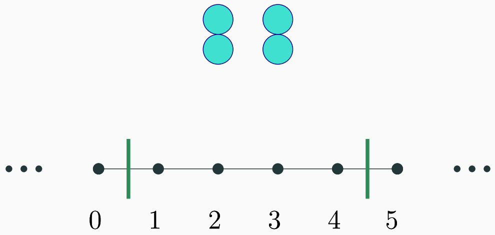


- $\mathbb{P}_c$ = Probability that balls stay between 1 and n.
- Normalised: $A_c = n!\mathbb{P}_c$.
- Proposition (Petrov 2021): A_c are well-defined integers.

Particle model: example



- Computing: $\mathbb{P}_{(0,2,2,0)}$

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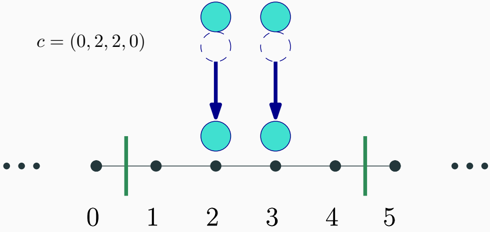

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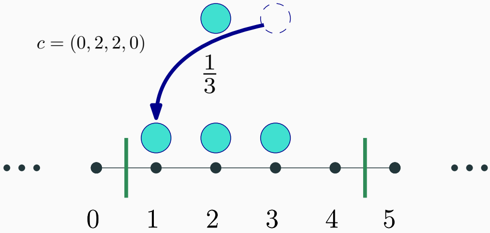

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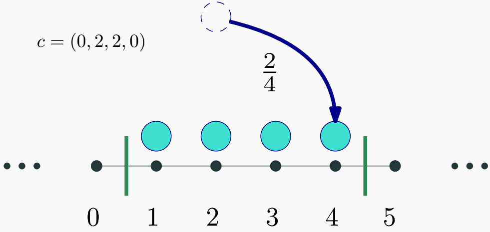

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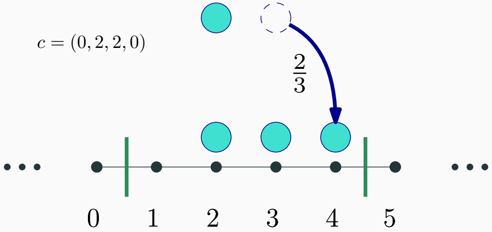

- Computing: $\mathbb{P}_{(0,2,2,0)} = \frac{1}{3}\frac{2}{4} + \frac{2}{3}$

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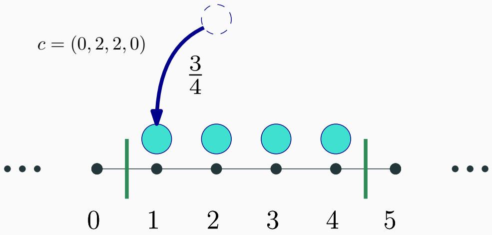

- Computing: $\mathbb{P}_{(0,2,2,0)} = \frac{1}{3}\frac{2}{4} + \frac{2}{3}\frac{3}{4}$.
- Therefore $A_{(0,2,2,0)} = 4!\mathbb{P}_{(0,2,2,0)} = 16$.

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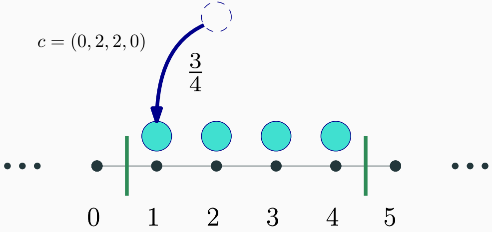

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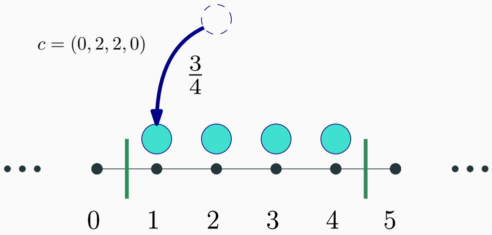

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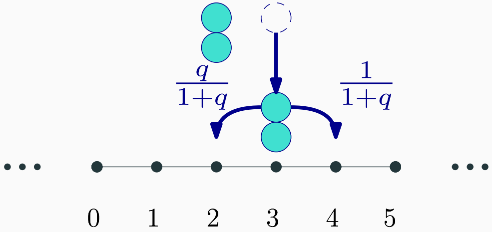
- Therefore $A_{(0,2,2,0)} = 4!\mathbb{P}_{(0,2,2,0)} = 16$.
- Note: $\mathbb{P}_c$ is well-defined!
- Formula for $\mathbb{P}_c$?

