Pattern avoiding 3-permutations and triangle bases

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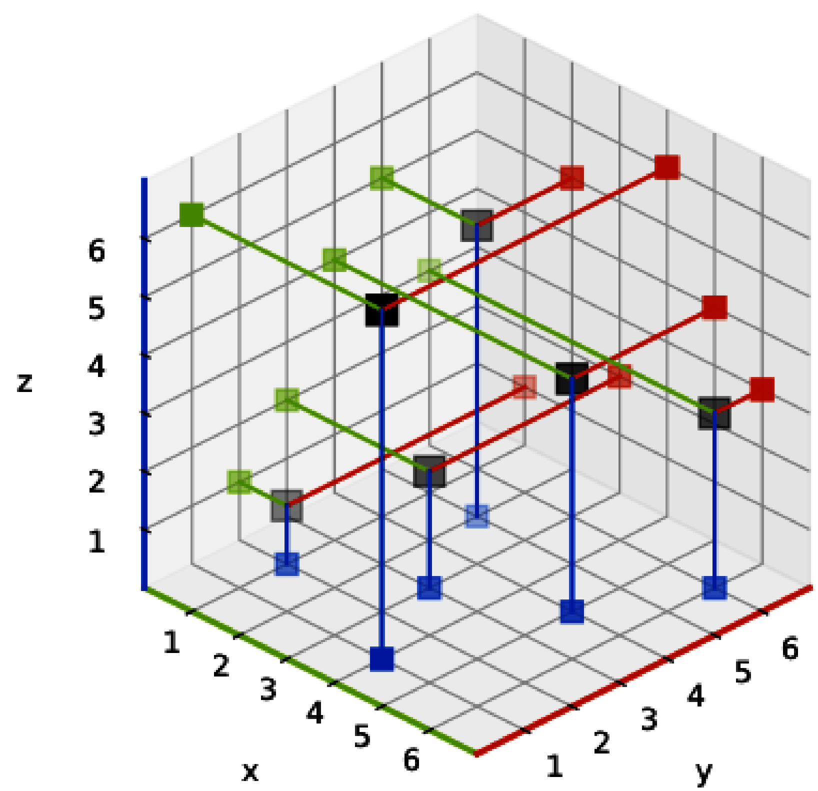
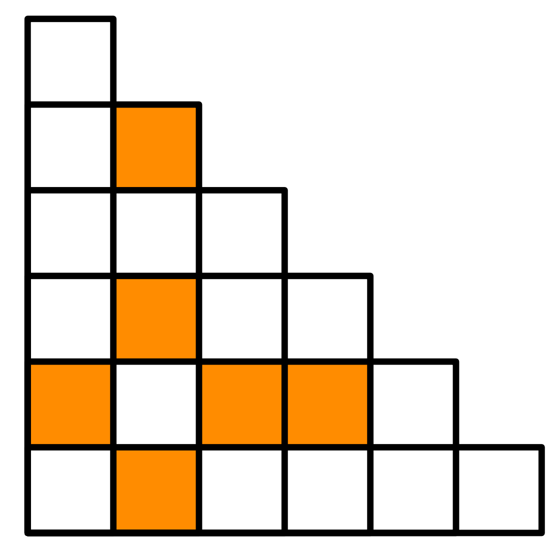
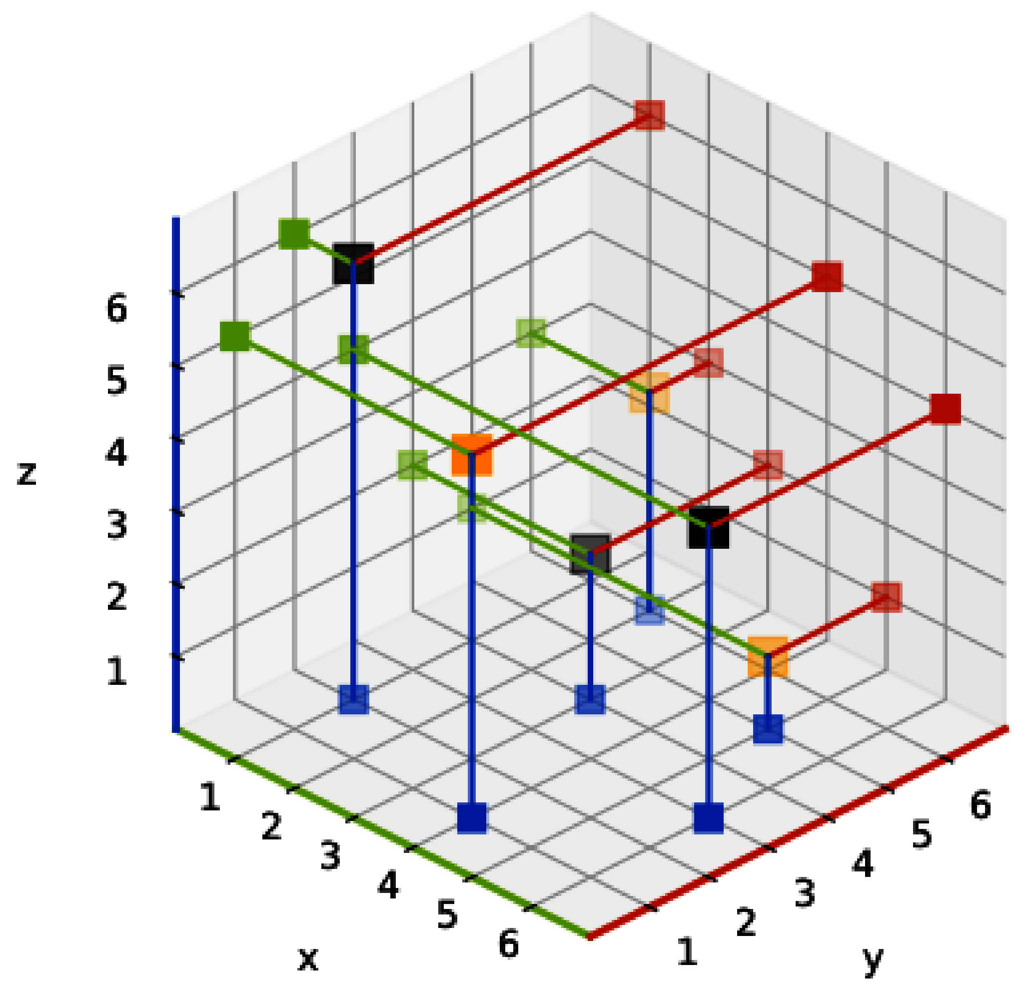




I- The objects

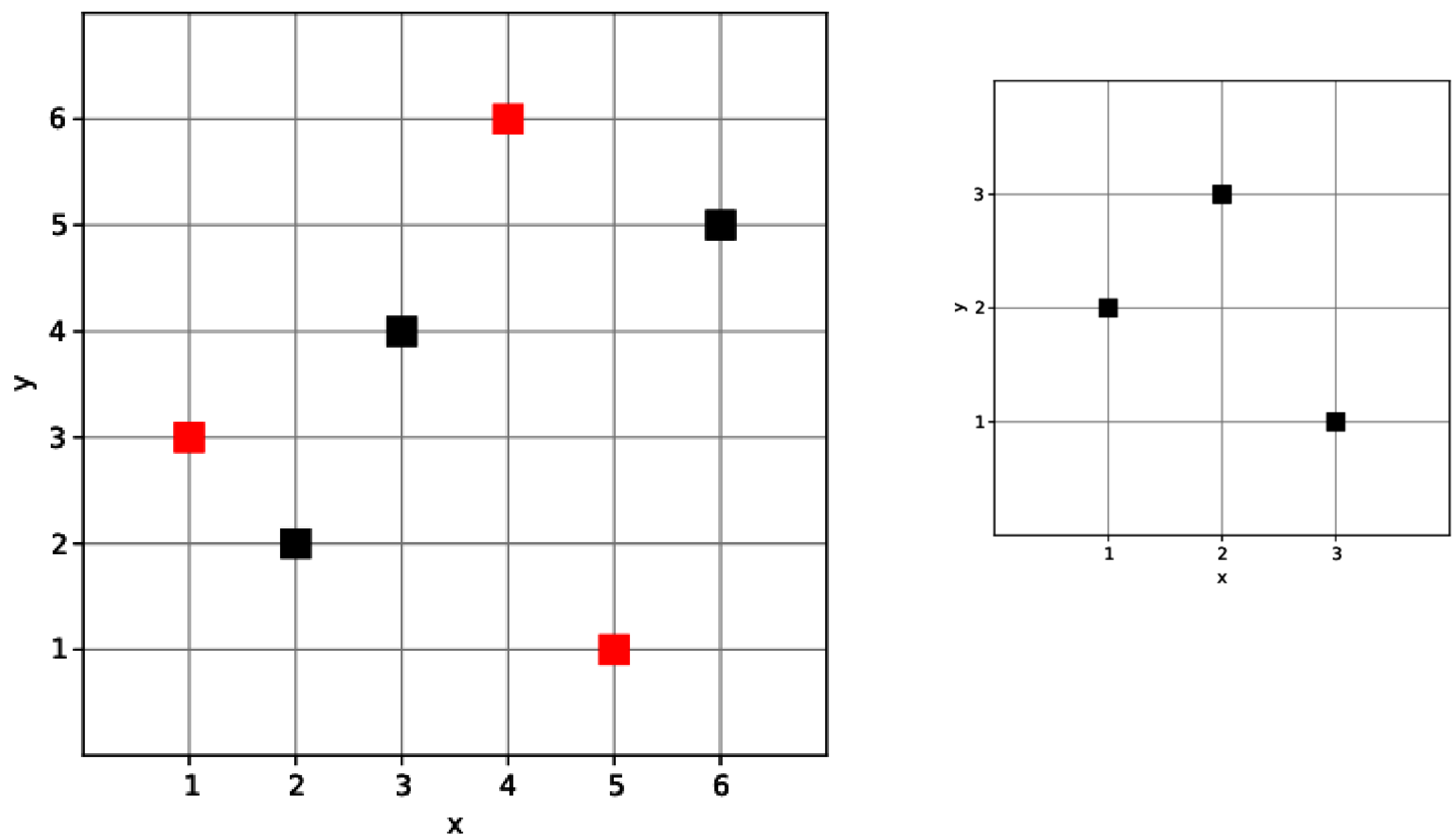
a) Pattern avoiding 3-permutations



Pattern avoidance in permutations

A **permutation** $\sigma = \sigma(1)\sigma(2)\dots\sigma(n)$ is a bijection from $[\![1,n]\!] = \{1,2,\dots,n\}$ to itself. Its **diagram** is the set of points $P_\sigma = \{(i,\sigma(i)) \mid 1 \leqslant i \leqslant n\}$.

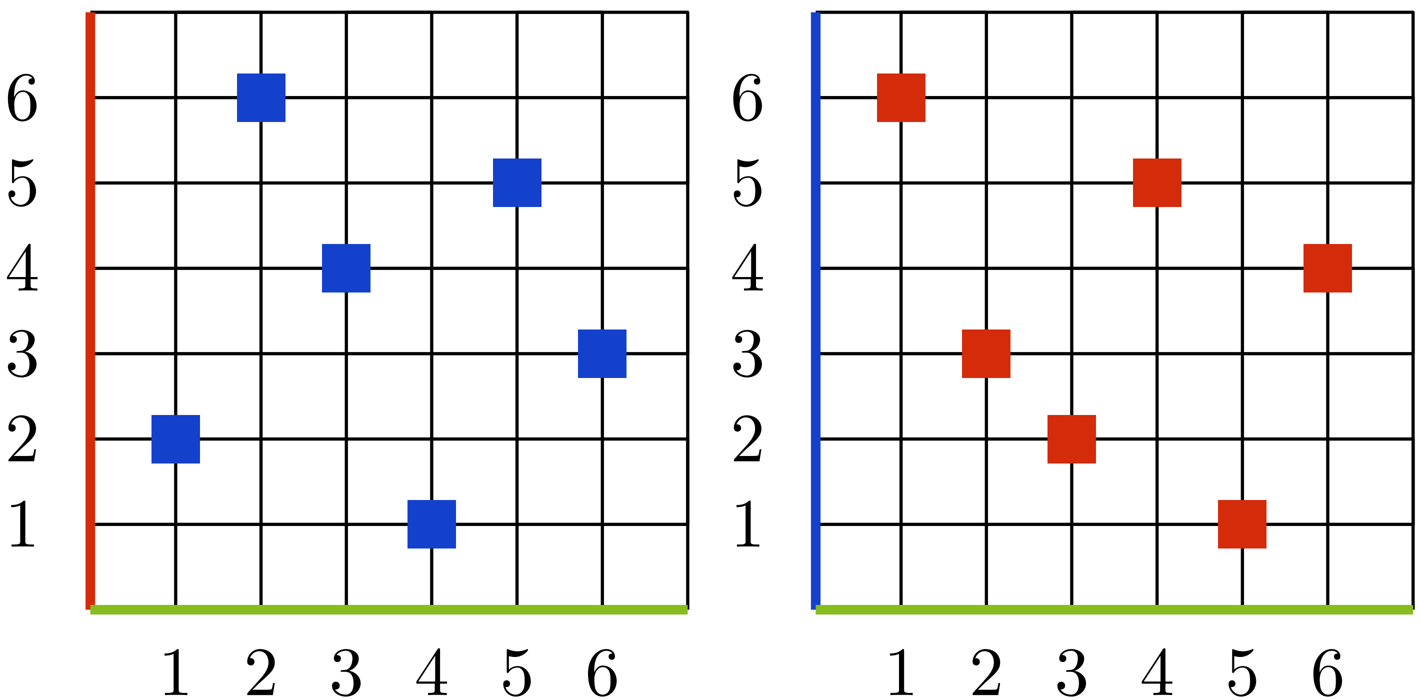
A permutation $\sigma \in \mathfrak{S}_n$ **contains** a pattern $\pi \in \mathfrak{S}_k$ if there is a set of indices I such that $\sigma_{|I} \simeq \pi$. Otherwise, it **avoids** it.



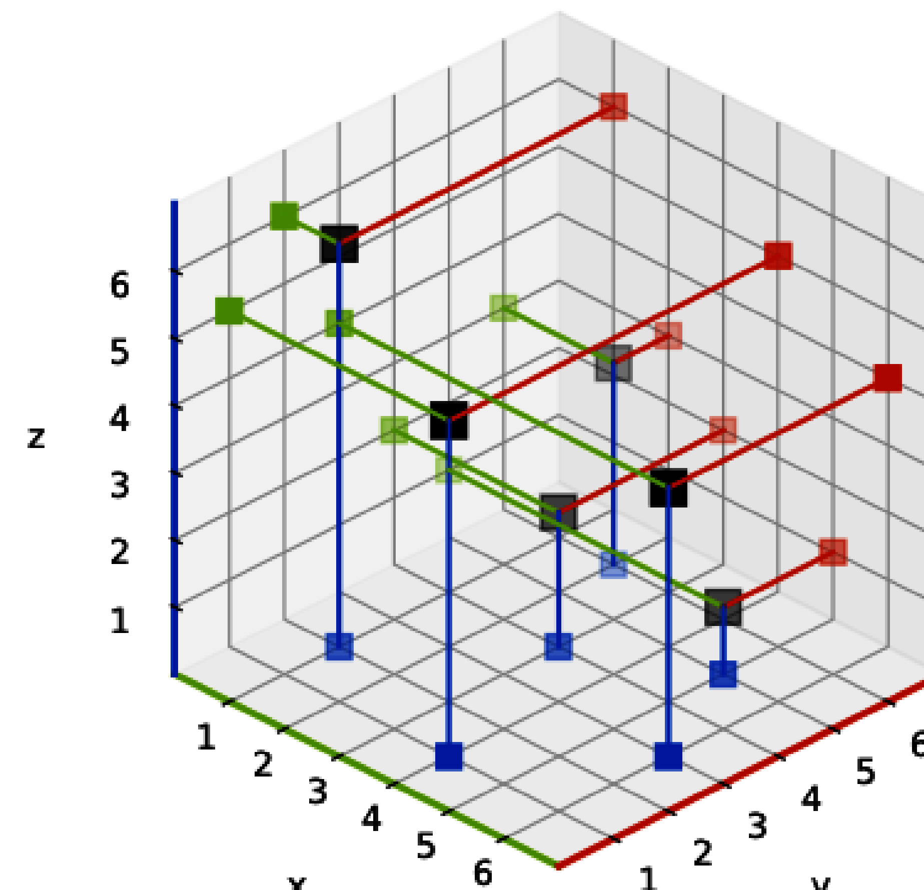
$\sigma = 324615$ contains the pattern $\pi = 231$.

Pattern avoidance in 3-permutations

A **3-permutation** is a couple of permutations $(\sigma, \tau) \in \mathfrak{S}_n^2$.
It is represented by the **diagram** $P_{(\sigma,\tau)} = \{(i, \sigma(i), \tau(i)) \mid 1 \leqslant i \leqslant n\}$.



