A multivariable generalization of Stirling-Eulerian polynomials

Chao Xu
Université Claude Bernard Lyon 1

March 11, 2025

Overview

Eulerian polynomials

For a permutation $\sigma \in \mathfrak{S}_n$, an index $1 \le i \le n-1$ is called

- a *descent* (des) of σ if $\sigma(i) > \sigma(i+1)$;
- an *ascent* (asc) of σ if $\sigma(i) < \sigma(i+1)$;
- an *excedance* (exc) of σ if $i < \sigma(i)$.

Example

For $\sigma = 54132$, $\mathrm{des}(\sigma) = \{1,2,4\}$, $\mathrm{asc}(\sigma) = \{3\}$, $\mathrm{exc}(\sigma) = \{1,2\}$.

It is well-known that the Eulerian polynomials have the following combinatorial interpretations:

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{asc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)}.$$

(α, β)-Eulerian polynomials

- A left-to-right maximum of σ is an element σ_i such that $\sigma_i > \sigma_j$ for every $j < i$.
- A right-to-left maximum of σ is an element σ_i such that $\sigma_i > \sigma_j$ for every $j > i$.

Let $\mathrm{LRmax}(\sigma)$ (resp. $\mathrm{RLmax}(\sigma)$) be the number of left-to-right (resp. right-to-left) maxima of σ.

Example

Given a permutation $\sigma = 27183654$, we have

$$\mathrm{LRmax}(\sigma) = |\{2, 7, 8\}| = 3 \quad \mathrm{RLmax}(\sigma) = |\{4, 5, 6, 8\}| = 4.$$

In the middle of 1970's Carlitz-Scoville considered the following multivariate Eulerian polynomials,

$$A_n(x, y \mid \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} \alpha^{\mathrm{LRmax}(\sigma)-1} \beta^{\mathrm{RLmax}(\sigma)-1}.$$

Two special cases of Carlitz-Scoville's formula

Theorem (Carlitz and Scoville)

$$\sum_{n\geq 0} A_n(x, y \mid \alpha, \beta)\frac{z^n}{n!} = (1 + xF(x, y; z))^{\alpha}(1 + yF(x, y; z))^{\beta},$$

where $F(x, y; z)$ is given by

$$F(x, y; z) = \frac{e^{xz} - e^{yz}}{xe^{yz} - ye^{ez}}.$$

Two special cases of Carlitz-Scoville's formula

Theorem (Carlitz and Scoville)

$$\sum_{n\geq 0} A_n(x,y\,|\,\alpha,\beta)\frac{z^n}{n!} = (1+xF(x,y;z))^{\alpha}(1+yF(x,y;z))^{\beta},$$

where $F(x,y;z)$ is given by

$$F(x,y;z) = \frac{e^{xz}-e^{yz}}{xe^{yz}-ye^{ez}}.$$

1. When $\alpha=0,\beta=1$ and $y=1$, the polynomials $A_n(x,1\,|\,0,\beta)$ reduces to the classical Eulerian polynomials $A_n(x)$.
2. When $x=y=1$ and $\beta=0$, it is not difficult to find that

$$A_n(1,1\,|\,\alpha,0) = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{\mathrm{LRmax}(\sigma)}.$$

We have

$$\sum_{n\geq 0} A_n(1,1\,|\,\alpha,0)\frac{z^n}{n!} = (1+F(1,1;z))^{\alpha} = \left(\frac{1}{1-t}\right)^{\alpha}.$$

Refinement of (α, β)-Eulerian polynomials

For $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with $\sigma_{\mathbf{0}} = \sigma_{\mathbf{n+1}} = \mathbf{0}$, the index $i \in [n]$ is called a

- *valley* (val) of σ if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$;
- *peak* (pk) of σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- *left double ascent* (da) of σ if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$;
- *right double descent* (dd) of σ if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Obviously we have $\mathrm{val}(\sigma) = \mathrm{pk}(\sigma) - 1$.