Side note: adding a parameter



- Biased case uses q-analogs:
  $(n)_q = 1 + q + q^2 + ... + q^{n-1}$, $(n)_q! = \prod_{i=1}^{n} (i)_q$.

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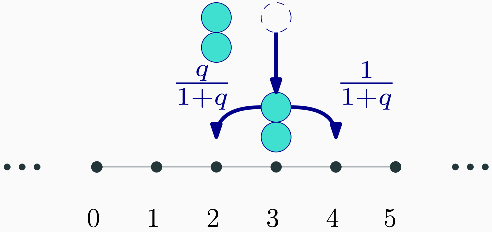

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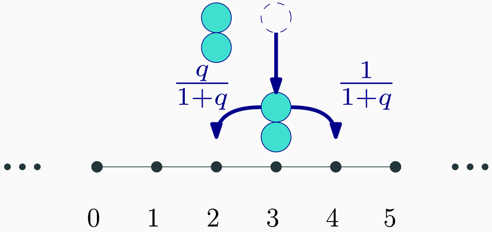

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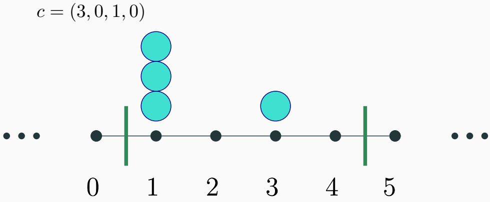
- Proposition (Nadeau-Tewari 2022): $A_c(q)$ are polynomials in q. Moreover, they are symmetric and unimodal.
- Example: $A_{(0,2,2,0)}(q) = q^5 + 4q^4 + 6q^3 + 4q^2 + q$.

Left-to-right configurations



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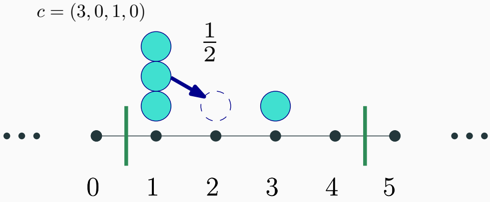

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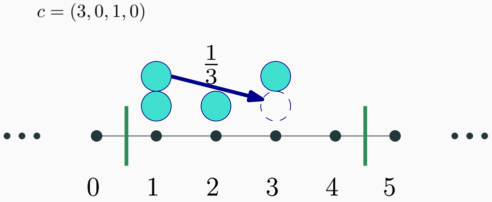

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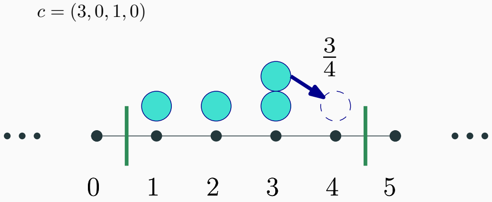

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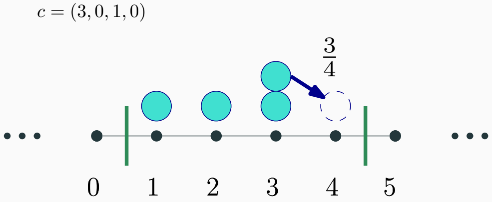

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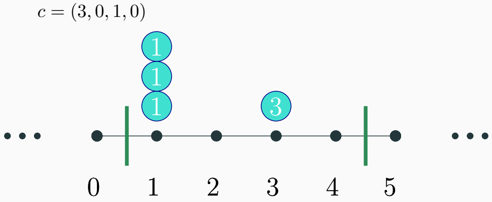

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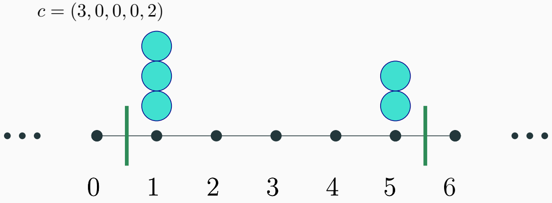
Binomial coefficients