$(264153, 632514)$



Points : $(1, 2, 6), (2, 6, 3), (3, 4, 2), (4, 1, 5), (5, 5, 1), (6, 3, 4)$

Pattern avoidance in 3-permutations

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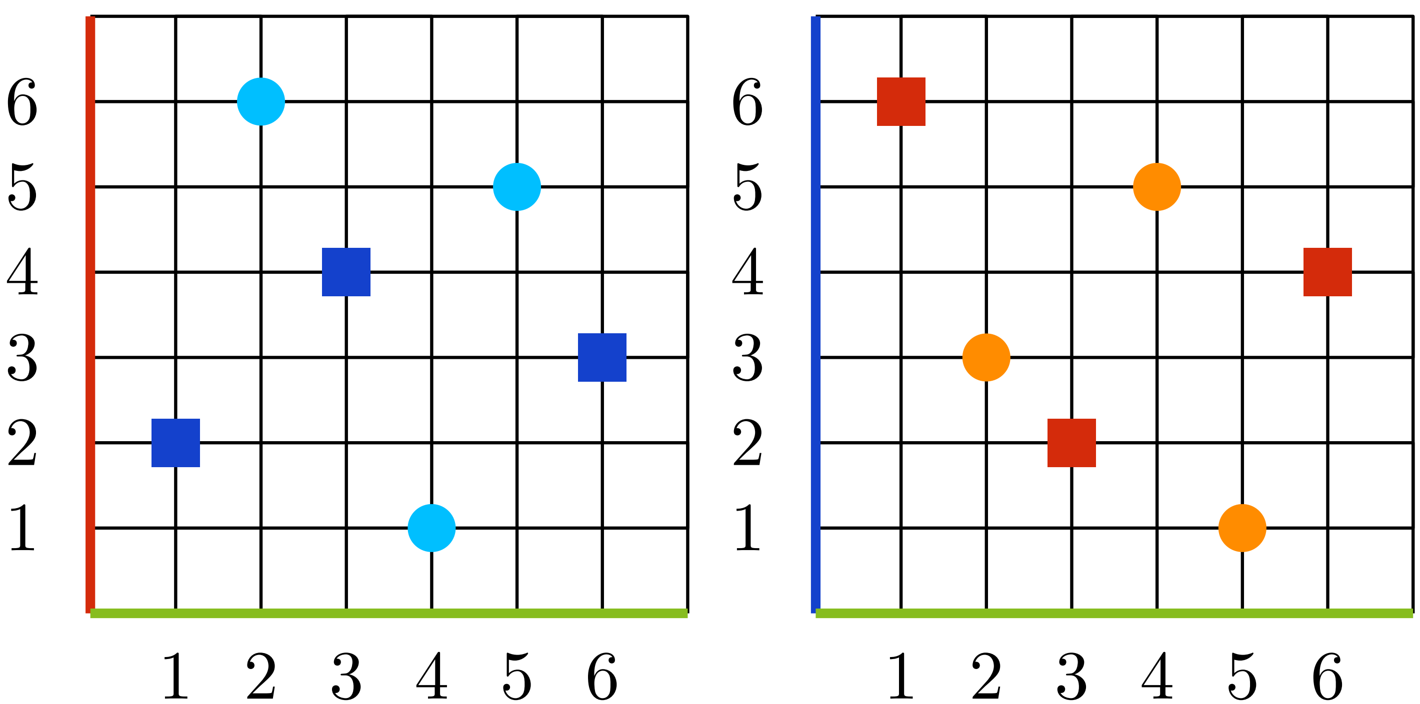
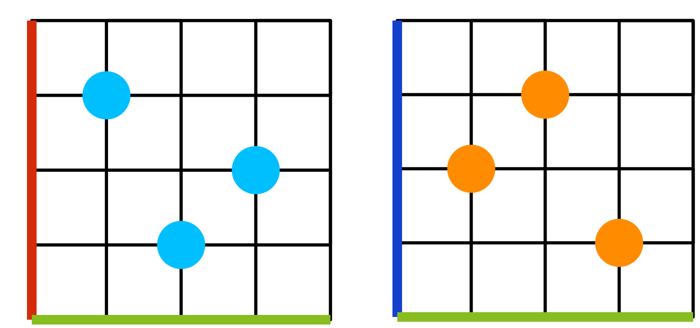
A 3-permutation $(\sigma, \tau) \in \mathfrak{S}_n^2$ **contains** a pattern $(\pi_1, \pi_2) \in \mathfrak{S}_k^2$ if there is a set of indices $I \subset [\![1, n]\!]$ such that $\sigma_{|I} \simeq \pi_1$ and $\tau_{|I} \simeq \pi_2$.
Otherwise it **avoids** it.



$(264153, 632514)$ contains the pattern $(312, 231)$.



It avoids the pattern $(12, 12)$ although both 264153 and 632514 contain 12.

Pattern avoidance classes

Patterns	TWE	Sequence	Comment
$(12, 12)$	4	$1, 3, 17, 151, 1899, 31711, \cdots$	weak-Bruhat intervals
$(12, 12), (12, 21)$	6	$n! = 1, 2, 6, 24, 120 \cdots$	$\sigma_1 \Rightarrow \sigma_2$
$(12, 12), (12, 21), (21, 12)$	4	$1, 1, 1, 1, 1, 1, \cdots$	1 diagonal
$(12, 12), (12, 21), (21, 12), (21, 21)$	1	$1, 0, 0, 0, 0, 0, \cdots$	
$(123, 123)$	4	$1, 4, 35, 524, 11774, 366352, \cdots$	*new*
$(123, 132)$	24	$1, 4, 35, 524, 11768, 365558, \cdots$	*new*
$(132, 213)$	8	$1, 4, 35, 524, 11759, 364372, \cdots$	*new*
$(12, 12), (132, 312)$	48	$(n+1)^{n-1} = 1, 3, 16, 125, 1296 \cdots$	[Atkinson et al. 93,95]
$(12, 12), (123, 321)$	12	$1, 3, 16, 124, 1262, 15898, \cdots$	distributive lattices inter.
$(12, 12), (231, 312)$	8	$1, 3, 16, 122, 1188, 13844, \cdots$	*A295928?*

[Bonichon & Morel, 22]

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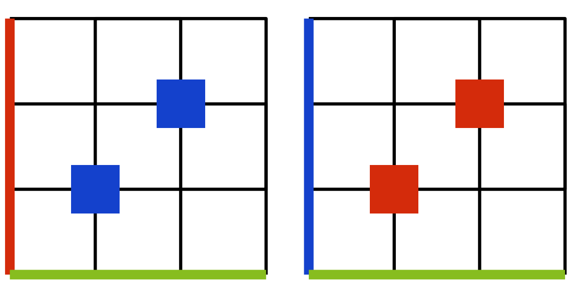
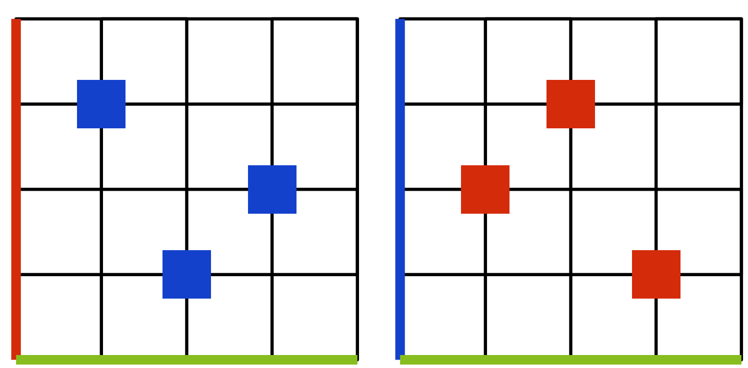


(12, 12)



(312, 231)

Pattern avoidance classes

A295928 Number of triangular matrices T(n,i,k), k <= i <= n, with entries "0" or "1" with the property that each triple {T(n,i,k), T(n,i,k+1), T(n,i-1,k)} containing a single "0" can be successively replaced by {1, 1, 1} until finally no "0" entry remains. 1

1, 3, 16, 122, 1188, 13844, 185448, 2781348, 45868268

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS A triple {T(n,i,k), T(n,i,k+1), T(n,i-1,k)} will be called a primitive triangle. It is easy to see that b(n) = n(n-1)/2 is the number of such triangles. At each step, exactly one primitive triangle is completed (replaced by {1, 1, 1}). So there are b(n) "0"- and n "1"-terms. Thus the starting matrix has no complete primitive triangle. Furthermore, any triangular submatrix T(m,i,k), k <= i <= m < n cannot have more than m "1"-terms because otherwise it would have less "0"-terms than primitive triangles. The replacement of at least one "0"-term would complete more than one primitive triangle. This has been excluded.

So T(n, i, k) is a special case of U(n, i, k), described in A101481: a(n) < A101481(n+1).

A start matrix may serve as a pattern for a number wall used on worksheets for elementary mathematics, see link "Number walls". That is why I prefer the more descriptive name "fill matrix".

The algorithm for the sequence is rather slow because each start matrix is constructed separately. There exists a faster recursive algorithm which produces the same terms and therefore is likely to be correct, but it is based on a conjecture. For the theory of the recurrence, see "Recursive aspects of fill matrices". Probable extension a(10)-a(14): 821096828, 15804092592, 324709899276, 7081361097108, 163179784397820.

The number of fill matrices with n rows and all "1"- terms concentrated on the last two rows, is A001960(n). See link "Reconstruction of a sequence".

LINKS Table of n, a(n) for n=1..9.

Gerhard Kirchner, Recursive aspects of fill matrices

Gerhard Kirchner, Number walls

Gerhard Kirchner, VB-program

Gerhard Kirchner, Reconstruction of a sequence

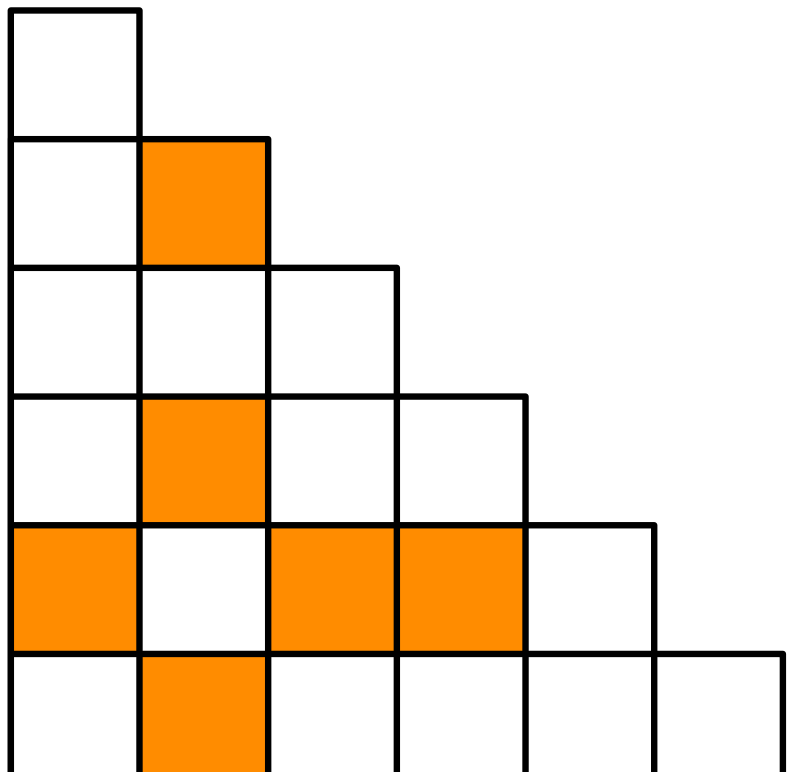
Ville Salo, Cutting Corners, arXiv:2002.08730 [math.DS], 2020.

Yuan Yao and Fedir Yudin, Fine Mixed Subdivisions of a Dilated Triangle, arXiv:2402.13342 [math.CO], 2024.

EXAMPLE
```
Example (n=2):    0     1    1
    a(2)=3       1 1   0 1  1 0
Example for completing a 3-matrix (3 bottom terms):
    1        1       1       1
   0 0  ->  1 0 ->  1 1 ->  1 1
  1 1 0    1 1 0   1 1 0   1 1 1
```

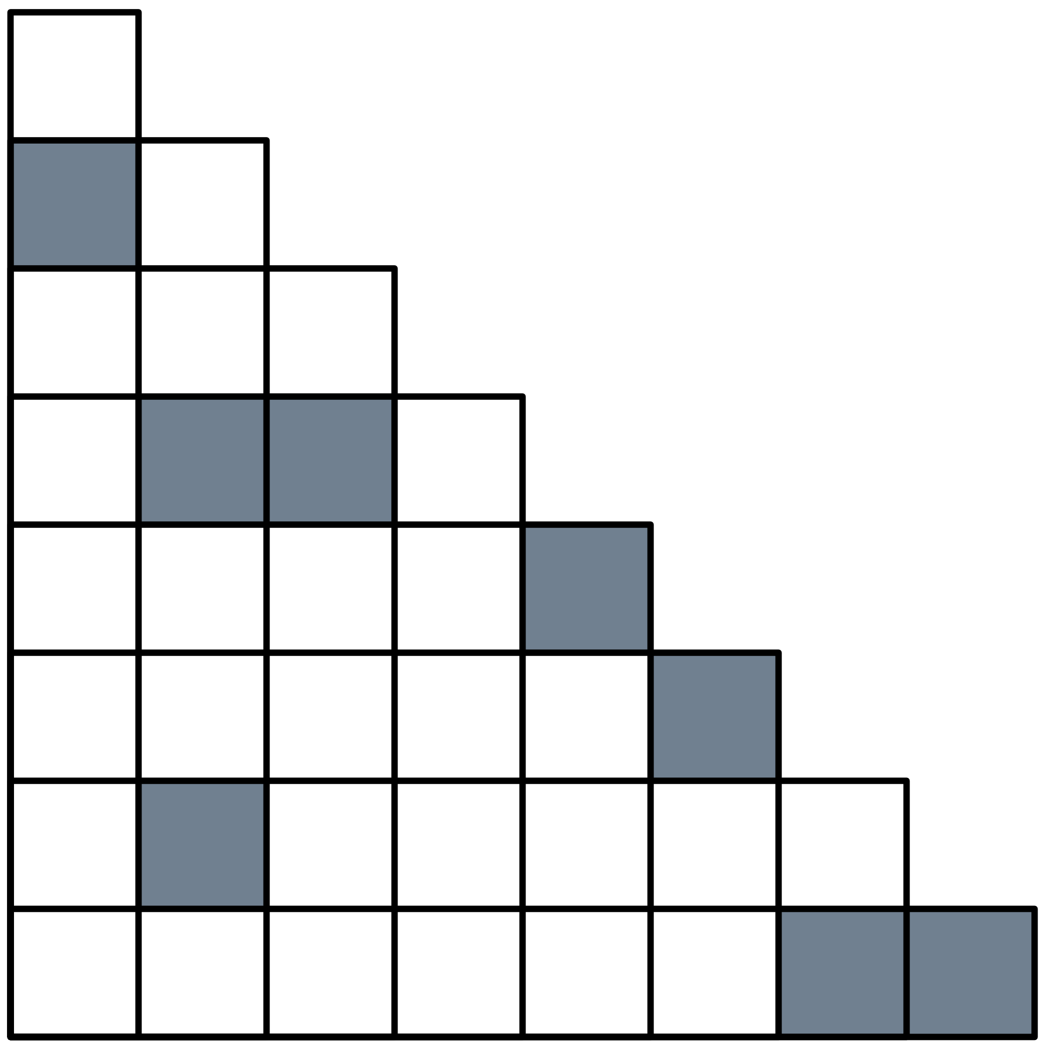
I- The objects

b) Triangle Bases



Filling configurations

A **configuration** of size n is a set of n points in the triangle $T_n = \{(a,b) \in \mathbb{N}^2 | a+b < n\}$.



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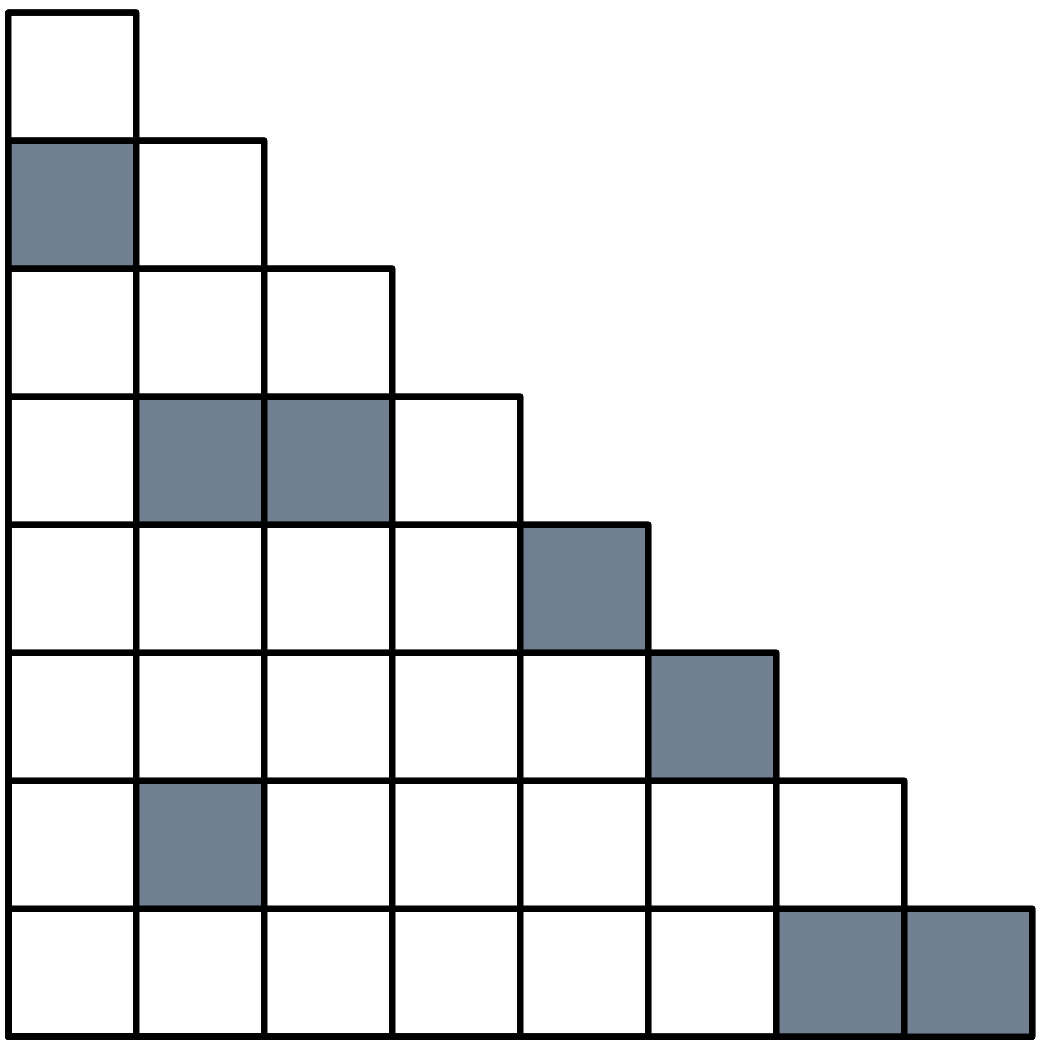
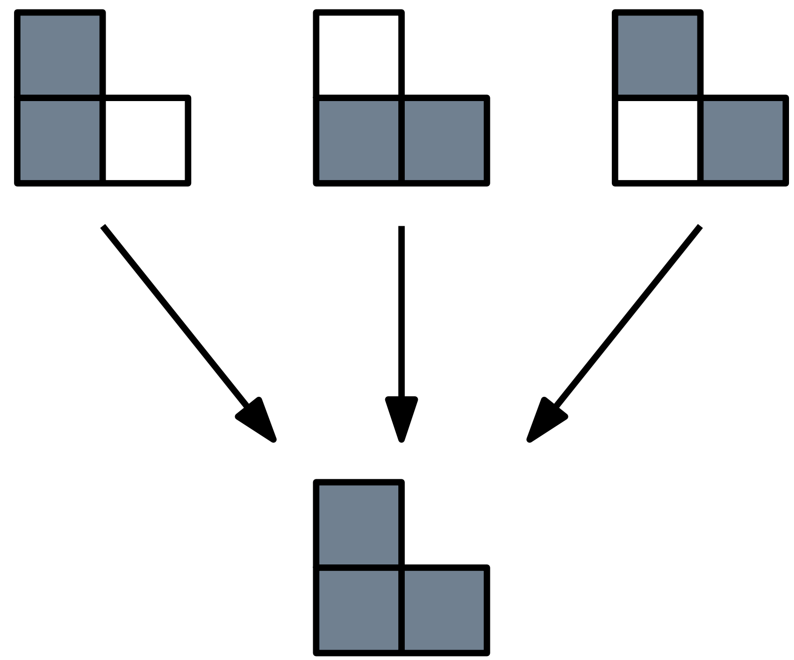
A **filling step** adds the missing point to a triangle with only 2 points.