Refinement of (α, β)-Eulerian polynomials

For $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with $\sigma_{\mathbf{0}} = \sigma_{\mathbf{n+1}} = \mathbf{0}$, the index $i \in [n]$ is called a

- *valley* (val) of σ if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$;
- *peak* (pk) of σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- *left double ascent* (da) of σ if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$;
- *right double descent* (dd) of σ if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Obviously we have $\mathrm{val}(\sigma) = \mathrm{pk}(\sigma) - 1$. Recently, Ji considered the following polynomials. Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$.

$$A_n(\mathbf{u} \,|\, \alpha, \beta) := \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{\mathrm{val}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} \alpha^{\mathrm{LRmax}(\sigma)-1} \beta^{\mathrm{RLmax}(\sigma)-1}.$$

She computed (context-free grammar) [Ji, 2023]

$$\sum_{n \geq 0} A_n(\mathbf{u} \,|\, \alpha, \beta) \frac{z^n}{n!} = (1 + xF(x, y; z))^{\frac{\alpha+\beta}{2}} (1 + yF(x, y; z))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)z},$$

where $x + y = u_3 + u_4, xy = u_1 u_2$.

Ji's follow-up work

Let $\mathcal{M}_n$ be the set of permutations $\sigma \in \mathfrak{S}_n$ such that the first descent (if any) of π appears at the letter n.

Example

$$\mathcal{M}_2 = \{12, 21\} \quad \text{and} \quad \mathcal{M}_3 = \{123, 132, 231, 312, 321\}.$$

The binomial-Eulerian polynomials, which are the h-polynomials of stellohedrons, are defined by

$$\widetilde{A}_n(x) := \sum_{\sigma \in \mathcal{M}_{n+1}} x^{\mathrm{des}(\sigma)} = 1 + x \sum_{m=1}^{n} \binom{n}{m} A_m(x). \tag{1}$$

1. Ji and Lin considered binomial-Stirling-Eulerian polynomials, using context-free grammar and group actions and proved the γ-positivity of two α-variants of Eulerian polynomials and also an α-analogue of Chung-Graham-Knuth's symmetric Euelrian indentity, see [Ji and Lin, 2024].
2. Dong-Lin-Pan gave a q-analogue of Ji's Stirling-Eulerian polynomials, see [Dong et al., 2024].

Cycle (α, t)-Eulerian polynomials

Define

$$A_n^{\mathrm{cyc}}(x, y, t \mid \alpha) := \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{drop}(\sigma)} t^{\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)},$$

where $\mathrm{cyc}(\sigma)$ denotes the number of cycles of σ and an index $i \in [n]$ is a

- *drop* (drop) of σ if $i > \sigma(i)$;
- *fixed point* (fix) of σ if $i = \sigma(i)$.

We notice that the Carlitz-Scoville's (α, β)-Eulerian polynomials and Ji's refinements are connected by

- $A_n(x, y \mid \alpha, \beta) = A_n^{\mathrm{cyc}}\left(x, y, \frac{\alpha x + \beta y}{\alpha + \beta} \mid \alpha + \beta\right)$;
- $A_n(\mathbf{u} \mid \alpha, \beta) = A_n^{\mathrm{cyc}}\left(x, y, \frac{\alpha u_3 + \beta u_4}{\alpha + \beta} \mid \alpha + \beta\right)$, where $xy = u_1 u_2$ and $x + y = u_3 + u_4$.

Cyclic permutation statistics

For a permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we say that an index $i \in [n]$ is a

- *cycle peak* (cpk) of σ if $\sigma^{-1}(i) < i > \sigma(i)$;
- *cycle valley* (cval) of σ if $\sigma^{-1}(i) > i < \sigma(i)$;
- *cycle double ascent* (cda) of σ if $\sigma^{-1}(i) < i < \sigma(i)$;
- *cycle double descent* (cdd) of σ if $\sigma^{-1}(i) > i > \sigma(i)$.

Note that $\mathrm{cpk}(\sigma) = \mathrm{cval}(\sigma)$.

Theorem 1

If $xy = u_1 u_2$ and $x + y = u_3 + u_4$, then

$$A_n^{\mathrm{cyc}}(x, y, t \mid \alpha) = \sum_{\sigma \in \mathfrak{S}_n} (u_1 u_2)^{\mathrm{cpk}(\sigma)} u_3^{\mathrm{cda}(\sigma)} u_4^{\mathrm{cdd}(\sigma)} t^{\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)}.$$

x-factorization

Fix a letter $x \in [n]$ and a permutation $\sigma \in \mathfrak{S}_n$, the *x-factorization* of σ is defined as the concatenation $\sigma = w_1 w_2\, x\, w_4 w_5$, where w_2 (resp. w_4) is the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all smaller than x.

Example (7-factorization of σ)

if $\sigma = 934278516$, then $w_1 = 9$, $w_2 = 342$, $w_4 = \emptyset$ and $w_5 = 8516$.