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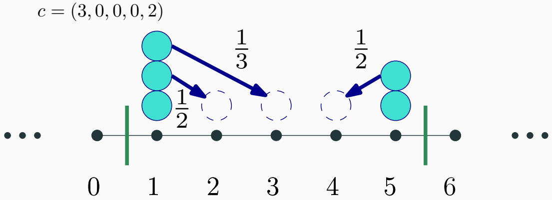

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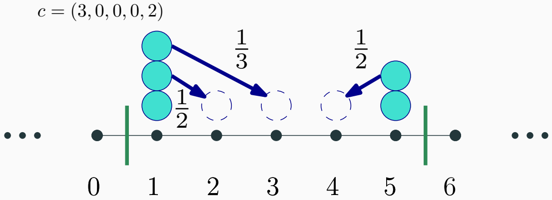

- $\mathbb{P}_c = \frac{1}{3!}\frac{1}{2!}$
- $A_c = \frac{5!}{3!2!} = \binom{5}{2}$.

Eulerian numbers

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  $$A(n-k,k) = (n-k)A(n-k,k-1)+(k+1)A(n-k-1,k).$$

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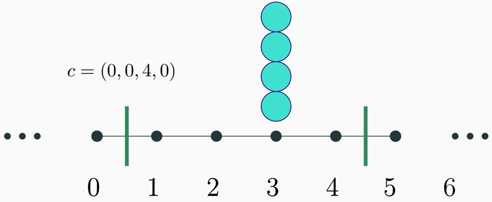

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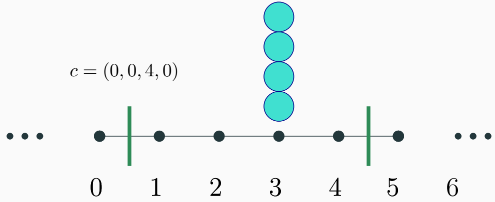
- One can prove: $A_{(0^k, n, 0^{n-k-1})} = A(n-k, k)$.
- We also have: $\frac{\sum_{k=0}^{n-1} t^k A(n-k,k)}{(1-t)^{n+1}} = \sum_{j \geq 0} t^j j^n$.



Carlitz-Scoville Eulerian numbers

Defined by $A(0,0|x,y) = 1$,

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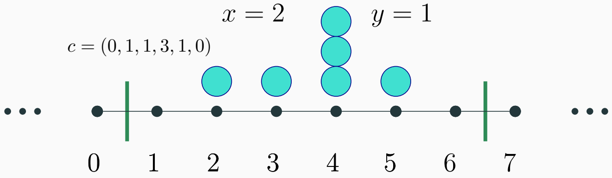

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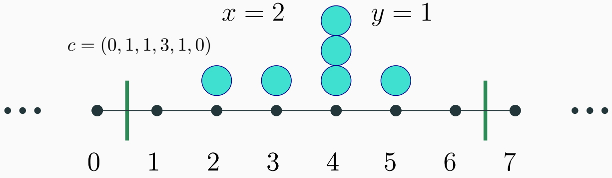
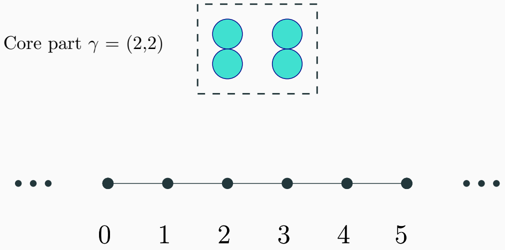
$$\frac{\sum_{k=0}^{r+s} t^k A(k, r+s-k|x,y)}{(1-t)^{r+s+x+y-1}} = \sum_{j\geq 0} t^j \binom{j+x+y-1}{j} (j+y)^{r+s}.$$



Connectedness



- *Connected configuration*: all the balls are next to each other, on m sites.

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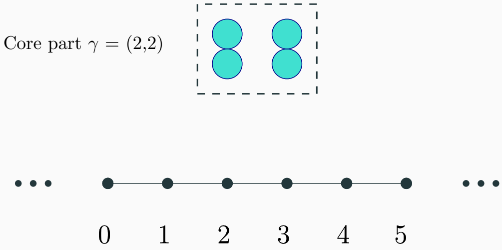

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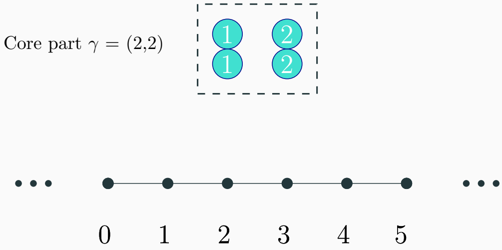

- *Connected configuration*: all the balls are next to each other, on m sites.
- Remark: they remain connected.
- $Mset(2,2) = \{\{1,1,2,2\}\}$.