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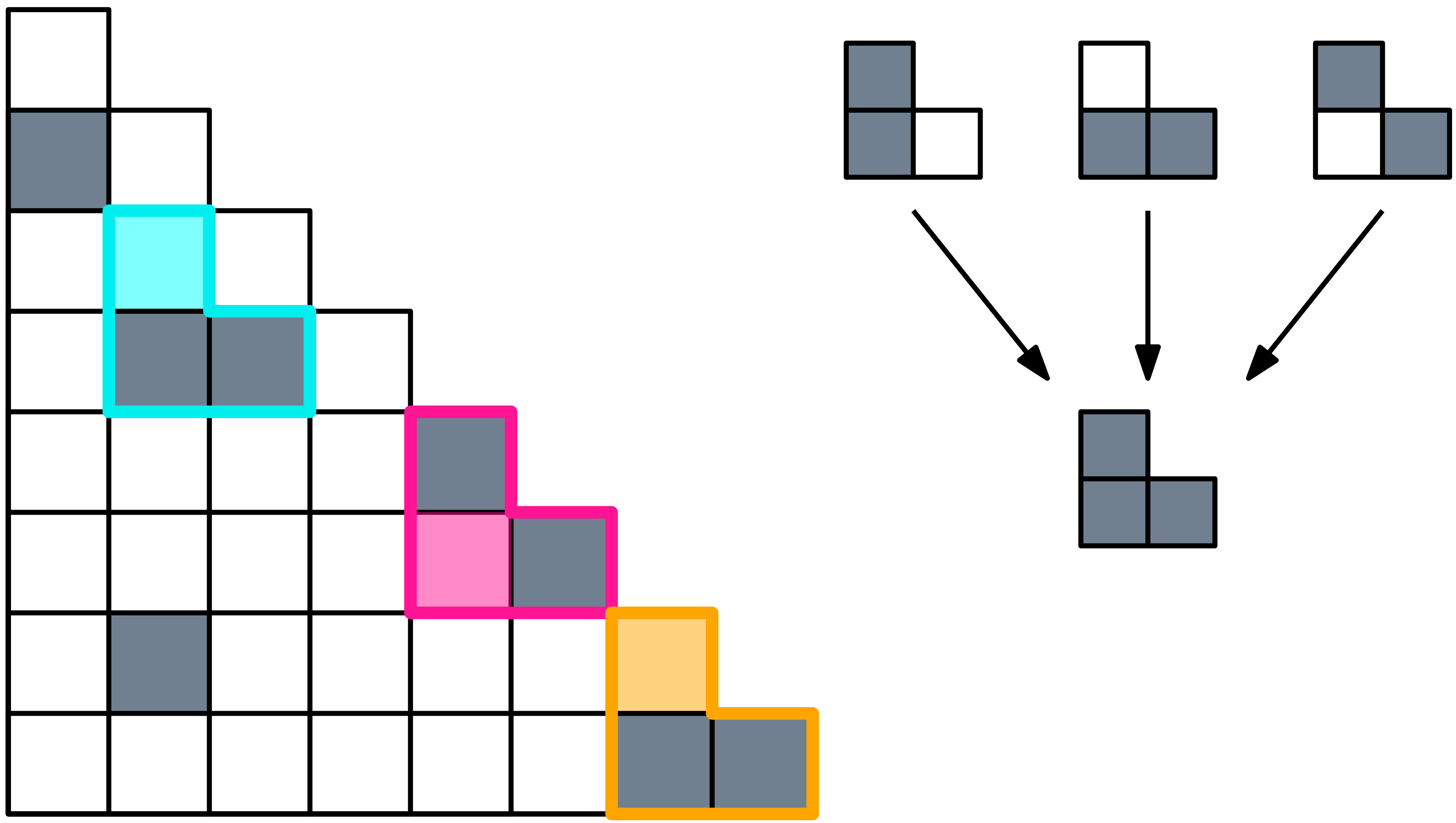

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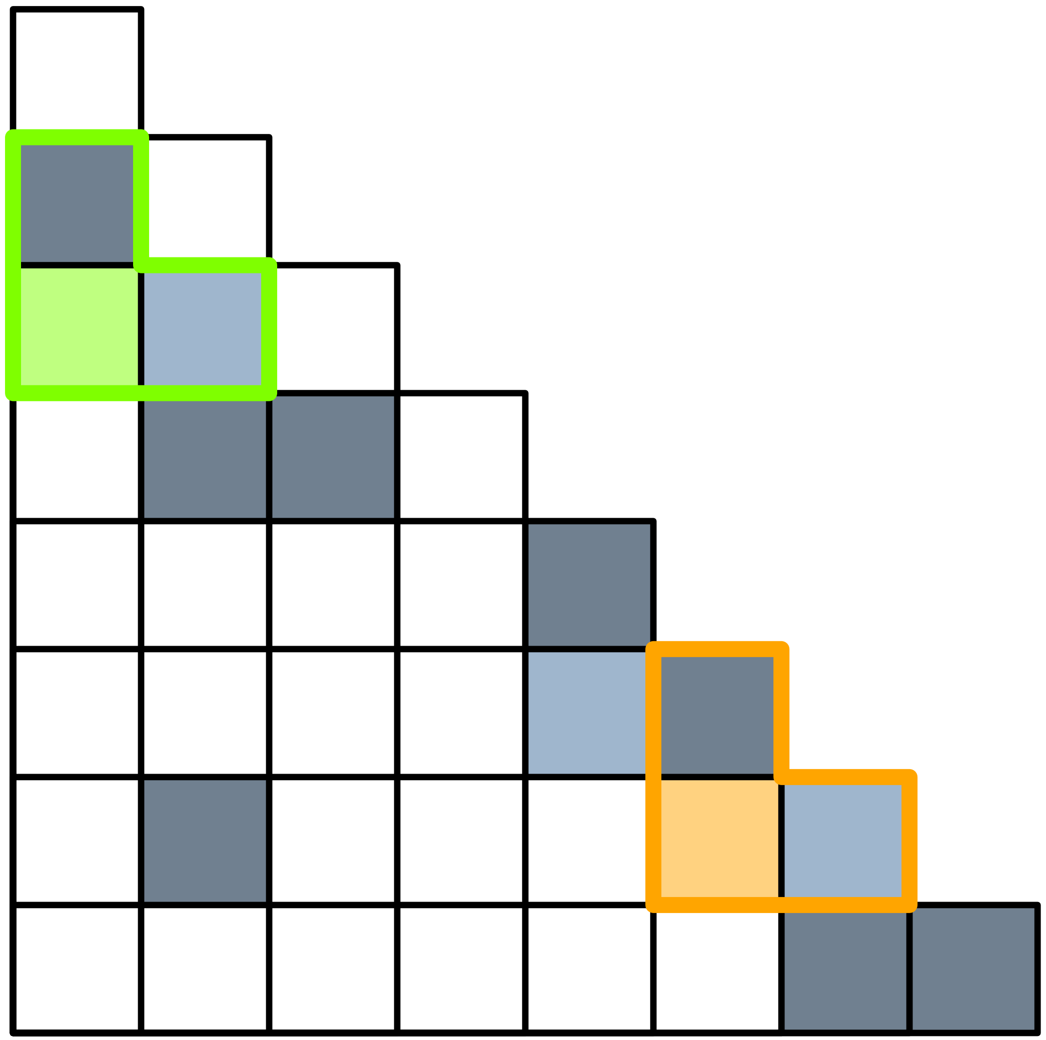

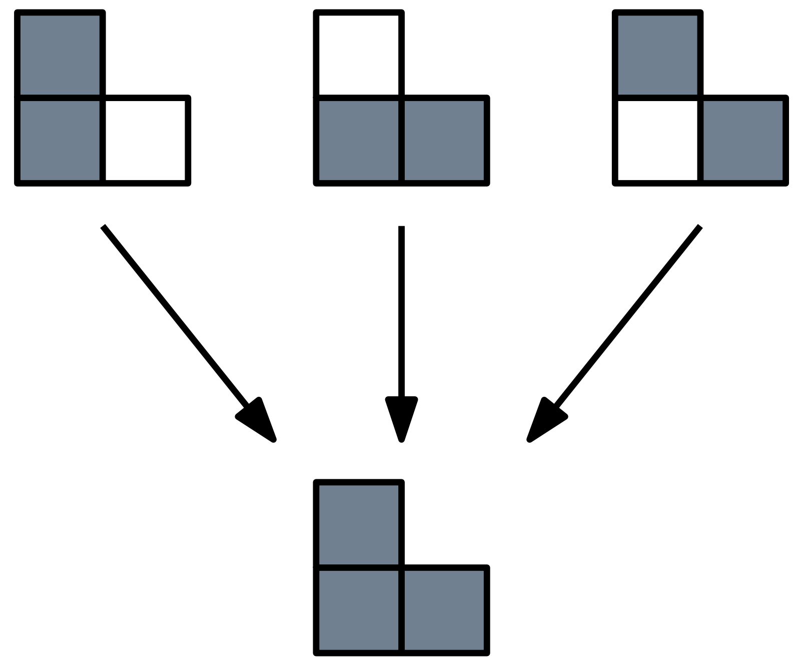

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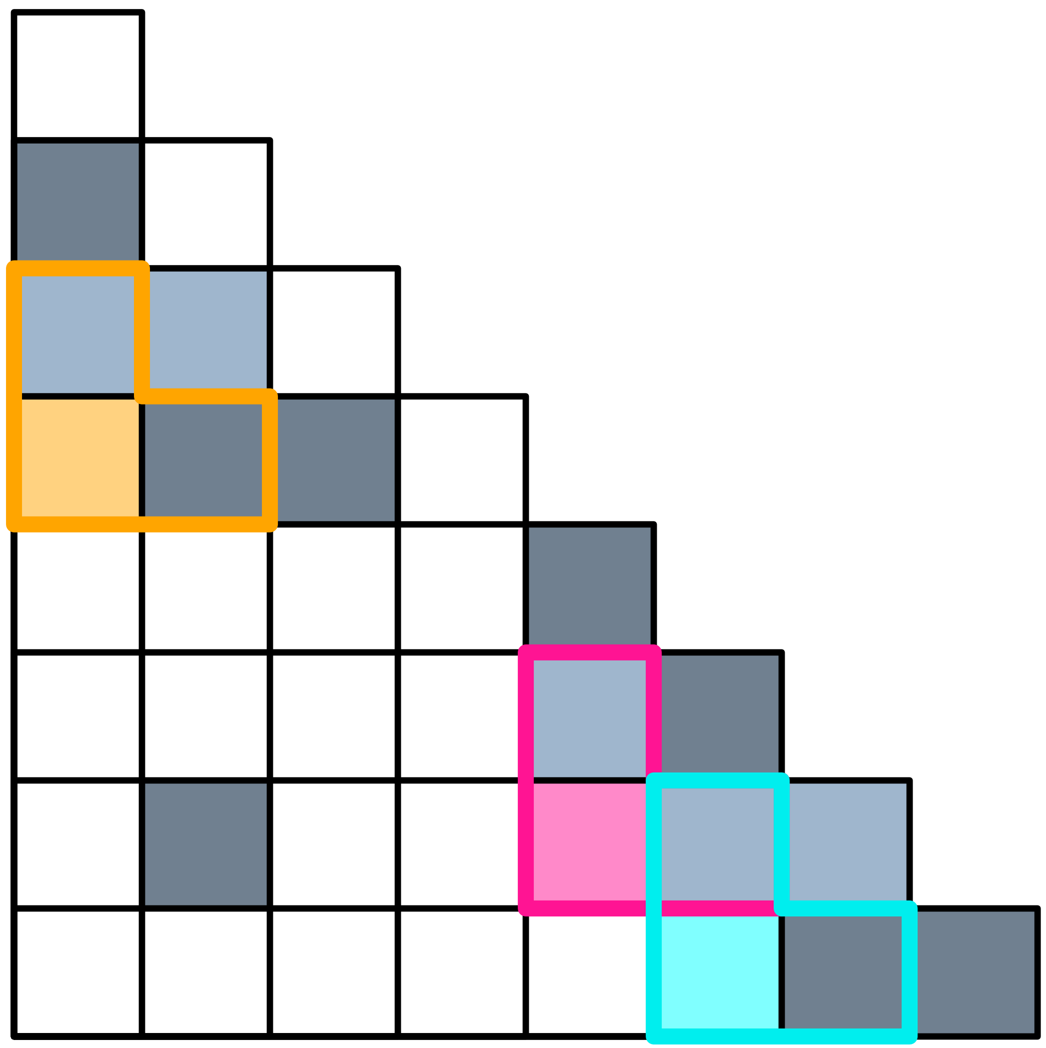

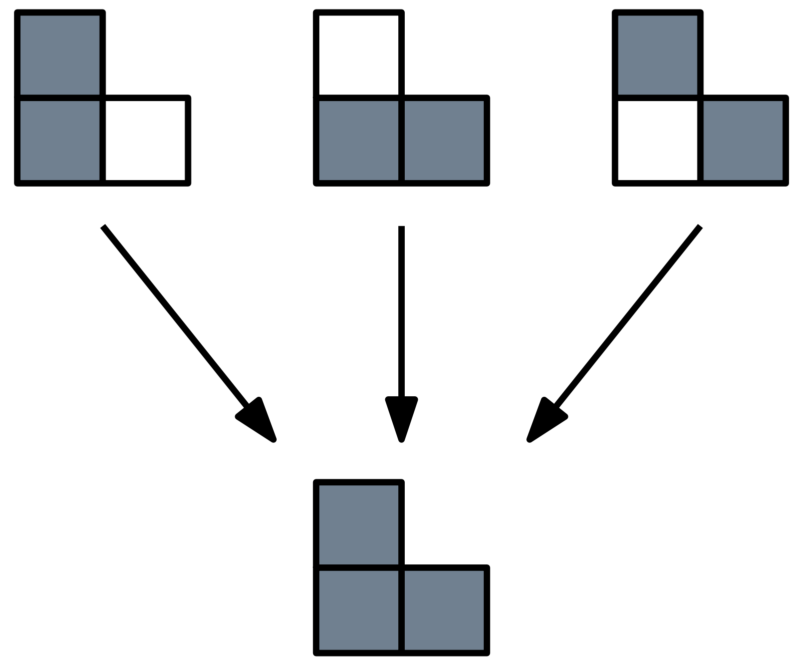

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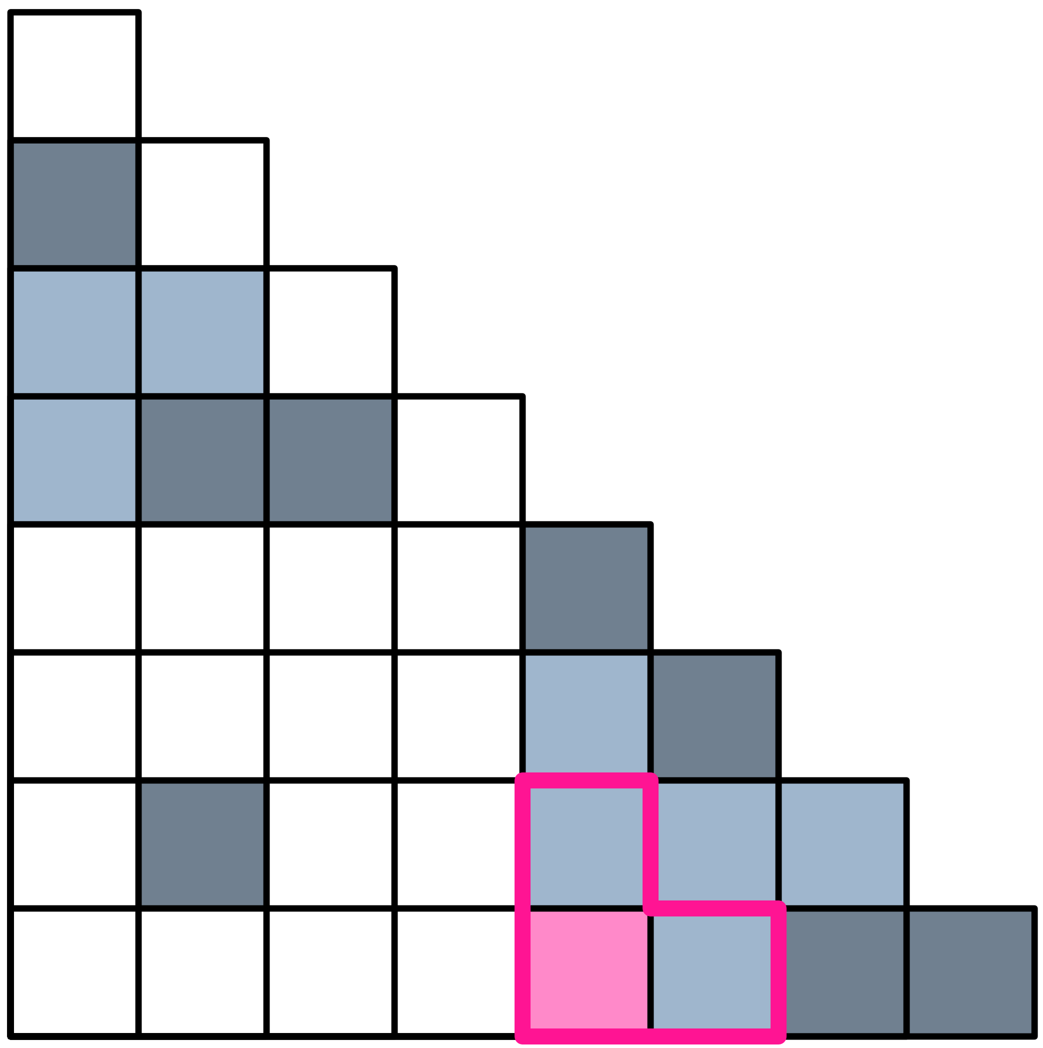