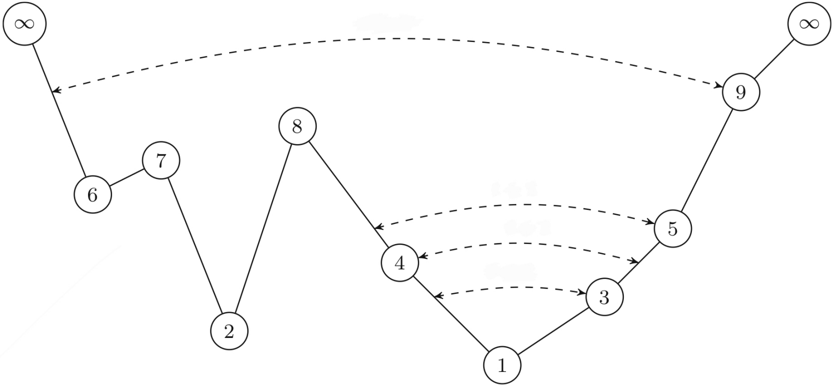
Define the involution $\varphi_x(\sigma) : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\varphi_x(\sigma) := \begin{cases} w_1 w_4\, x\, w_2 w_5, & \text{if } x \text{ is a double ascent or a double descent of } \sigma; \\ \sigma, & \text{if } x \text{ is a valley or a peak of } \sigma. \end{cases}$$

Here, we use conventions $\sigma(0) = \sigma(n+1) = \infty$. It is easy to see that the involutions φ_x and φ_y commute with each other. Given a subset $S \subseteq [n]$, we define $\varphi_S := \prod_{x \in S} \varphi_x$. The involutions $\{\varphi_S\}_{S \subseteq [n]}$ define a $\mathbb{Z}_2^n$-action on $\mathfrak{S}_n$, which is a modified group action of Foata and Strehl and called the *MFS*-action by Bränden. A geometric version of this action was given by Shapiro et al.

Example: valley hopping

The mountain range view of the permutation $\sigma = 672841359$.



Cyclic valley hopping

Simply define $\psi_x(\sigma) : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\psi_x(\sigma) := \begin{cases} \theta^{-1} \circ \varphi_x \circ \theta(\sigma), & \text{if } x \text{ is not a fixed point of } \sigma; \\ \sigma, & \text{if } x \text{ is a fixed point of } \sigma, \end{cases}$$

where we treat the 0-th letter of $\theta(\sigma)$ as 0 and the (n+1)-th letter as ∞. Given a subset $S \subseteq [n]$, define $\psi_S := \prod_{x \in S} \psi_x$. We will call the $\mathbb{Z}_2^n$-action defined by the involutions $\{\psi_S\}_{S \subseteq [n]}$ *cyclic valley-hopping*. Satisfying

(1) $\mathrm{Cval}(\psi_S(\sigma)) = \mathrm{Cval}(\sigma)$;
(2) $\mathrm{Cpk}(\psi_S(\sigma)) = \mathrm{Cpk}(\sigma)$;
(3) $\mathrm{Cda}(\psi_S(\sigma)) = (\mathrm{Cda}(\sigma) \setminus S) \cup (S \cap \mathrm{Cdd}(\sigma))$;
(4) $\mathrm{Cdd}(\psi_S(\sigma)) = (\mathrm{Cdd}(\sigma) \setminus S) \cup (S \cap \mathrm{Cda}(\sigma))$;
(5) $\mathrm{Fix}(\psi_S(\sigma)) = \mathrm{Fix}(\sigma)$;
(6) $\mathrm{cyc}(\psi_S(\sigma)) = \mathrm{cyc}(\sigma)$.

Example

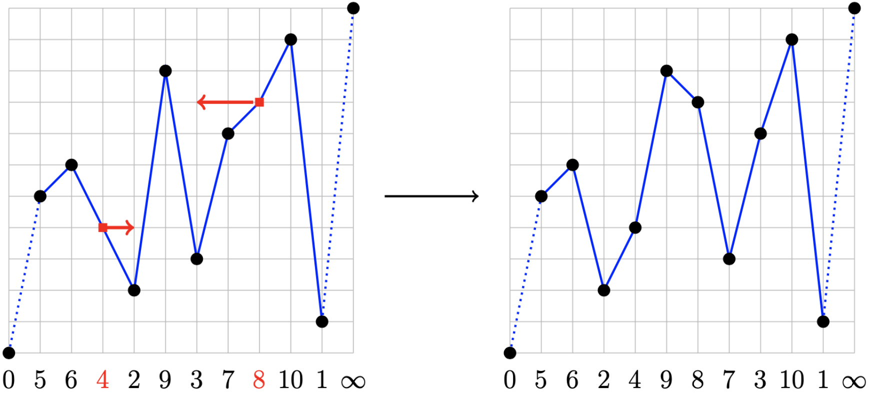


FIGURE 1. Cyclic valley-hopping on $\pi = (5)(6\,4\,2)(9\,3\,7\,8)(10\,1)$ with $S = \{4, 8\}$ yields $\psi_S(\pi) = (5)(6\,2\,4)(9\,8\,7\,3)(10\,1)$.