Link with hit numbers

For any γ without hole:

$$\frac{\sum_{k=0}^{n-m} t^k A_{(0^k,\gamma,0^{n-m-k})}}{(1-t)^{n+1}} = \sum_{j\geq 0} t^j \prod_{a\in MSet(\gamma)} (j+a).$$

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Formula of Garsia and Remmel:

$$\frac{\sum_{k=0}^{n} t^k H_k(\lambda, q)}{\prod_{i=0}^{n}(1-tq^i)} = \sum_{j\geq 0} t^j \prod_{i=1}^{n} (j+i-\lambda_{n+1-i})_q.$$

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For $q=1$:

$$\frac{\sum_{k=0}^{n} t^k H_k(\lambda)}{(1-t)^{n+1}} = \sum_{j\geq 0} t^j \prod_{i=1}^{n} (j+i-\lambda_{n+1-i}).$$

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For all $j \geq 0$:

$$\sum_{k=0}^{j} \binom{n+k}{k} A_{(0^{j-k},\gamma,0^{n-m-(j-k)})} = \prod_{a\in MSet(\gamma)} (j+a).$$

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For all $k \in [0; n-m]$:

$$A_{(0^k,\gamma,0^{n-m-k})} = \sum_{j=0}^{k} (-1)^{k+j} \binom{n+1}{k-j} \prod_{a\in MSet(\gamma)} (j+a).$$

Link with the formula: example

- For all $j \geq 0$:

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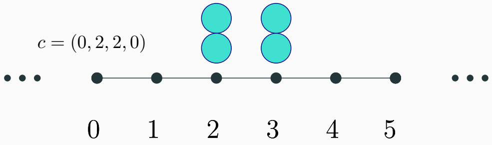

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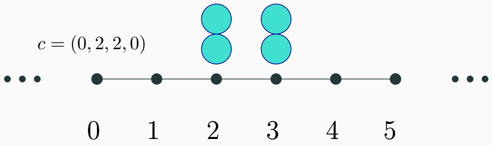
$$A_{(0^1,\gamma,0^{2-1})} = \sum_{j=0}^{1} (-1)^{j+1} \binom{5}{1-j} \prod_{a \in \{\{1,1,2,2\}\}} (j+a)$$



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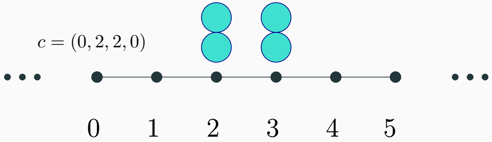
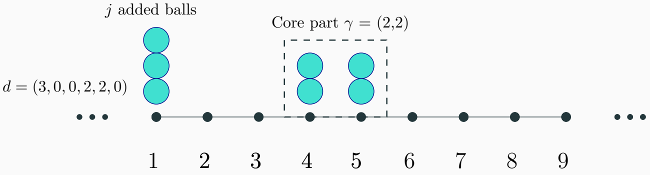
$$A_{(0,2,2,0)} = -5 \times 1^2 \times 2^2 + 2^2 \times 3^2 = 16.$$



New connected proof: definitions

- Take any integer $j \geq 0$.
- Why is there:

$$\sum_{k=0}^{j} \binom{n+k}{k} A_{(0^{j-k},\gamma,0^{n-m-(j-k)})} = \prod_{a \in MSet(\gamma)} (j+a)?$$



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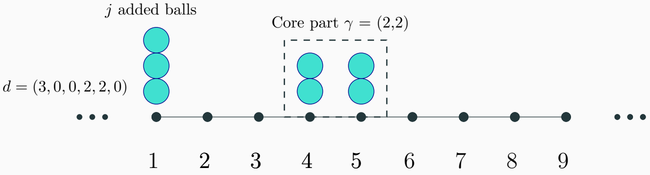
- Add j balls to γ, as in this figure.