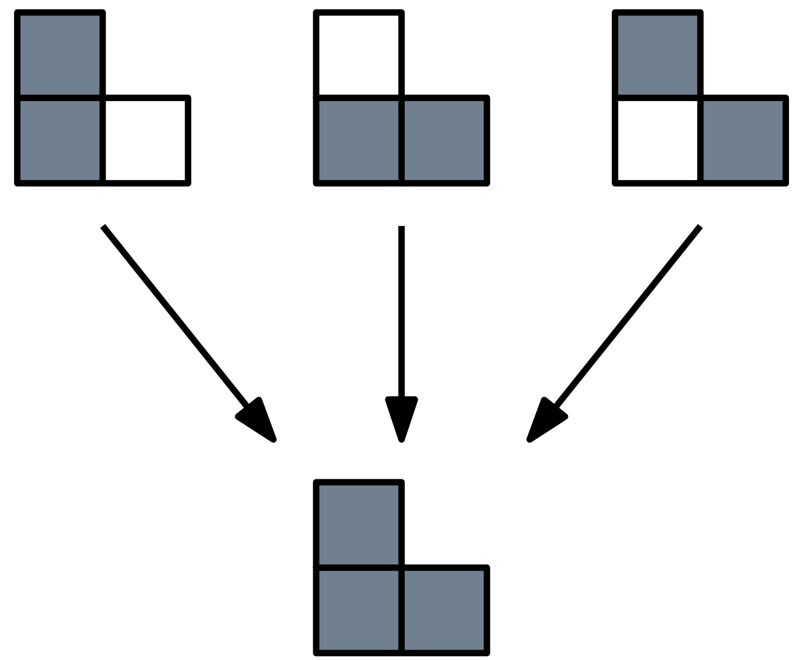

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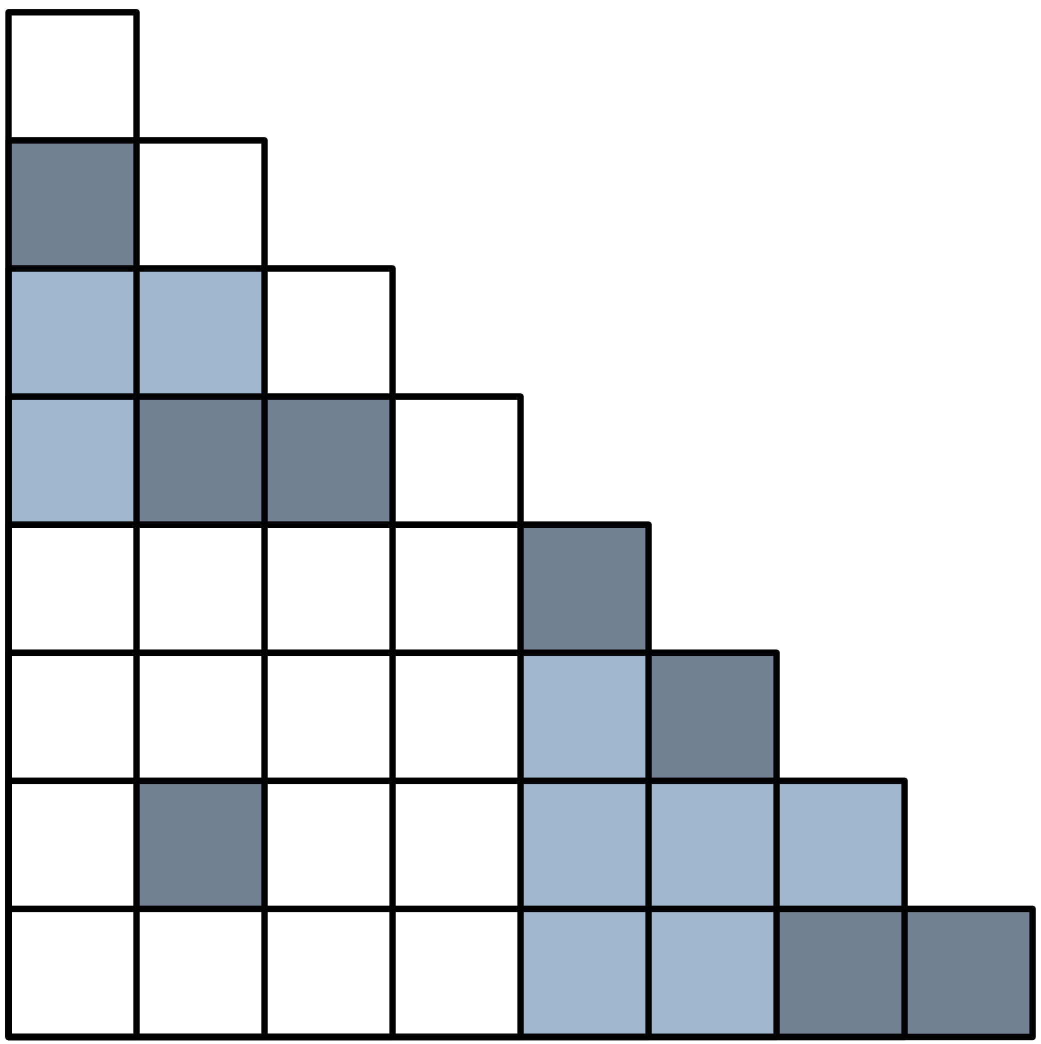
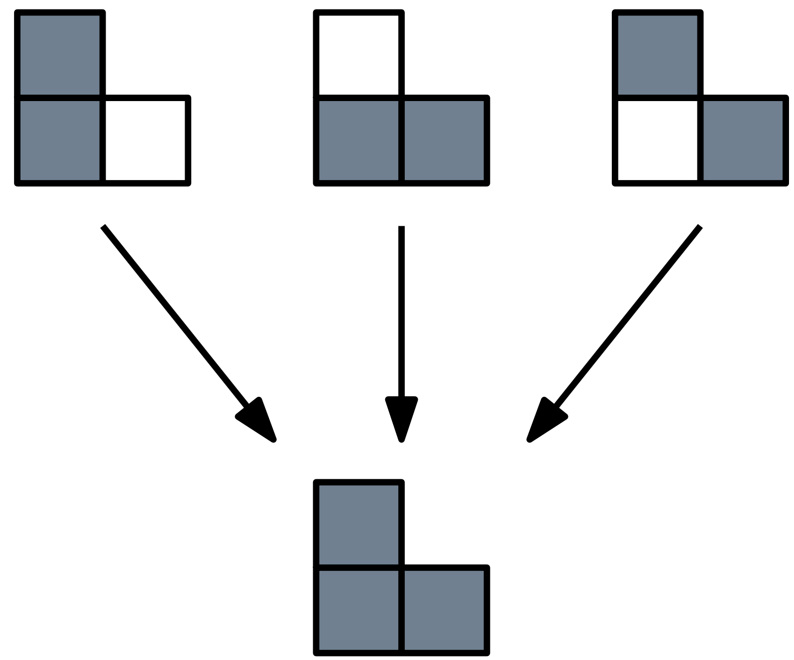
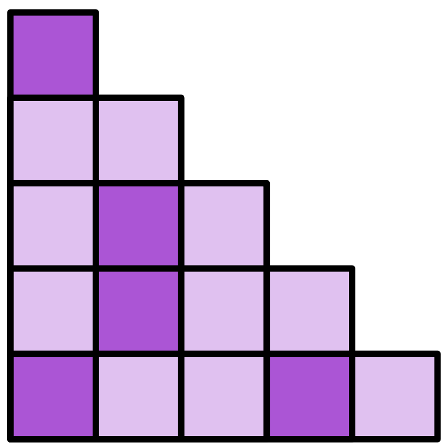




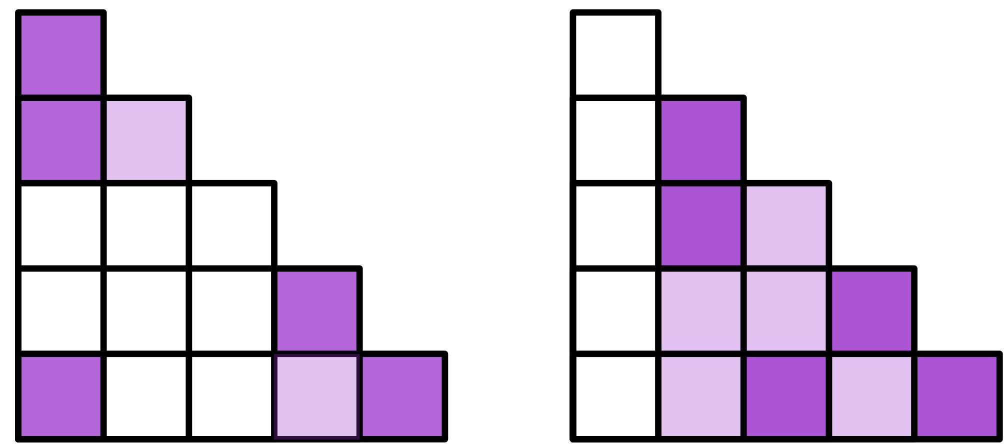
Triangle bases

$T_n = \{(a, b) \in \mathbb{N}^2 \mid a + b < n\}$ **triangle** of size n.

A **triangle basis** of size n is a configuration of n points that fills T_n. Denote $\mathcal{B}_n$ their set.



A basis.



Not a basis.

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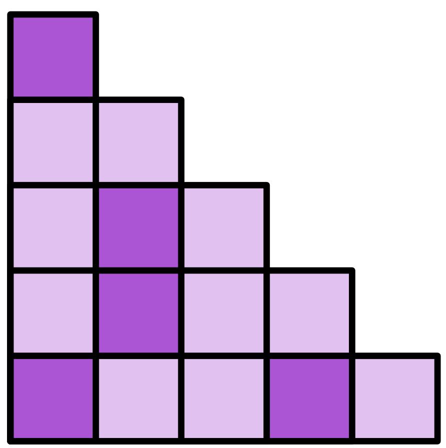
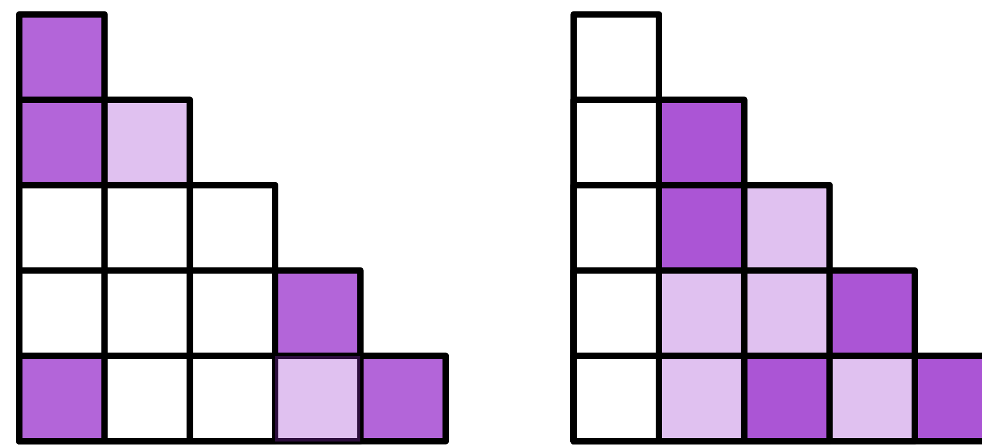


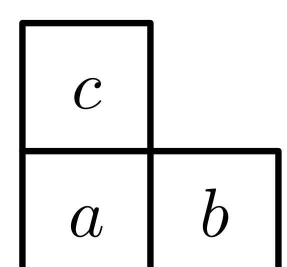
A basis.



Not a basis.

- Used to study the family of tilings such that if two cells of a small triangle are colored, there is a unique valide choice for the last.

For instance, the XOR automaton :



$$c = a + b \mod 2$$

Triangle bases

$T_n = \{(a,b) \in \mathbb{N}^2 \mid a+b<n\}$ **triangle** of size n.

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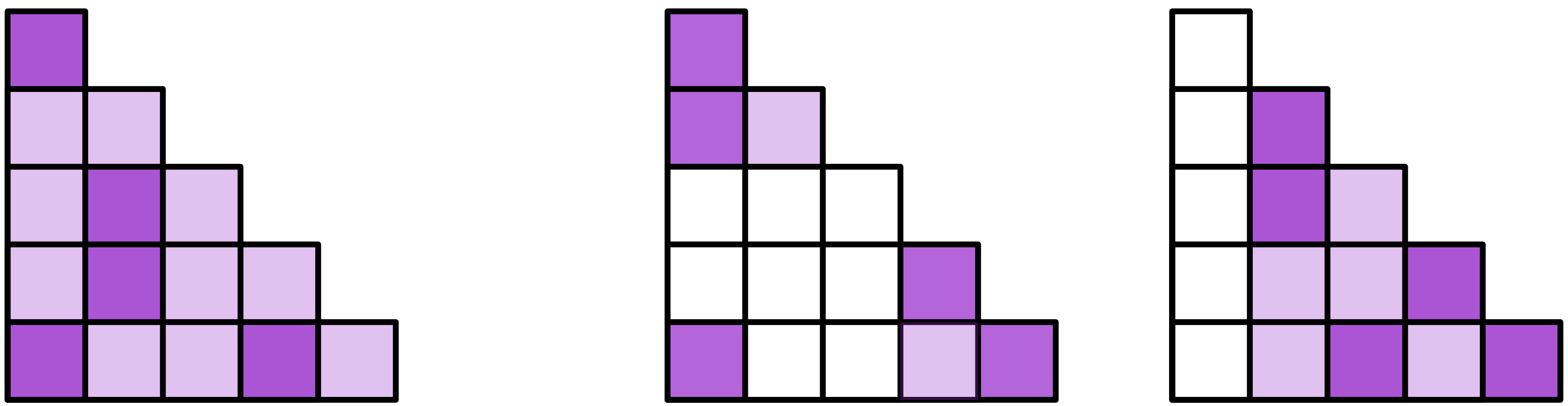
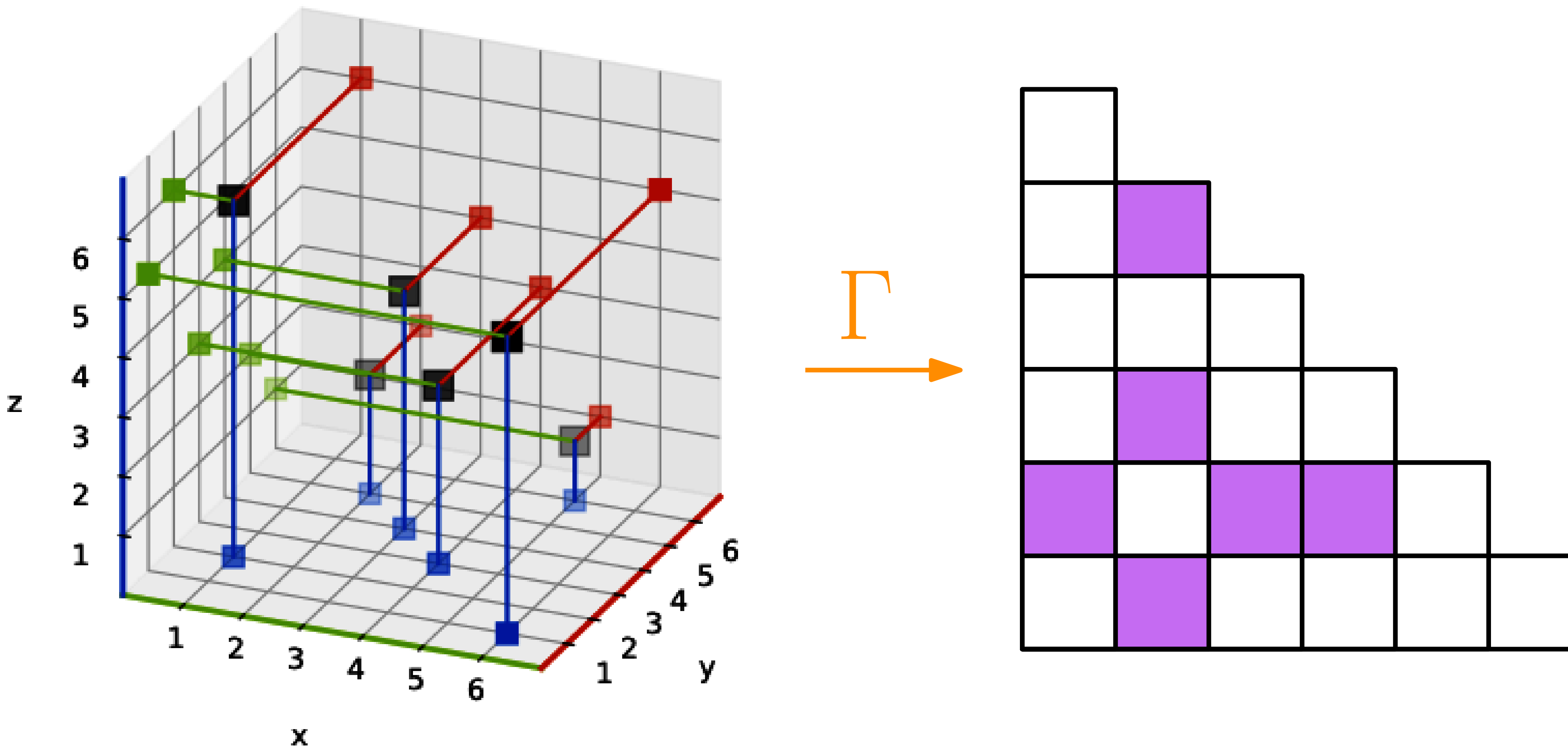


A basis. Not a basis.

Theorem. [S. 25] For all n, the set of triangle bases of size n is in bijection with $Av_n((12,12),(312,231))$.