Proof of Theorem 1

$\mathrm{Orb}(\sigma) := \{\psi_S(\sigma) | S \subseteq [n]\}$: the orbit of σ under cyclic valley-hopping. The cyclic MFS-action divides the set $\mathfrak{S}_n$ into disjoint orbits. There is a unique permutation $(\bar{\sigma})$ in each orbit which has no cyclic double descents.

$$\sum_{\pi \in \mathrm{Orb}(\sigma)} (u_1 u_2)^{\mathrm{cpk}(\pi)} u_3^{\mathrm{cda}(\pi)} u_4^{\mathrm{cdd}(\pi)} t^{\mathrm{fix}(\pi)} \alpha^{\mathrm{cyc}(\pi)} = (u_1 u_2)^{\mathrm{cpk}(\bar{\sigma})} (u_3 + u_4)^{\mathrm{cda}(\bar{\sigma})} t^{\mathrm{fix}(\bar{\sigma})} \alpha^{\mathrm{cyc}(\bar{\sigma})}.$$

By definition, it is clear that for $\sigma \in \mathfrak{S}_n$,

$$\mathrm{exc}(\sigma) = \mathrm{cda}(\sigma) + \mathrm{cpk}(\sigma) \quad \text{and} \quad \mathrm{drop}(\sigma) = \mathrm{cdd}(\sigma) + \mathrm{cval}(\sigma).$$

Setting $u_1 = u_3 = x$ and $u_2 = u_4 = y$ in above identity, we obtain

$$\sum_{\pi \in \mathrm{Orb}(\sigma)} x^{\mathrm{exc}(\pi)} y^{\mathrm{drop}(\pi)} t^{\mathrm{fix}(\pi)} \alpha^{\mathrm{cyc}(\pi)} = (xy)^{\mathrm{cpk}(\bar{\sigma})} (x + y)^{\mathrm{cda}(\bar{\sigma})} t^{\mathrm{fix}(\bar{\sigma})} \alpha^{\mathrm{cyc}(\bar{\sigma})}.$$

Thus

$$\sum_{\pi \in \mathrm{Orb}(\sigma)} (u_1 u_2)^{\mathrm{cpk}(\pi)} u_3^{\mathrm{cda}(\pi)} u_4^{\mathrm{cdd}(\pi)} t^{\mathrm{fix}(\pi)} \alpha^{\mathrm{cyc}(\pi)} = \sum_{\pi \in \mathrm{Orb}(\sigma)} x^{\mathrm{exc}(\pi)} y^{\mathrm{drop}(\pi)} t^{\mathrm{fix}(\pi)} \alpha^{\mathrm{cyc}(\pi)}.$$

where $u_1 u_2 = xy$ and $u_3 + u_4 = x + y$. Then summing over all the orbits of $\mathfrak{S}_n$.

Generalized Eulerian polynomials

An index $i \in [n]$ is a

- *left-to-right-maximum-peak* (LRmaxp) if it is a peak and σ_i is a left-to-right maximum;
- *right-to-left-maximum-peak* (RLmaxp) if it is a peak and σ_i is a right-to-left maximum;
- *left-to-right-maximum-da* (LRmaxda) if it is a double ascent and σ_i is a left-to-right maximum;
- *right-to-left-maximum-dd* (RLmaxdd) if it is a double descent and σ_i is a right-to-left maximum.

Define the enumerative polynomial

$$A_n(\mathbf{u}, f, g, t \,|\, \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{\mathrm{V}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} f^{\mathrm{LRmaxpk}(\sigma)-1} g^{\mathrm{RLmaxpk}(\sigma)-1} \times t^{\mathrm{LRmaxda}(\sigma)+\mathrm{RLmaxdd}(\sigma)} \alpha^{\mathrm{LRmax}(\sigma)-1} \beta^{\mathrm{RLmax}(\sigma)-1}.$$

Ji's (α, β)-Eulerian polynomial is $A_n(\mathbf{u}, 1, 1, 1 \,|\, \alpha, \beta)$ and Ji-Lin's binomial Stirling-Eulerian polynomial is $A_n(\mathbf{u}, 0, 1, 1 \,|\, \alpha, \alpha)$.