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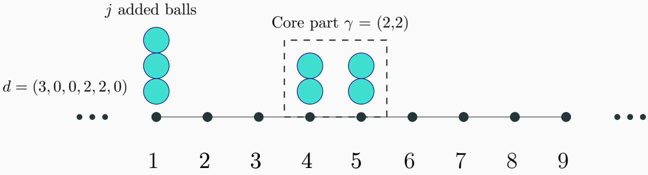
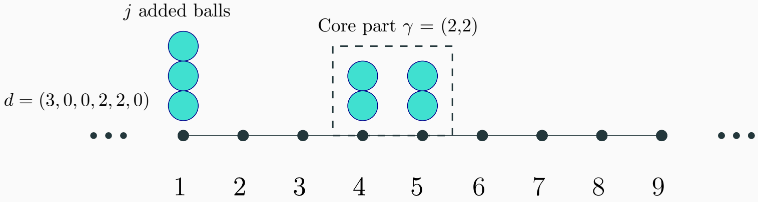
- Add j balls to γ, as in this figure.
- Two ways to study this bigger configuration.

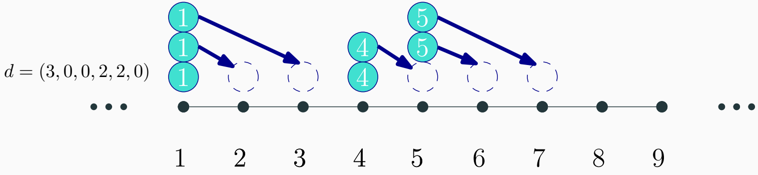


New connected proof: first count



- d is left-to-right.

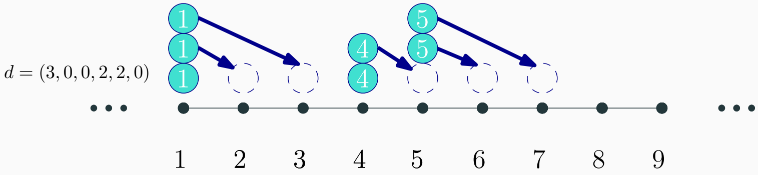
New connected proof: first count



- d is left-to-right.

$$A_d = \prod_{a \in MSet(d)} a.$$

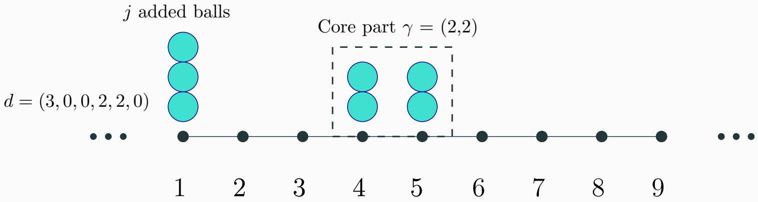
New connected proof: first count



- d is left-to-right.

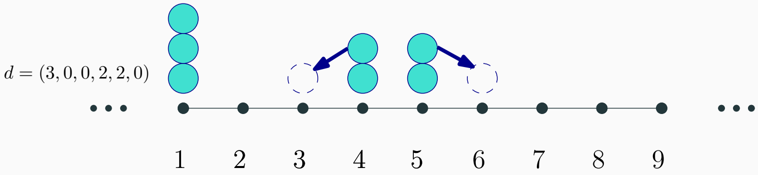
$$A_d = \prod_{a \in MSet(d)} a$$
$$= \prod_{a \in MSet(\gamma)} (j + a).$$

New connected proof: second count



- Resolving the core part first, with k balls falling on the left:

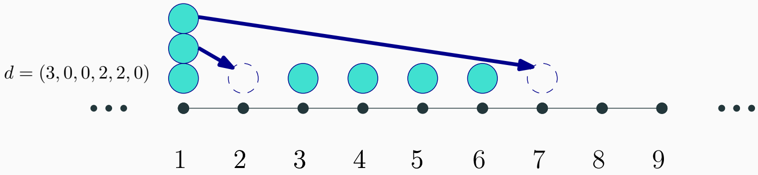
New connected proof: second count



- Resolving the core part first, with k balls falling on the left:

$$\mathbb{P}_d = \sum_{k=0}^{n-m} \mathbb{P}_{(0^k, \gamma, 0^{n-m-k})} \times$$

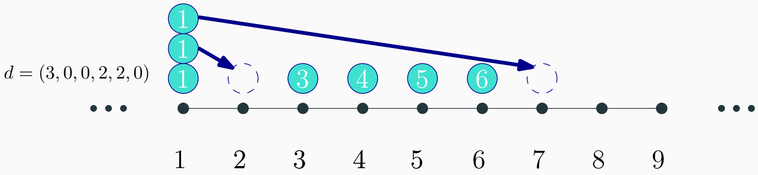
New connected proof: second count



- Resolving the core part first, with k balls falling on the left:

$$\mathbb{P}_d = \sum_{k=0}^{n-m} \mathbb{P}_{(0^k,\gamma,0^{n-m-k})} \times \mathbb{P}_{(j,0^{j-1-k},1^n,0^{n-m-k+j})}.$$

