A Conjecture on descents, inversions and the weak order

Christophe Hohlweg – **Viviane Pons**

LACIM UQAM – **LISN, Univ. Paris-Saclay**

$$
\begin{array}{ccccccccc}
 & & & & {\color{red}(1,6)} & & & & \\
 & & & {\color{red}(1,5)} & & (2,6) & & & \\
 & & {\color{red}(1,4)} & & {\color{blue}(2,5)} & & {\color{blue}(3,6)} & & \\
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Coxeter Groups

A group W generated by a set $S = \{s_1, \ldots, s_n\}$ such that

- $s_i^2 = 1$
- $(s_i s_j)^{m_{i,j}} = 1$ with $m_{i,j} \in \mathbb{N} \cup +\infty$

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Type A : the group $\mathfrak{S}_n = A_{n-1}$

Permutations os size n, generated by simple transpositions $\{s_1, \ldots, s_{n-1}\}$

- $s_i^2 = 1$
- $(s_i s_{i+1})^3 = 1$
- $(s_i s_j)^2 = 2$ if $|i - j| > 1$

Reflections

- $T :=$ reflections of W, all possible conjugates of elements from S
- the generating set S are called the *simple reflections*

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In type A

- T : all transpositions
- S : simple transpositions $(i, i+1)$

Inversion set

Let $w \in W$, the (left) inversion set of w is given by

$$\mathrm{invs}(w) := \{t \in T \mid \ell(tw) < \ell(t)\}.$$

where T is the set of *reflections* of W

Example in Type A

$w = 526341$

$$
\begin{array}{ccccccccc}
 & & & & (1,6) & & & & \\
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Example in type A : a bipartition of an element w

$w = 526341$

$u = 234561$

$v = 152634$

$$\mathrm{invs}(w) = \mathrm{invs}(u) \sqcup \mathrm{invs}(v)$$

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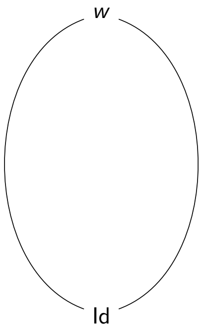
Bipartitions and the (right) weak order

(right) Weak order: order elements of W by inclusion of (left) inversion sets.



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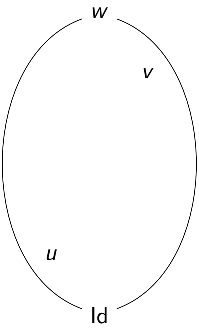

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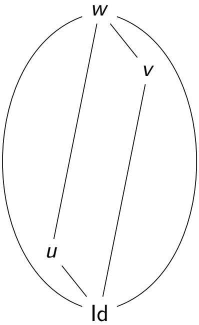

$$w = u \vee v$$

$$d = u \wedge v$$

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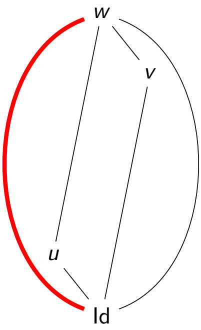

$$w = u \vee v$$
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$d(id, w)$

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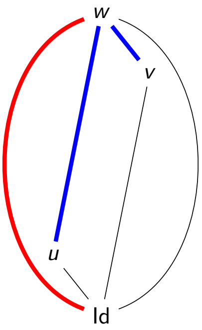

$$w = u \vee v$$
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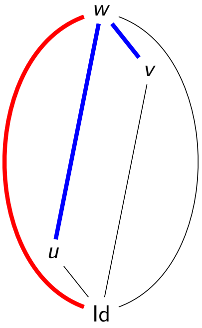
$d(id, w)$

$d(u, v)$

Bipartitions and the (right) weak order

(right) Weak order: order elements of W by inclusion of (left) inversion sets.



$$w = u \vee v$$
$$d = u \wedge v$$

$d(id, w)$

$d(u, v)$

u, v bipartition of $w \Leftrightarrow$

$$d(u, v) = d(id, w)$$

Some numbers

	$A2$	$A3$	$A4$	$A5$
Elements which admit bipartition	1	11	81	554
Partition-indecomposable elements	5	13	39	166

Inversion partitions in the literature

Appear in a geometric problem

Eigenvalue problem and a new product in cohomology of flag varieties,
Belkale and Kumar, 2006

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Also

Decomposing sets of inversions, Katthän, 2013

Decomposing inversion sets of permutations and applications to faces of the Littlewood-Richardson cone, Dewji, Dimitrov, McCabe, Roth, Wehlau and Wilson, 2017

Our motivation

A conjecture formulated by Ressayre on the number of descents

(right) descents of elements

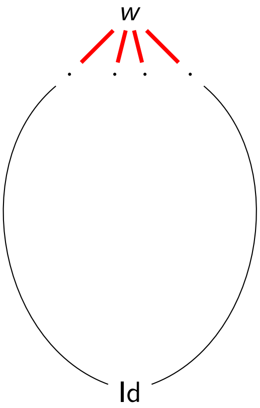
Let $w \in W$, a *(right) descent* s is a simple reflection such that $\ell(ws) = \ell(w) - 1$. We write $d_R(w)$ the number of right descents.