Connection

Theorem 2

If $xy = u_1 u_2$ and $x + y = u_3 + u_4$, then

$$A_n^{\mathrm{cyc}}\left(x, y, \frac{\alpha u_3 + \beta u_4}{\alpha f + \beta g}\, t \,\middle|\, \alpha f + \beta g\right) = A_n(\mathbf{u}, f, g, t \,|\, \alpha, \beta).$$

We will give a bijective proof based on Foata's fundamental transformation and Theorem 1 (just proved).
Note that

$$\sum_{n\geq 0} A_n(\mathbf{u}, f, g, t \,|\, \alpha, \beta) \frac{z^n}{n!} = e^{(\alpha u_3 + \beta u_4)tz} \left(\frac{x - y}{xe^{yz} - ye^{xz}}\right)^{\alpha f + \beta g},$$

where $xy = u_1 u_2$ and $x + y = u_3 + u_4$.

Proof of Theorem 2

Let $\mathfrak{S}_n^\star$ be the set of permutations in $\mathfrak{S}_n$ of which each cycle has a color in $\{$*Red*, *Blue*$\}$. Define

$$w(\sigma;\alpha,a,b) = (u_1u_2)^{\mathrm{cpk}(\sigma)} u_3^{\mathrm{cda}(\sigma)} u_4^{\mathrm{cdd}(\sigma)} a^{\mathrm{fix}(\sigma)} b^{\mathrm{cyc}(\sigma)-\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)},$$
$$w_1(\sigma;\alpha,a,b,t) = w(\sigma;\alpha,a,b)\, t^{\mathrm{LRmaxda}(\sigma)},$$
$$w_2(\tau;\alpha,a,b,t) = w(\sigma;\alpha,a,b)\, t^{\mathrm{RLmaxdd}(\tau)}.$$

Equality can be written as

$$\begin{aligned} &A_n^{\mathrm{cyc}}\left(x,y,t(\alpha u_3+\beta u_4)(\alpha f+\beta g)^{-1} \,|\, \alpha f+\beta g\right) \\ &= \sum_{\pi\in\mathfrak{S}_n} (u_1u_2)^{\mathrm{cpk}(\pi)} u_3^{\mathrm{cda}(\pi)} u_4^{\mathrm{cdd}(\pi)} (t(\alpha u_3+\beta u_4))^{\mathrm{fix}(\pi)} (\alpha f+\beta g)^{\mathrm{cyc}(\pi)-\mathrm{fix}(\pi)} \\ &= \sum_{(\sigma,\tau)\in\mathfrak{S}_n^\star} w_1(\sigma;\alpha,u_3,f,t)\, w_2(\tau;\beta,u_4,g,t). \end{aligned}$$

Any permutation $w = w_1\ldots w_{n+1} \in \mathfrak{S}_{n+1}$ can be written uniquely as $w = u\,(n+1)\,v$. Define a mapping $\rho : \mathfrak{S}_n^\star \to \mathfrak{S}_{n+1}$ as follows:

$$(\sigma,\tau) \mapsto \widetilde{\pi} := \theta(\sigma)\, x\, \theta'(\tau) \quad \text{with } x = n+1,$$

where θ is the FFT and θ' its variant.

Example

More precisely, we arrange the cycles of σ in increasing order from left to right of their maximum, which are put at the beginning of their cycles. Erasing the parentheses gives $\theta(\sigma)$. Similarly, we arrange the cycles of τ in increasing order from right to left of their maxima, which are put at the end of their cycles. Erasing the parentheses gives $\theta'(\tau)$.

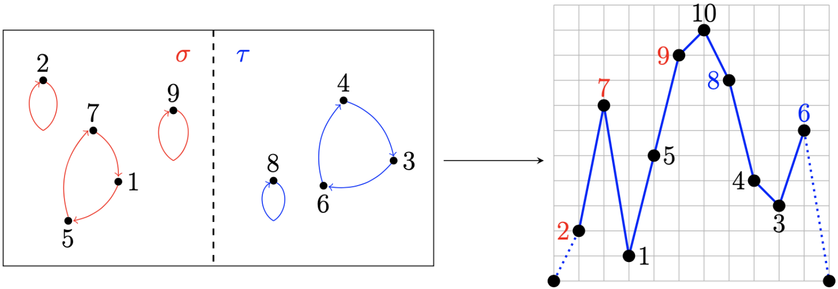


FIGURE 2. The mapping $\rho : (\sigma, \tau) \mapsto \widetilde{\pi} = 2\,7\,1\,5\,9\,\mathbf{10}\,8\,4\,3\,6$.