New connected proof: computations



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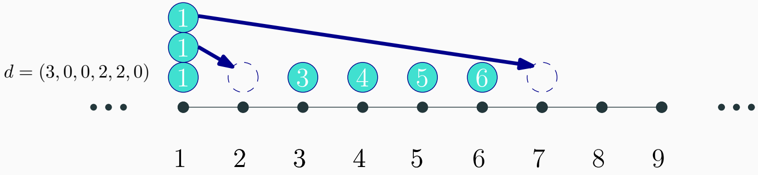

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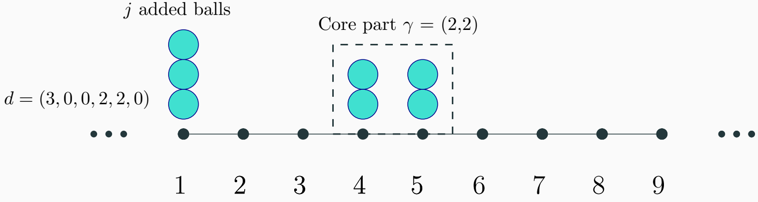

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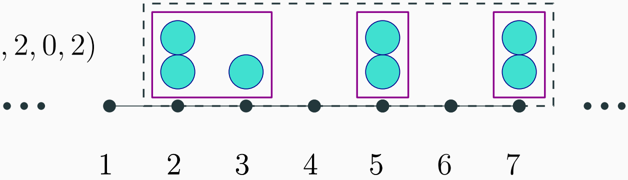
Near-connected configurations: definition

$\gamma = (2, 1, 0, 2, 0, 2)$



- Some holes $h \in H$ within the central part.

Near-connected configurations: definition

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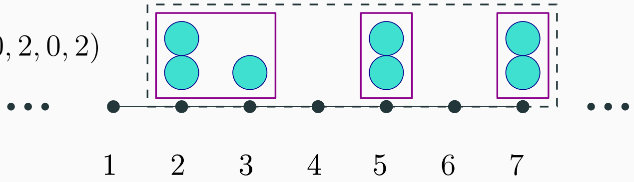

- Some holes $h \in H$ within the central part, yet more balls than holes.

Near-connected configurations: formula

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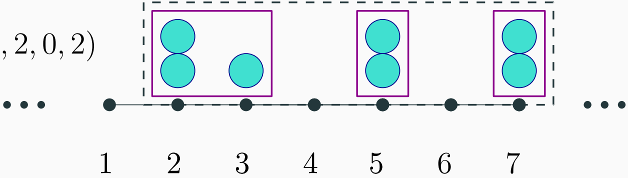

- Set $\ell_h = \sum_{i \leq h} c_i$, and $D_h = \prod_{a \in MSet(c)} |a - h|$.

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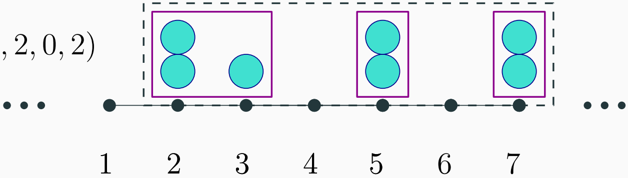

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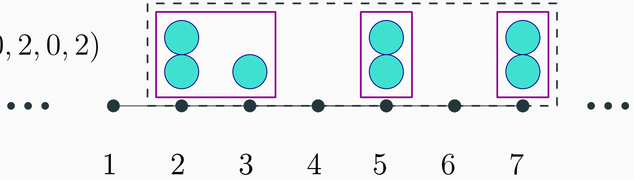

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For all $j \geq 0$:

$$\prod_{a \in MSet(\gamma)} (j + a) = \sum_{k=0}^{j} \binom{n+k}{k} A_{(0^{j-k}, \gamma, 0^{n-m-(j-k)})} + \sum_{h \in H} D_h \binom{j + h + n - \ell_h}{j + h} \binom{j + h - 1}{\ell_h}$$

Thanks

Thank you for listening!