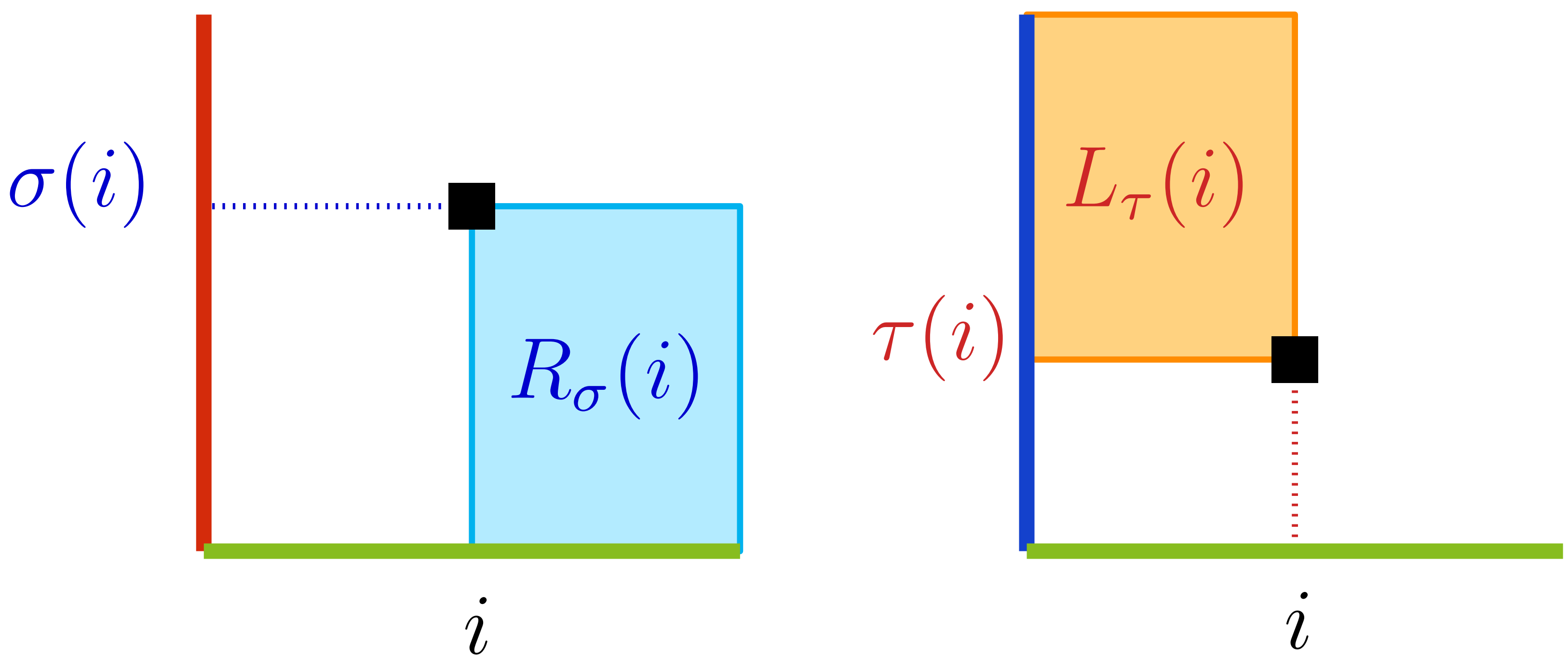
II- A bijection



Inversions

An **inversion** for $\sigma \in \mathfrak{S}_n$ is $(i,j) \in [\![1,n]\!]$ with $i < j$ and $\sigma(i) > \sigma(j)$.

The **inversion sets** are $R_\sigma(i) = \{j > i \mid \sigma(j) < \sigma(i)\}$ (right inversions) and $L_\sigma(i) = \{j < i \mid \sigma(j) > \sigma(i)\}$ (left inversions).
We denote $r_\sigma(i) = |R_\sigma(i)|$ and $\ell_\sigma(i) = |L_\sigma(i)|$.



The bijection: $\Gamma : (\sigma, \tau) \mapsto \{(r_\sigma(i), \ell_\tau(i)) \mid i \in [\![1,n]\!]\}$

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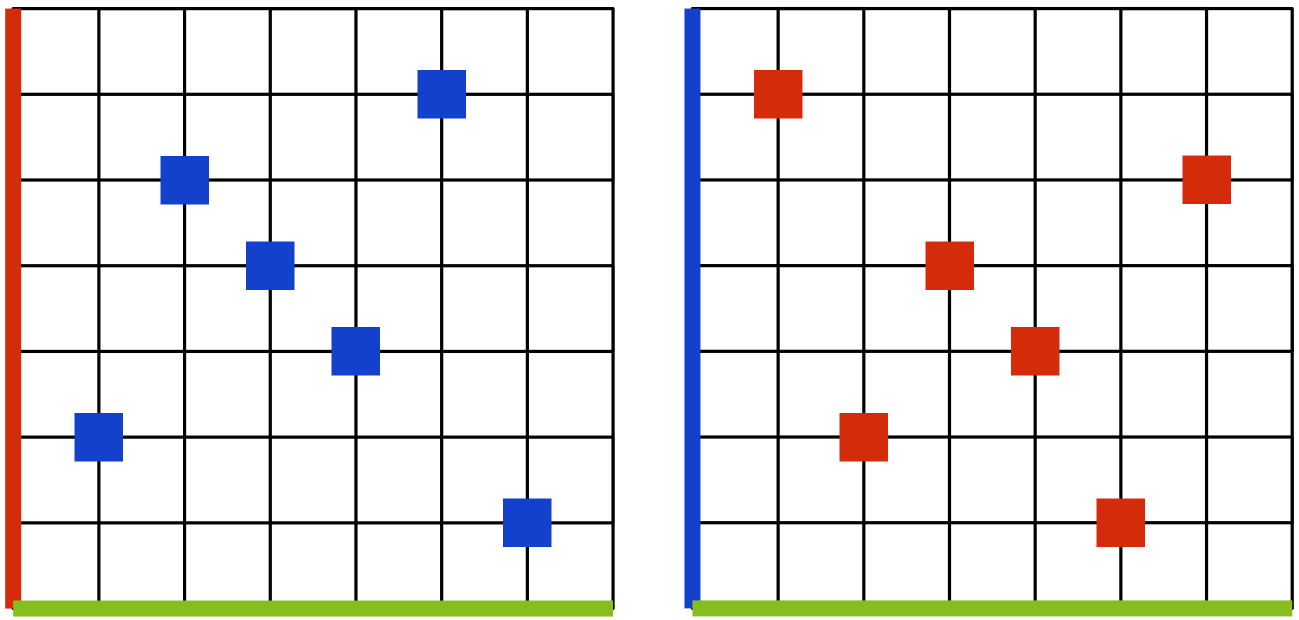

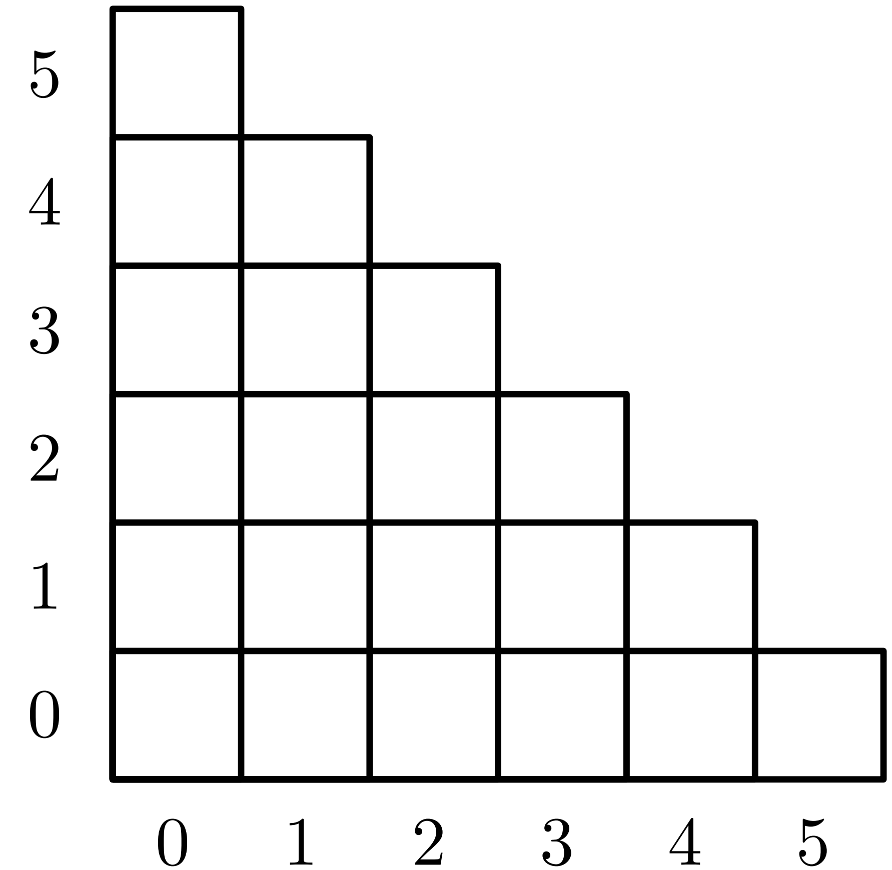

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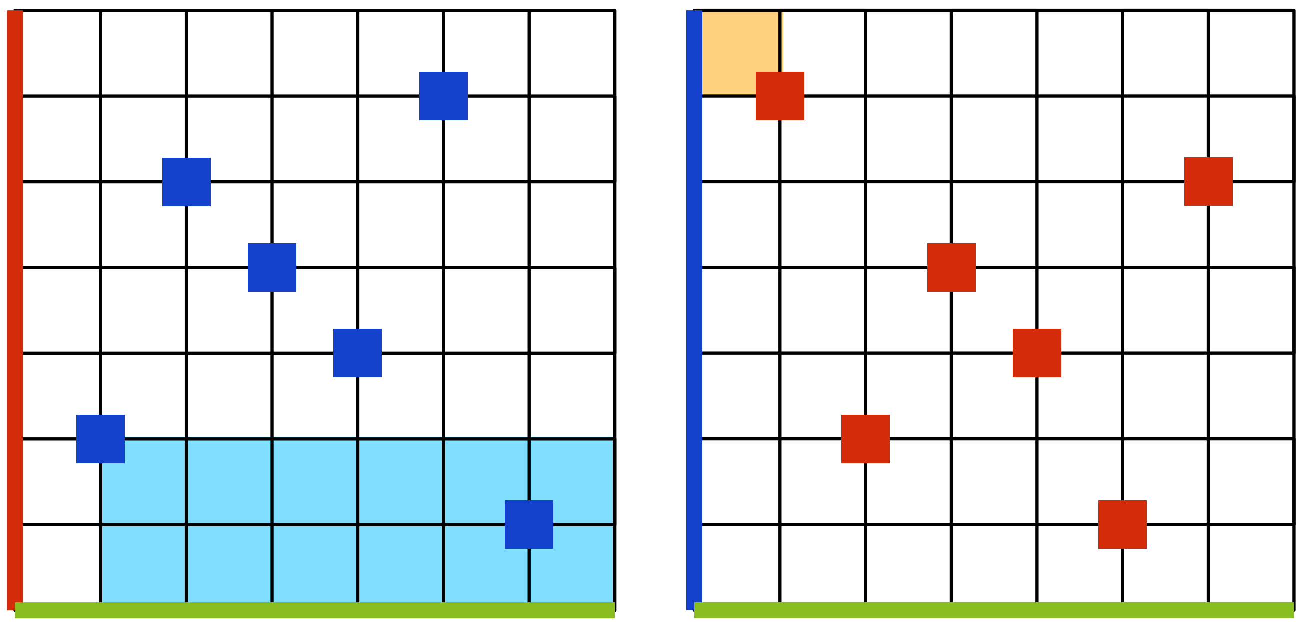

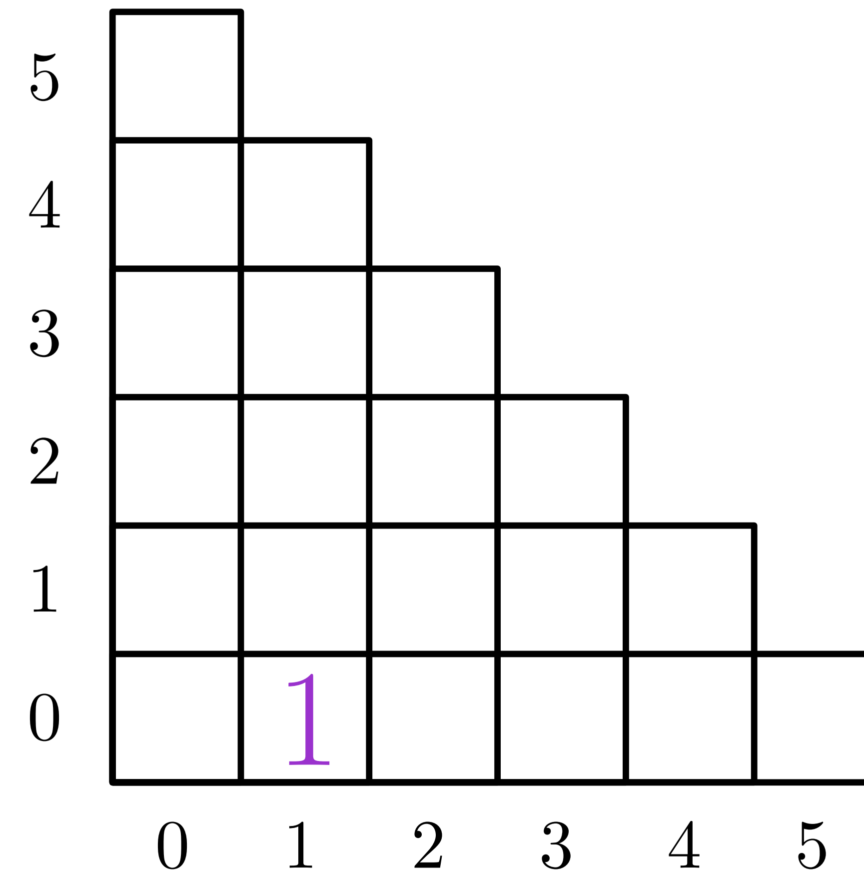

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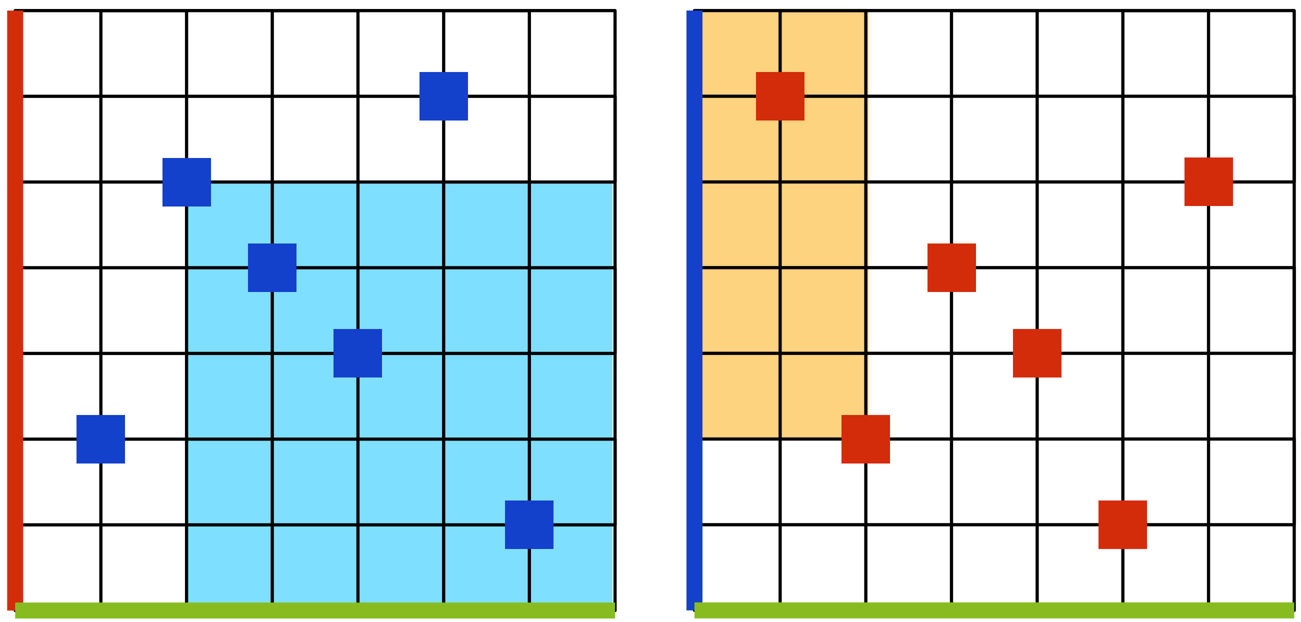

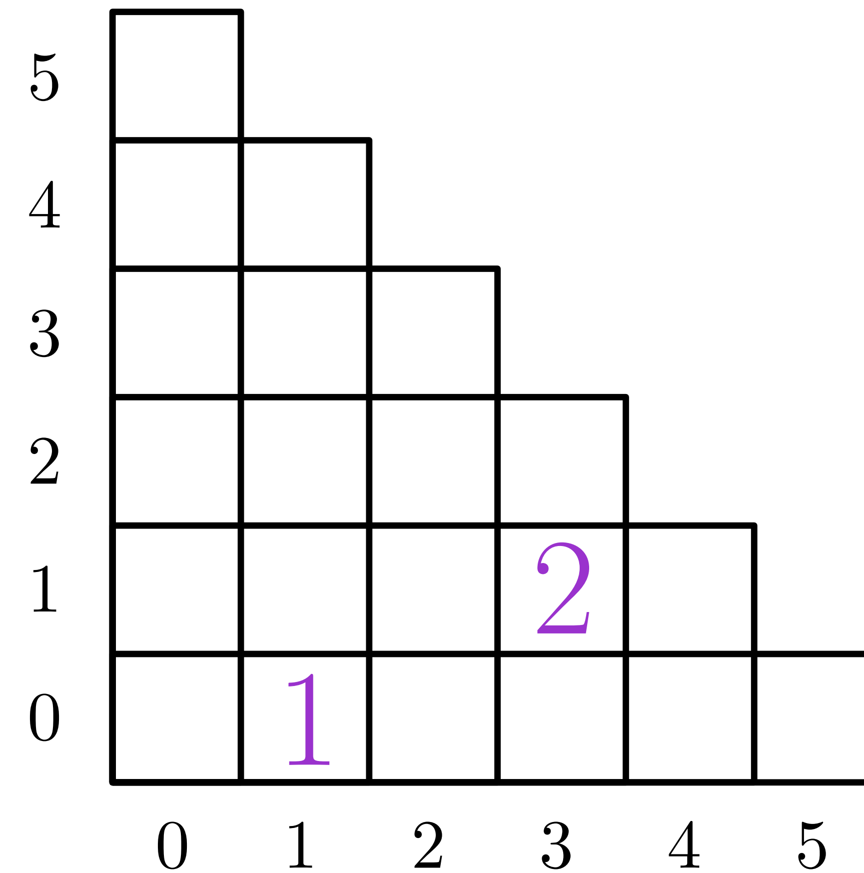

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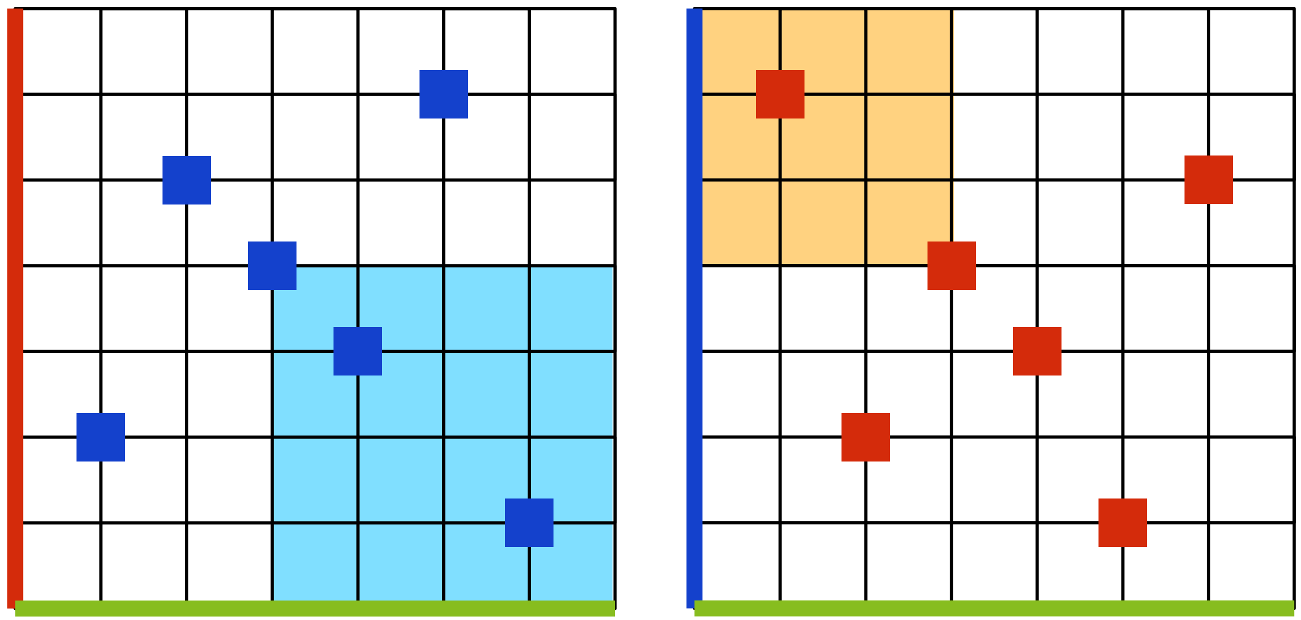