Example in type A

$$526341 \times (1,2) = 256341$$

$$d_R(5.26.34.1) = 3$$

On the weak order



descents of w = coatoms of w

The Conjecture

If u, v is a bipartition of w, then

$$d_R(w) = d_R(u) + d_R(v)$$

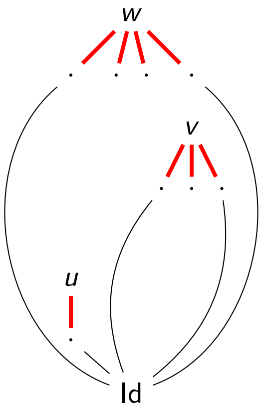
Example in type A

$w = 5.26.34.1 \quad d_R(w) = 3$

$u = 23456.1 \quad d_R(u) = 1$

$v = 15.26.34 \quad d_R(v) = 2$

On the weak order



The Conjecture

True for Weyl groups (through algebraic geometry, case by case)

Intersection multiplicity one for the belkale-kumar product, Ressayre and Francone, 2023

Direct (elementary) proof in type A and B

Hohlweg and P., 2024

Still open for other Coxeter groups

Idea of the proof on the example

Recursive decomposition

"left words" "right words"

$w = 5.26.34.1$

$u = 23456.1$

$v = 15.26.34$

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$w = 5.26.34.1$

$u = 23456.1$

$v = 15.26.34$

Idea of the proof on the example

Recursive decomposition

	"left words"	"right words"
$w = 5.26.34.1$	5.2	6.34.1
$u = 23456.1$	25	346.1
$v = 15.26.34$	5.2	16.34

Idea of the proof on the example

Recursive decomposition

	"left words"	"right words"
$w = 5.26.34.1$	2.1	$4.23.1$
$u = 23456.1$	12	234.1
$v = 15.26.34$	2.1	14.23

Idea of the proof on the example

Recursive decomposition

	"left words"	"right words"	"decreased right words"
$w = 5.26.34.1$	2.1	4.23.1	23.1
$u = 23456.1$	12	234.1	
$v = 15.26.34$	2.1	14.23	

Idea of the proof on the example

Recursive decomposition

	"left words"	"right words"	"decreased right words"
$w = 5.26.34.1$	2.1	4.23.1	23.1
$u = 23456.1$	12	234.1	23.1
$v = 15.26.34$	2.1	14.23	123

Also in type B

“left words” “right words”

$$w = 1.\bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}|5.26.34.\bar{1}$$
$$u = \bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}\bar{1}|15.26.34$$
$$v = 1.\bar{6}\bar{5}\bar{4}\bar{3}\bar{2}|23456.\bar{1}$$

Also in type B

	"left words"	"right words"
$w = 1.\bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}\vert 5.26.34.\bar{1}$	$\bar{2}.\bar{5}\vert 5.2$	$6.34.\bar{1}$
$u = \bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}\bar{1}\vert 15.26.34$	$\bar{2}.\bar{5}\vert 5.2$	$\bar{1}6.34$
$v = 1.\bar{6}\bar{5}\bar{4}\bar{3}\bar{2}\vert 23456.\bar{1}$	$\bar{5}\bar{2}\vert 25$	$346.\bar{1}$

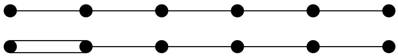
Also in type B

	"left words"	"right words"
$w = 1.\bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}\vert 5.26.34.\bar{1}$	$\bar{1}.\bar{2}\vert 2.1$	$4.23.1$
$u = \bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}\bar{1}\vert 15.26.34$	$\bar{1}.\bar{2}\vert 2.1$	14.23
$v = 1.\bar{6}\bar{5}\bar{4}\bar{3}\bar{2}\vert 23456.\bar{1}$	$\bar{2}\bar{1}\vert 12$	234.1

Coxeterize?

A_n

B_n



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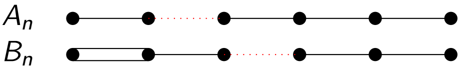
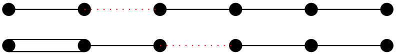


Coxeterize?

A_n
B_n



Goal: find a group generic recursive decomposition of elements that "keeps" the descents

Thank you!

A conjecture on descents, inversions and the weak order, Hohlweg and P., arXiv:2412.09227, 2024