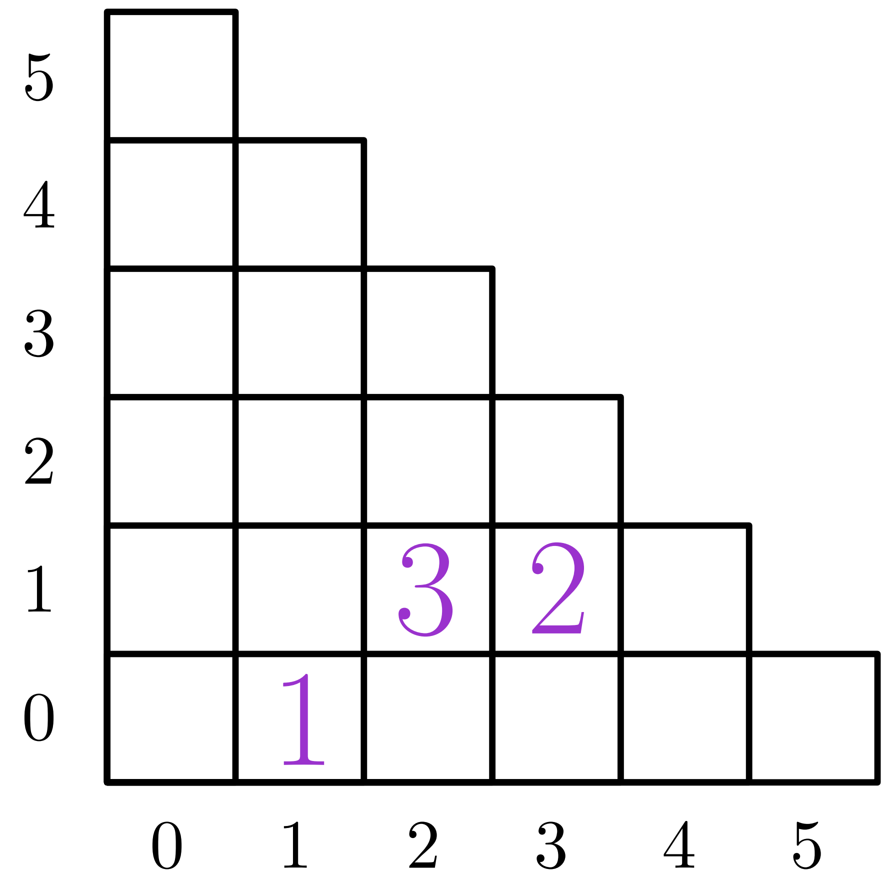

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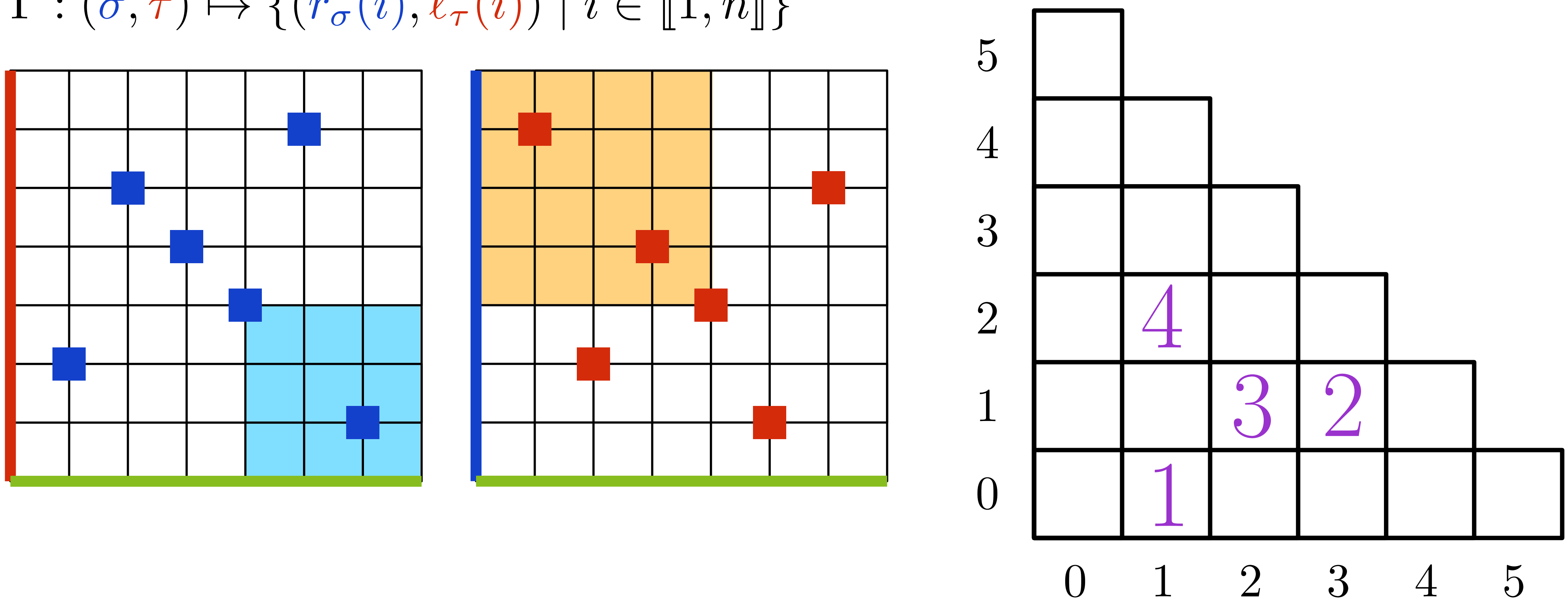

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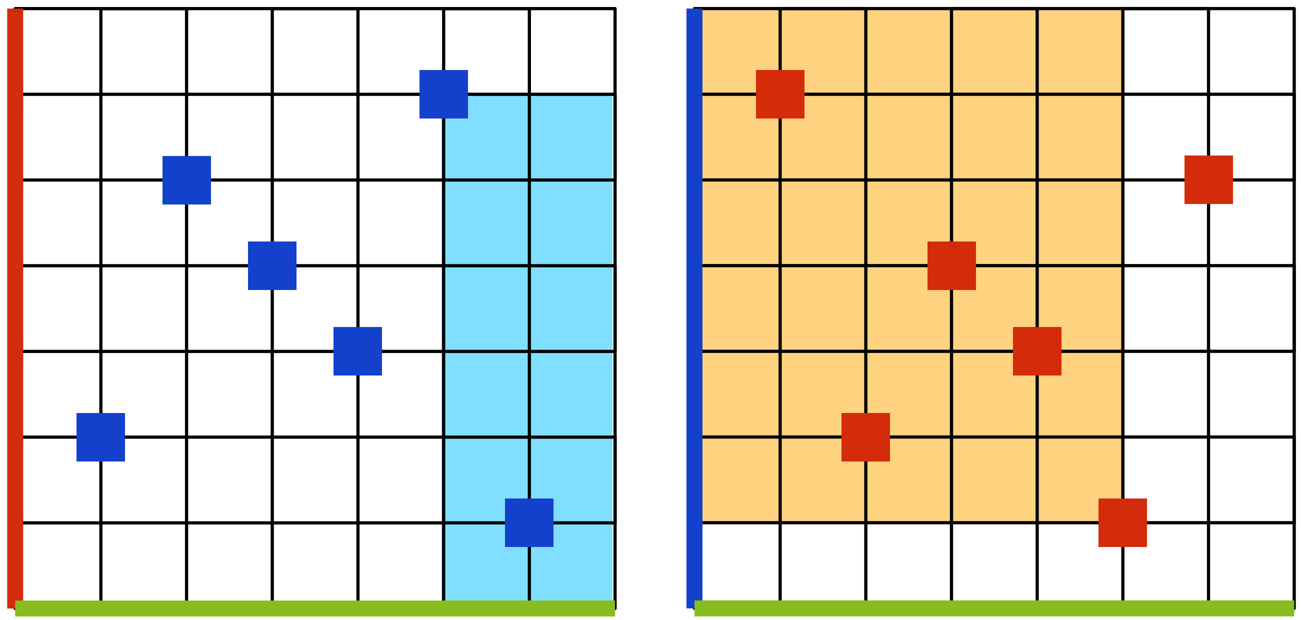

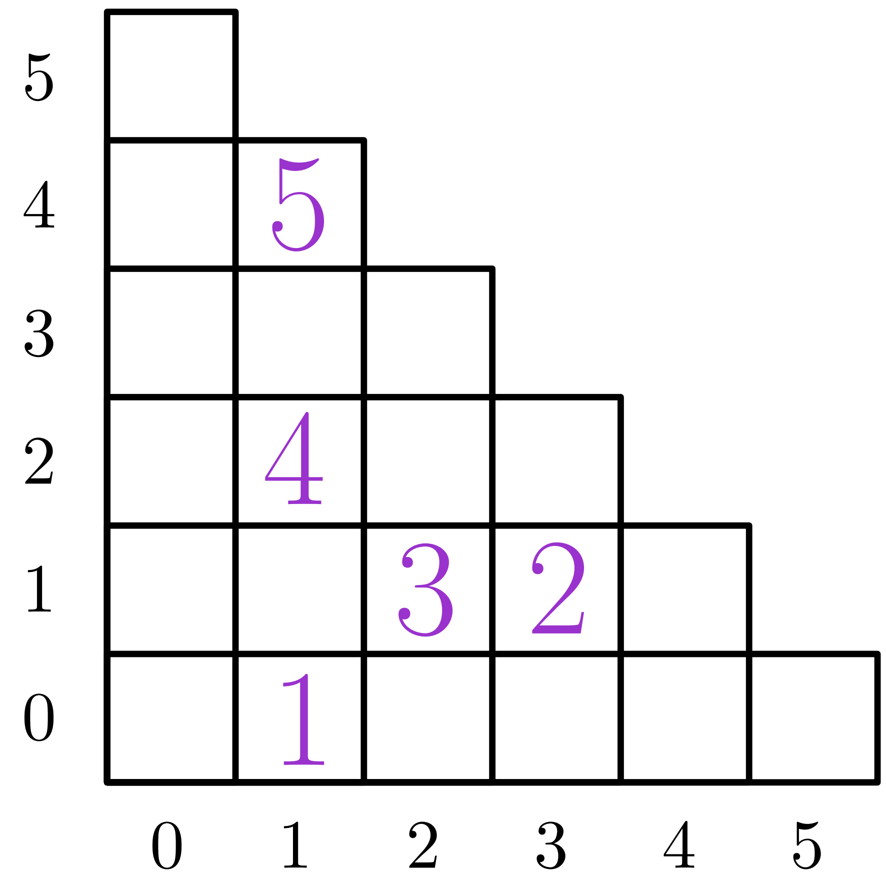

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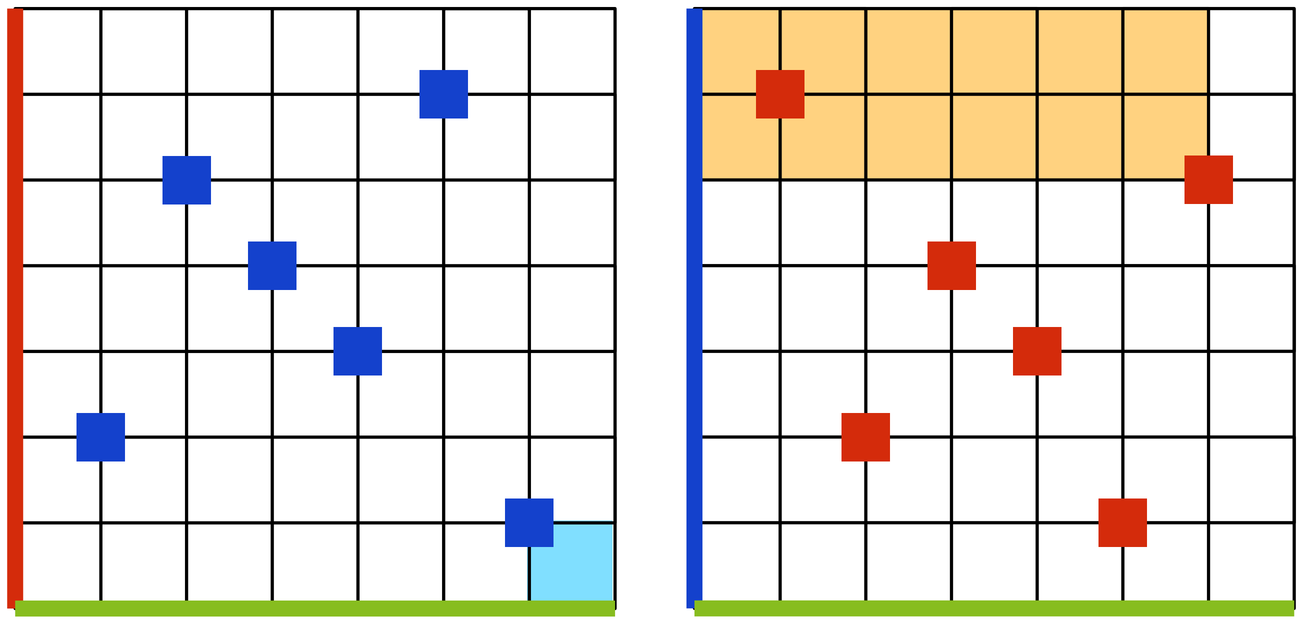
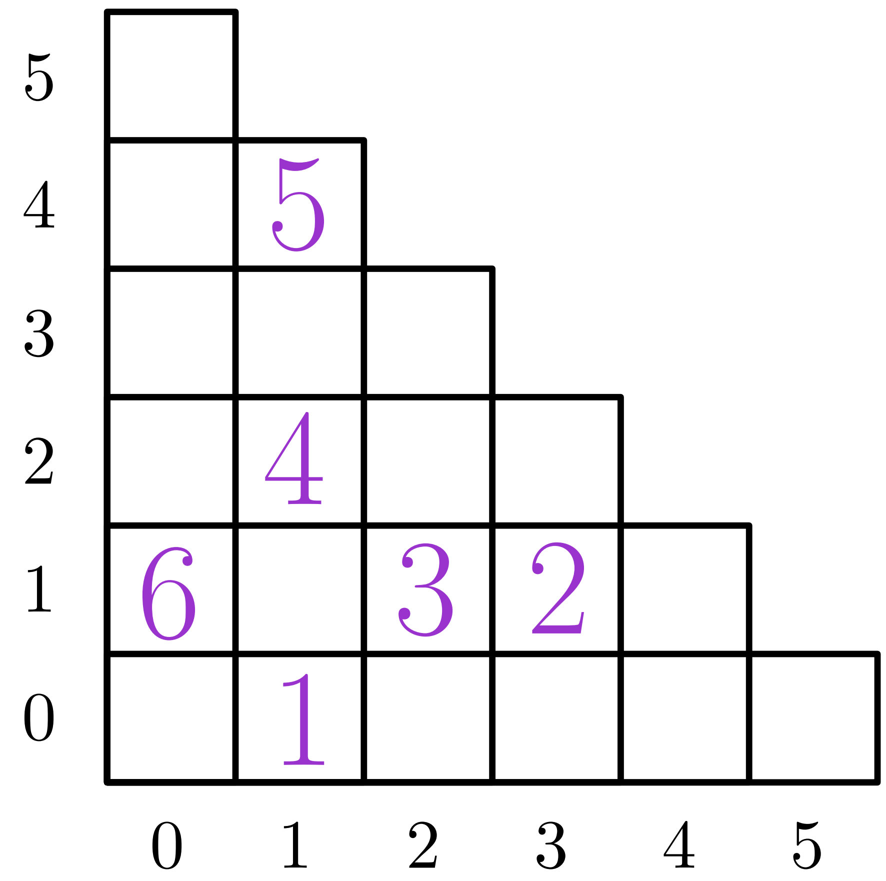
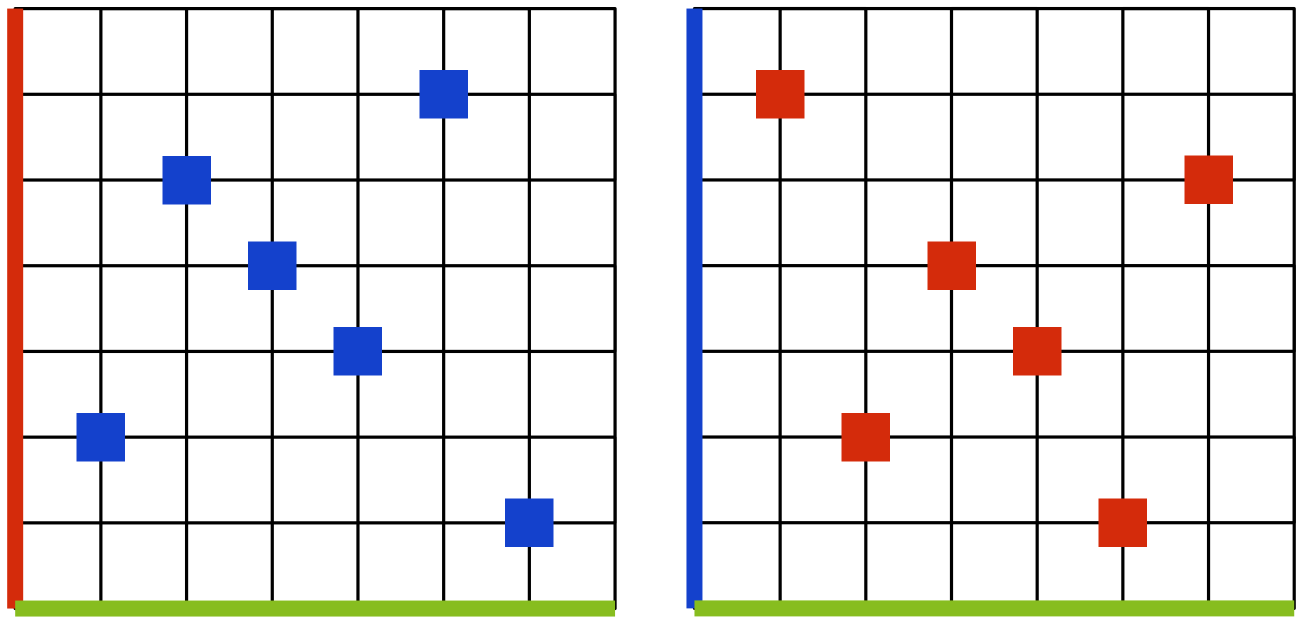
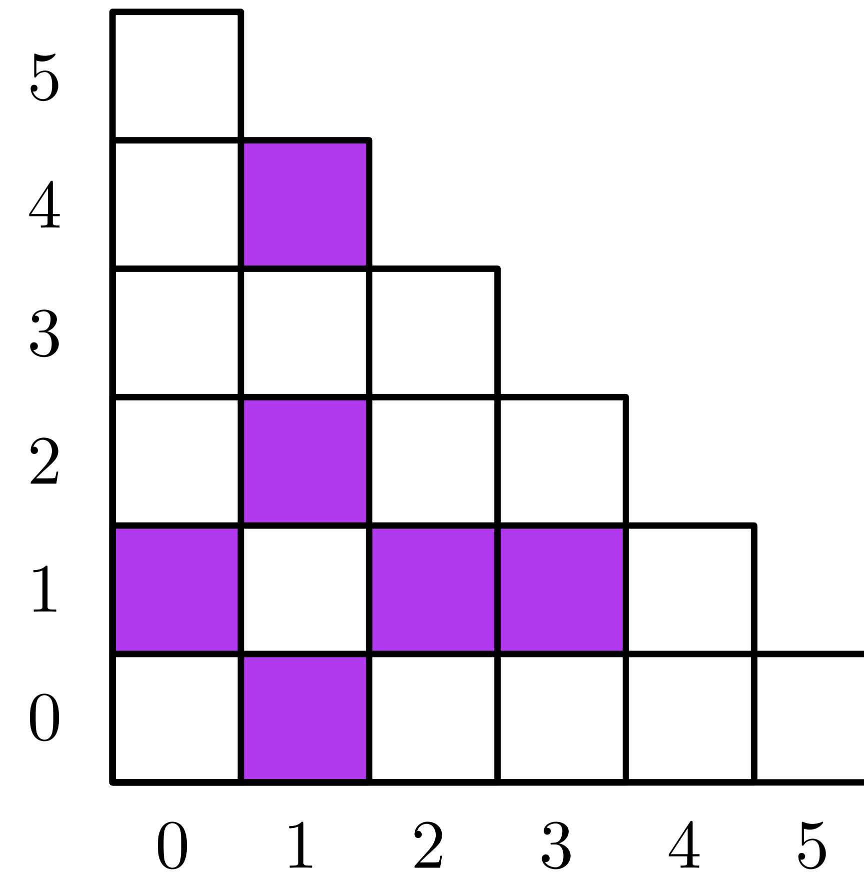




The bijection

$\Gamma : (\sigma, \tau) \mapsto \{(r_\sigma(i), \ell_\tau(i)) \mid i \in [\![1, n]\!]\}$





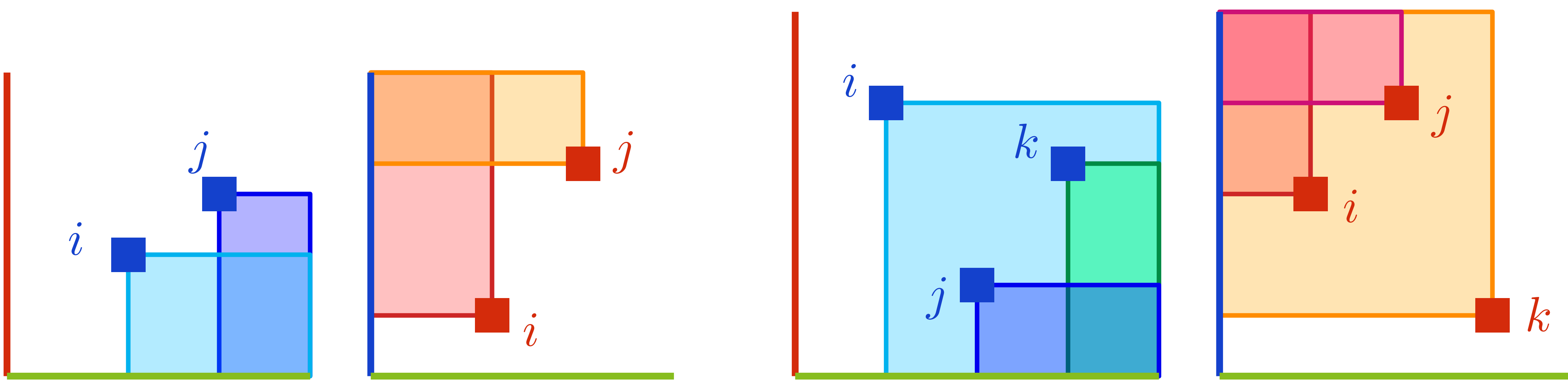
Theorem. [S. 25] For all n, Γ is a bijection between $Av_n((12, 12), (312, 231))$ and the triangle bases of size n.

Essentially, we need to prove that:

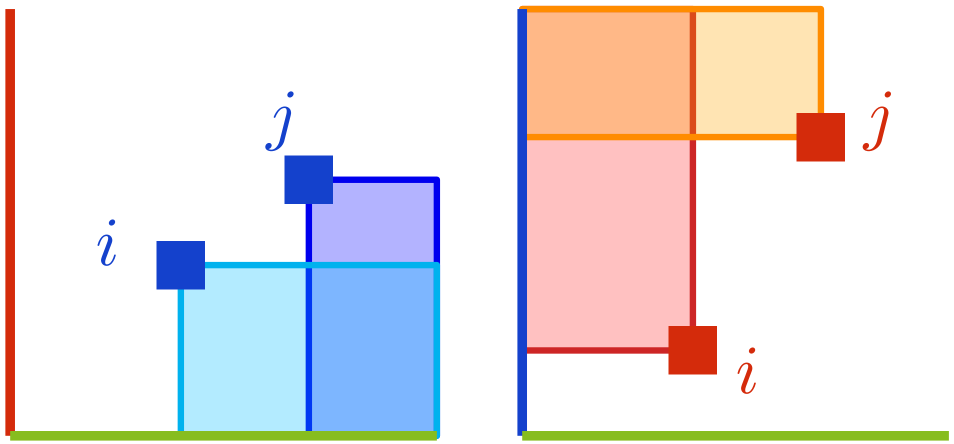
- If $(\sigma, \tau) \in Av_n((12, 12), (312, 231))$ then $\Gamma(\sigma, \tau)$ is a basis.
- Γ is bijective. (How to recover the label ?)

Why are $(\color{blue}{12}, \color{red}{12})$ and $(\color{blue}{312}, \color{red}{231})$ the right patterns to avoid?

Intuition



Avoiding $(12, 12)$: no "points too close"



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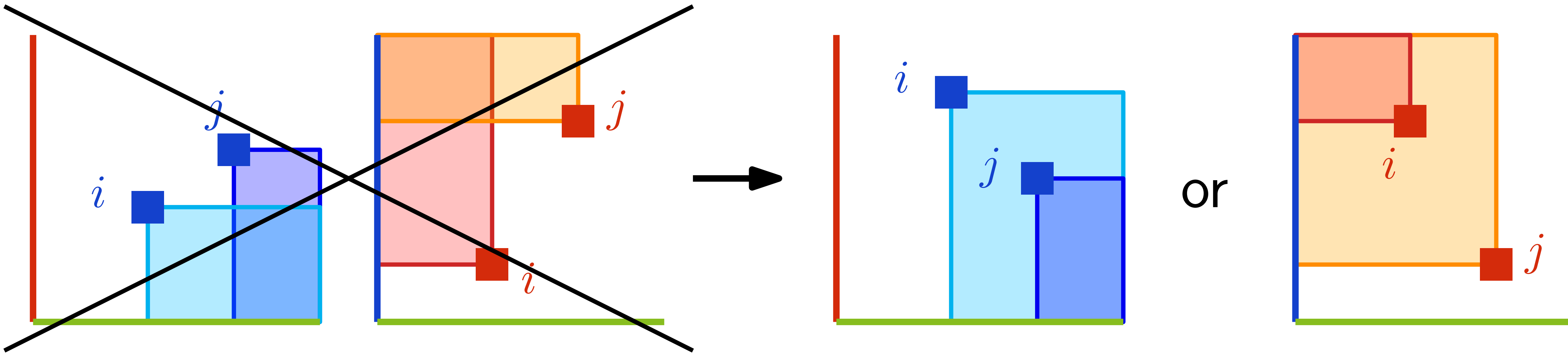

$r_\sigma(i) > r_\sigma(j)$ $\quad$ $\ell_\tau(i) < \ell_\tau(j)$

Consequence: If (σ, τ) avoids $(12, 12)$ then

- all points $(r_\sigma(i), \ell_\tau(i))$ are distinct
- the points are well spread.

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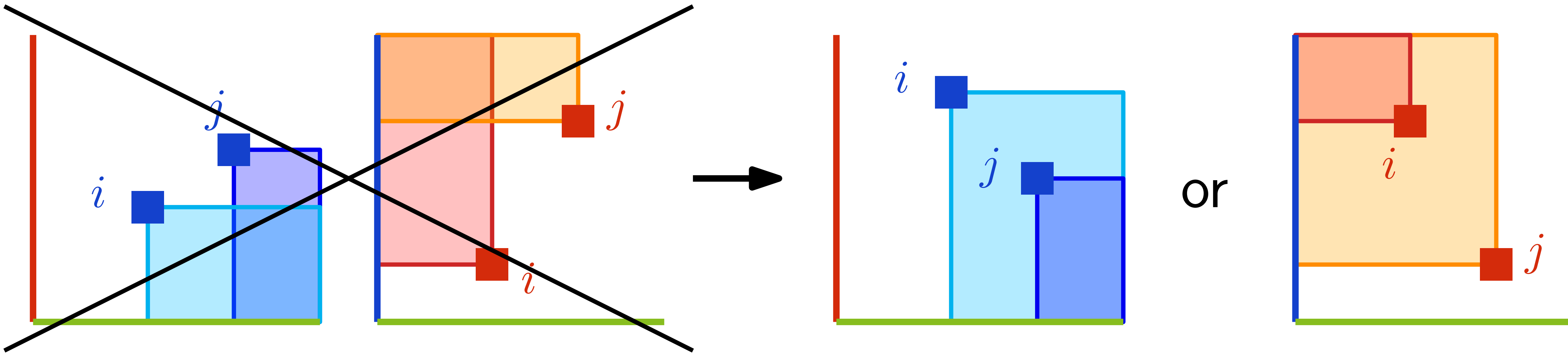

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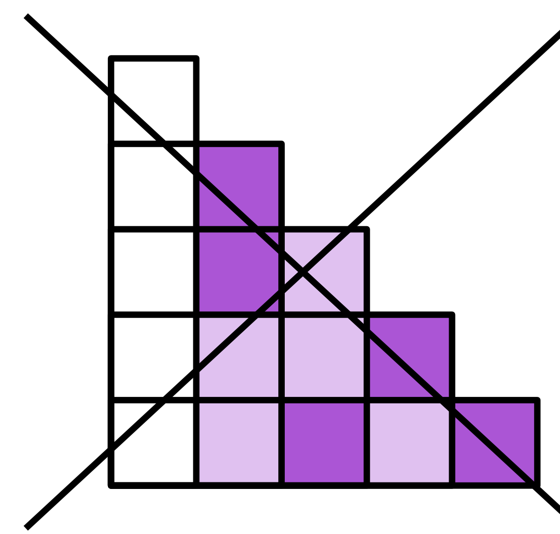
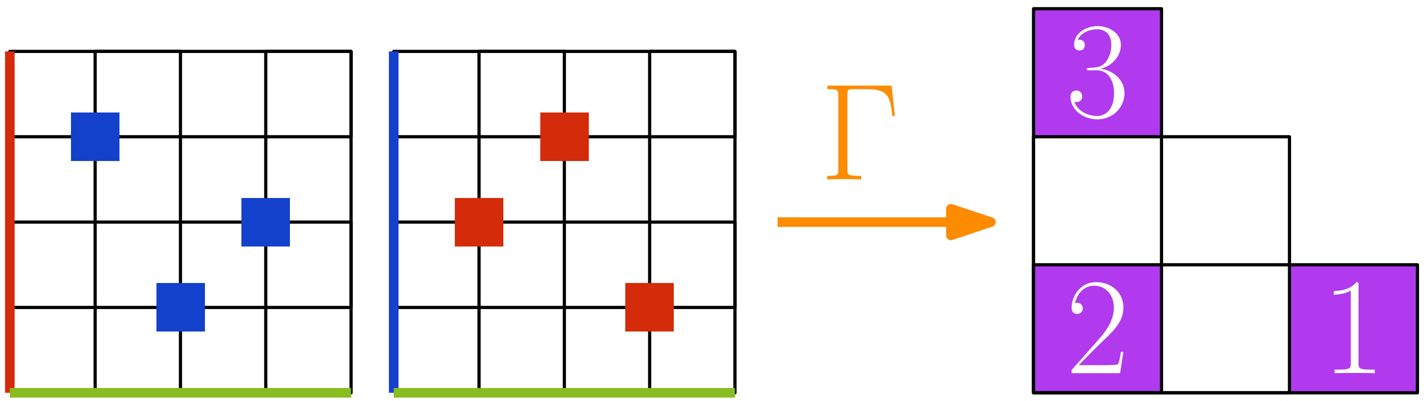


Formally, a configuration C is **sparse** if there is no triangle T of size k such that $|C \cap T| > k$.

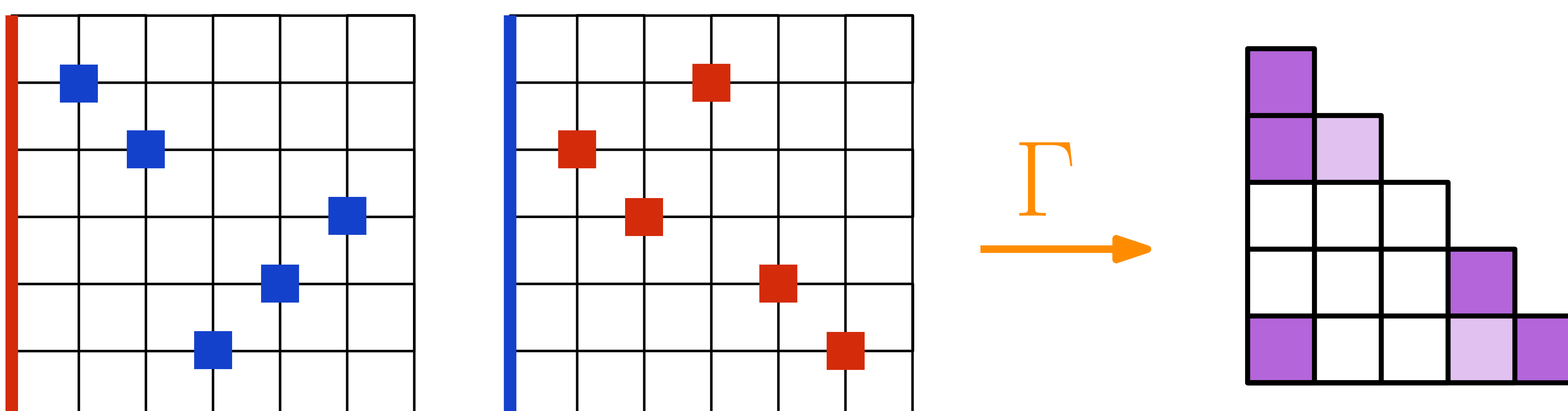
Proposition. If (σ, τ) avoids $(12, 12)$ then $\Gamma(\sigma, \tau)$ is sparse.

Avoiding $(312, 231)$: no "points too far"

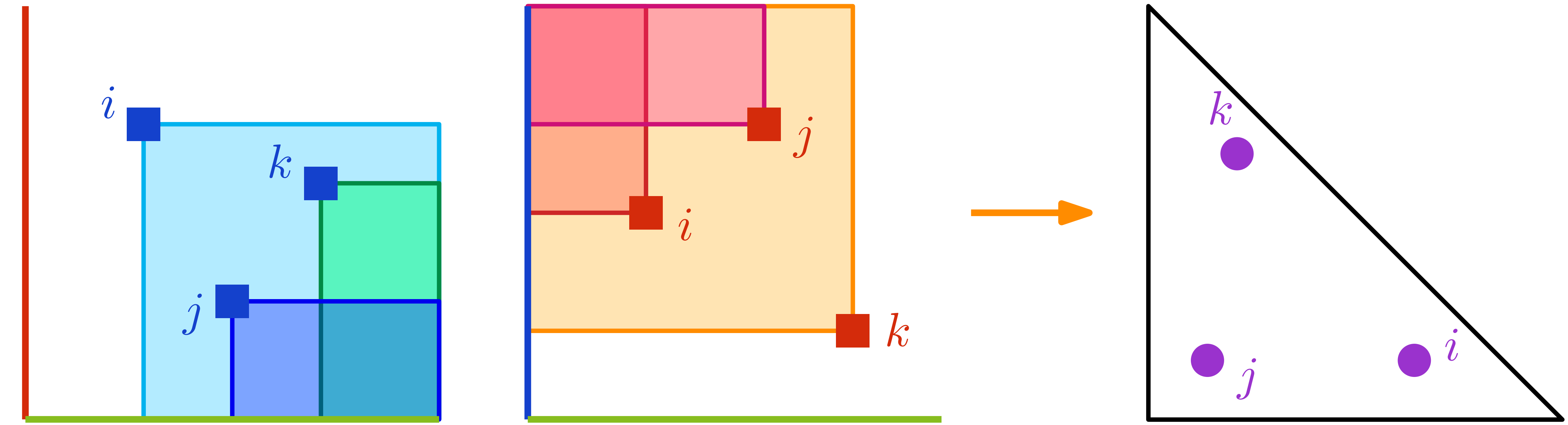


the only sparse configuration of size 3 that does not fill.

Intuition: Avoiding $(312, 231)$ prevents "filling gaps".

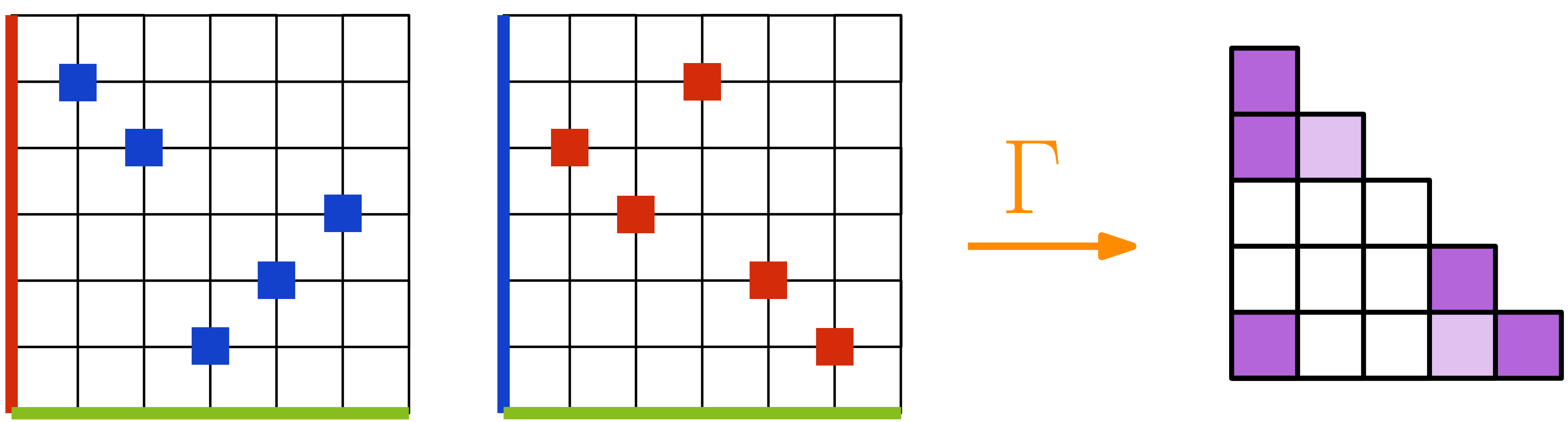


Avoiding $(312, 231)$: no "points too far"



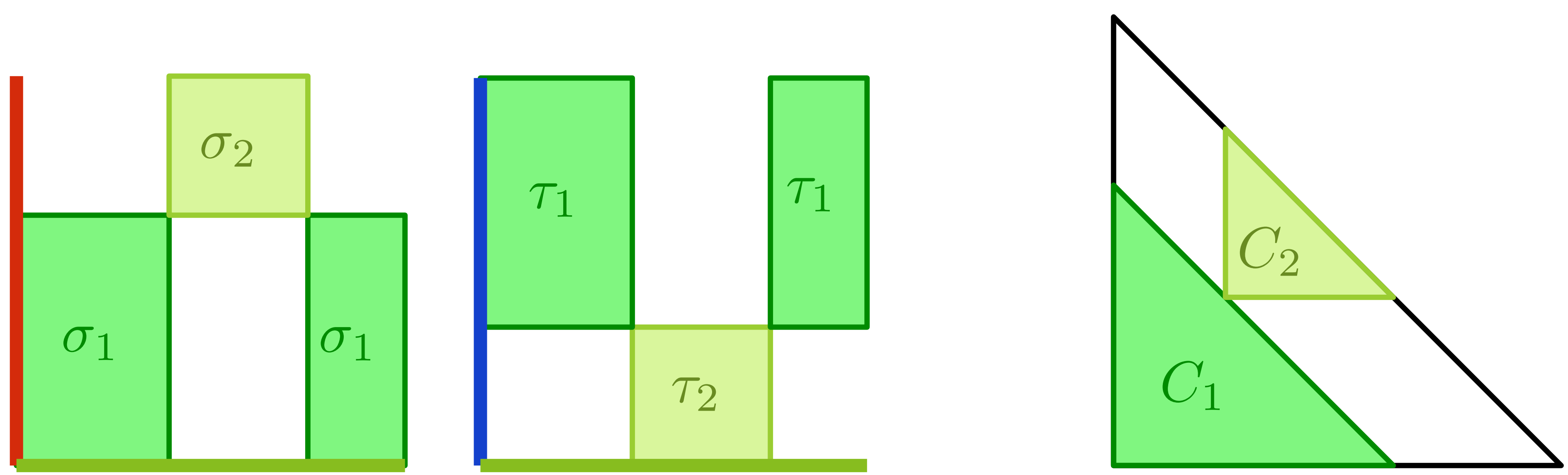
$r_\sigma(i) \gg r_\sigma(j) \approx r_\sigma(k)$ $\ell_\tau(i) \approx \ell_\tau(j) \ll \ell_\tau(k)$

Points too far to fill



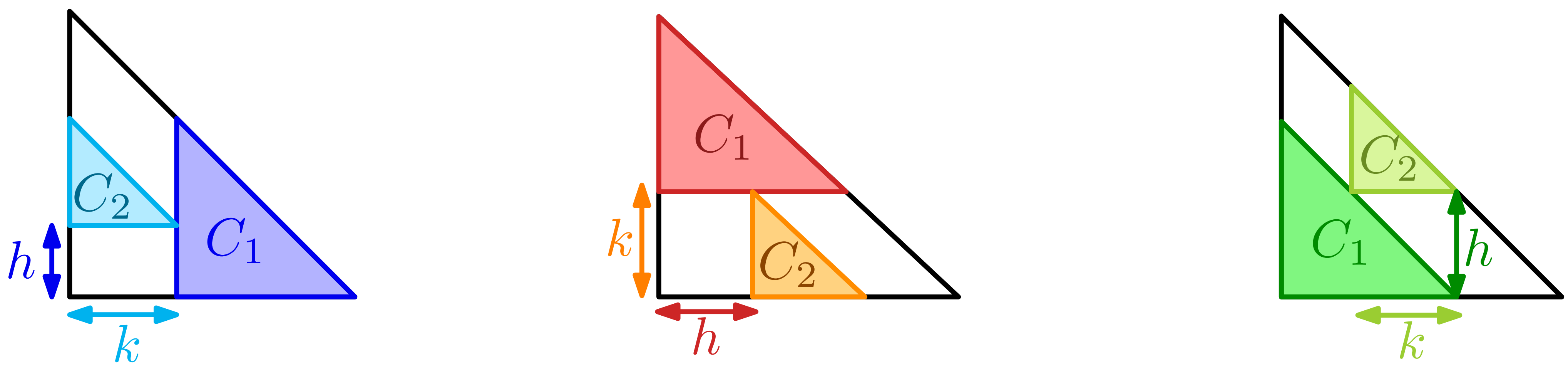
The key tool of the proof:

A recursive decomposition



Key of the proof: a recursive decomposition

Lemma. [Salo, S. 22] Any basis of size $n \geqslant 2$ can be cut into two smaller bases in one of the 3 following ways.



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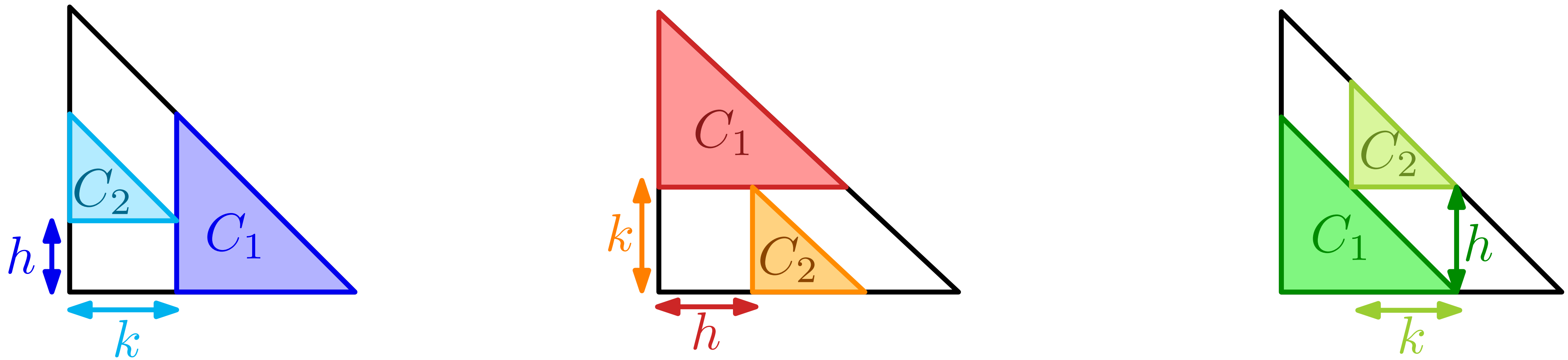
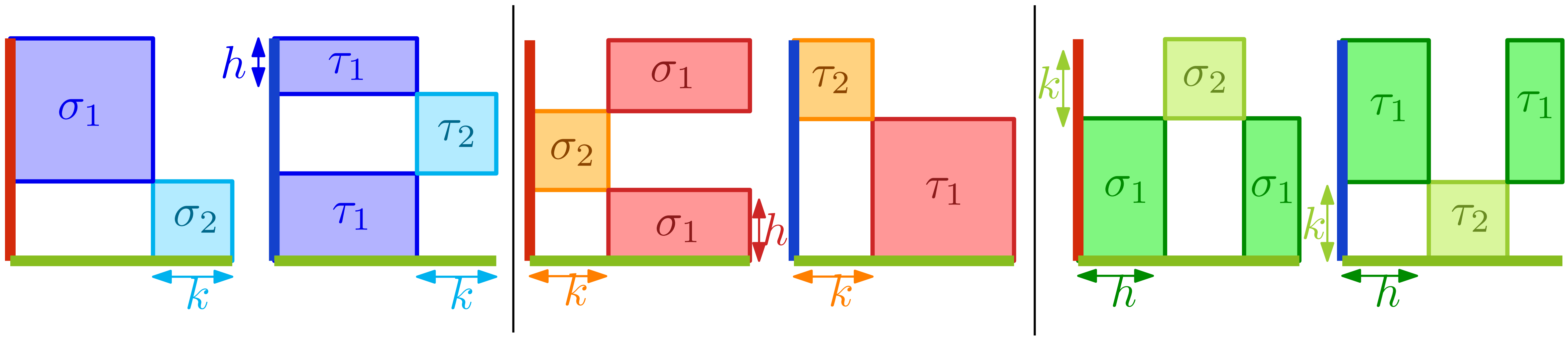


Lemma. Γ transports the cuts: $\Gamma(\sigma, \tau)$ admits a given cut if and only if (σ, τ) does.



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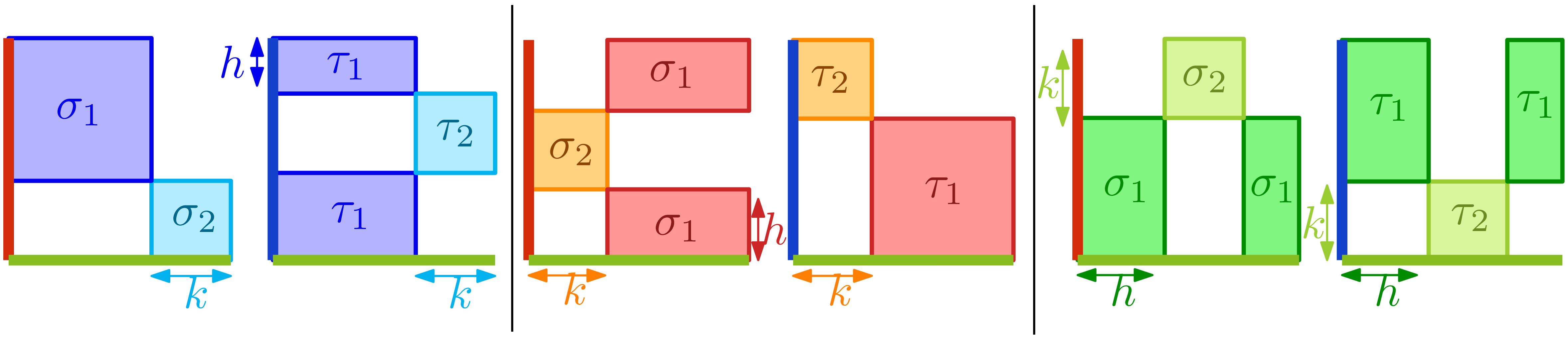

Lemma. All 3-permutations of $Av_n((12, 12), (312, 231))$ admit a cut.

Key of the proof: a recursive decomposition

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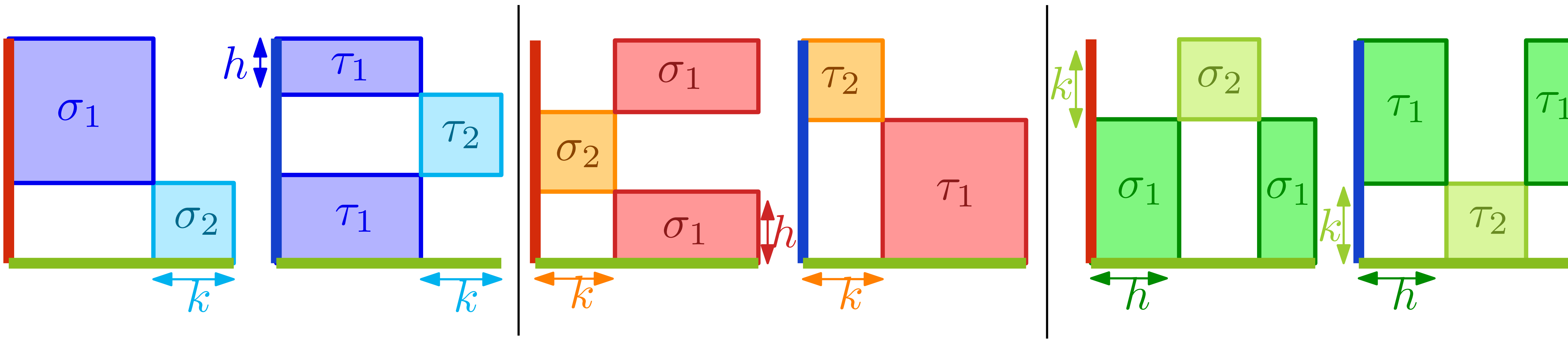


Lemma. All 3-permutations of $Av_n((12, 12), (312, 231))$ admit a cut.

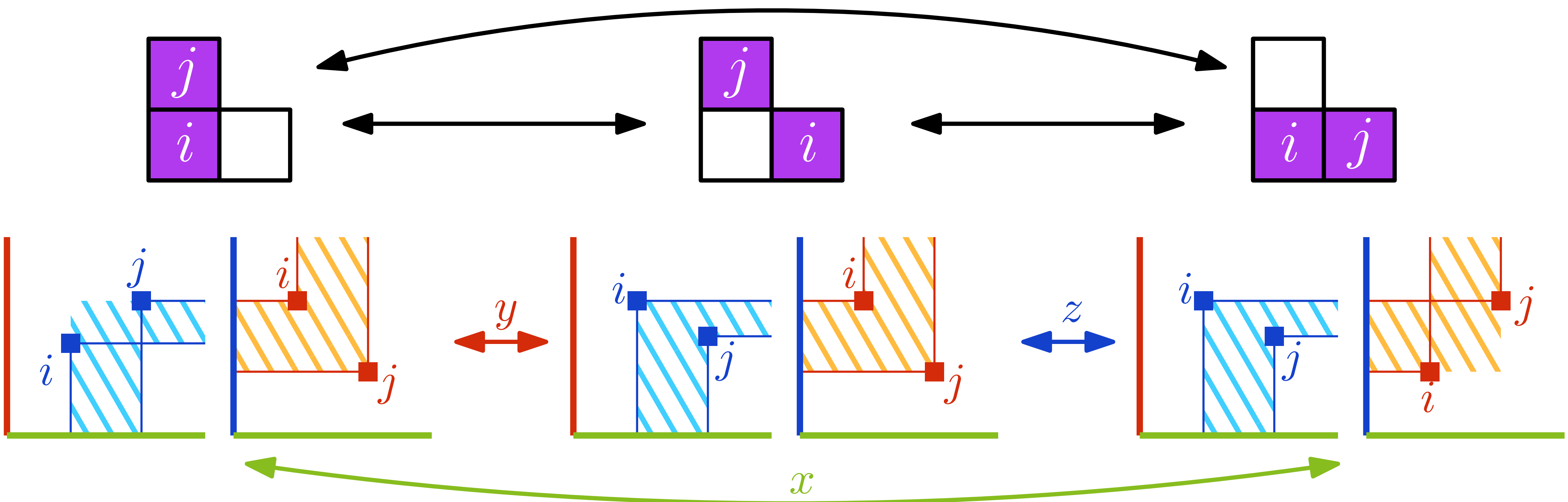
- We can now prove everything by induction!
  - $\Gamma(Av_n((12, 12), (312, 231))) \subset \mathcal{B}_n$
  - Γ is surjective
  - Γ is injective.

Corollary. Γ is a bijection between $Av_n((12, 12), (312, 231))$ and $\mathcal{B}_n$.

Nice properties and consequences

- Simple construction that transports symmetries.
- Links two objects that are understood very differently $\implies$ tools transfer.

  ▶ On bases: a canonical labelling on bases, maybe a characterisation by forbidden patterns.
  ▶ On permutations: a dynamical system on 3-permutations (and others!) which could allow sampling.



What's next?

- No enumerative result.
  - Best known bounds : $3n! \leqslant |\mathcal{B}_n| \leqslant c\left(\frac{e}{2}\right)^n n^{n-\frac{5}{2}}$ with $c > 0$.

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Thank you!