Application 1: Exponential generating functions

The exponential generating function of polynomials $A_n^{\mathrm{cyc}}(x, y, t \mid \alpha)$ is well-known and reads as follows

$$\sum_{n\geq 0} A_n^{\mathrm{cyc}}(x, y, t \mid \alpha)\frac{z^n}{n!} = \left(\frac{(x-y)e^{tz}}{xe^{yz} - ye^{xz}}\right)^{\alpha}. \tag{2}$$

From the connection formula,

Theorem

Let $xy = u_1 u_2$ and $x + y = u_3 + u_4$. We have

$$\sum_{n\geq 0} A_n(\mathbf{u}, f, g, t \mid \alpha, \beta)\frac{z^n}{n!} = e^{(\alpha u_3 + \beta u_4)tz}\left(\frac{x-y}{xe^{yz} - ye^{xz}}\right)^{\alpha f + \beta g}. \tag{3}$$

Application 2: Gamma positivity in two special cases

Lemma

For any variable f, we have

$$\begin{aligned} &A_n(x,y,\textcolor{red}{0},x+y,f,1,t\,|\,\alpha,\alpha) \\ &\quad= \sum_{\substack{\sigma\in\mathfrak{S}_{n+1}\\ \mathrm{da}(\sigma)=0}} (xy)^{\mathrm{asc}(\sigma)}(x+y)^{n-2\mathrm{asc}(\sigma)} f^{\mathrm{LRmaxp}(\sigma)-1} t^{\mathrm{RLmaxdd}(\sigma)} \alpha^{\mathrm{LRmax}(\sigma)+\mathrm{RLmax}(\sigma)-2} \\ &\quad= \sum_{\substack{\sigma\in\mathfrak{S}_{n}\\ \mathrm{cda}(\sigma)=0}} (xy)^{\mathrm{exc}(\sigma)}(x+y)^{n-2\mathrm{exc}(\sigma)} t^{\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)} (f+1)^{\mathrm{cyc}(\sigma)-\mathrm{fix}(\sigma)}. \end{aligned}$$

We define three types of subsets of $\mathfrak{S}_n$:

$$\mathfrak{S}^{\mathrm{cda}=0}_{n,\mathrm{exc}=j} := \{\sigma\in\mathfrak{S}_n : \mathrm{cda}(\sigma)=0 \text{ and } \mathrm{exc}(\sigma)=j\};$$
$$\mathfrak{S}^{\mathrm{da}=0}_{n,\mathrm{asc}=j} := \{\sigma\in\mathfrak{S}_n : \mathrm{da}(\sigma)=0 \text{ and } \mathrm{asc}(\sigma)=j\};$$
$$\mathcal{M}^{\mathrm{da}=0}_{n,\mathrm{asc}=j} := \{\sigma\in\mathcal{M}_n : \mathrm{da}(\sigma)=0 \text{ and } \mathrm{asc}(\sigma)=j\}.$$

The (α, t)-Eulerian polynomials $A_n(x, y, t \,|\, \alpha)$ by

$$A_n\,(x, y, t \,|\, \alpha) := \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} t^{\mathrm{LRmaxda}(\sigma)+\mathrm{RLmaxdd}(\sigma)} \alpha^{\mathrm{LRmax}(\sigma)+\mathrm{RLmax}(\sigma)-2},$$

which is equal to $A_n(x, y, x, y, 1, 1, t \,|\, \alpha, \alpha)$.

Theorem

For $0 \le j \le \lfloor n/2 \rfloor$, we have

$$A_n(x, y, t \,|\, \alpha) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j}(\alpha, t)(xy)^j (x + y)^{n-2j}, \tag{5}$$

where

$$\gamma_{n,j}(\alpha, t) = \sum_{\sigma \in \mathfrak{S}^{\mathrm{da}=0}_{n+1,\mathrm{asc}=j}} \alpha^{\mathrm{LRmax}(\sigma)+\mathrm{RLmax}(\sigma)-2} t^{\mathrm{RLmaxdd}(\sigma)} \tag{6a}$$

$$= \sum_{\sigma \in \mathfrak{S}^{\mathrm{cda}=0}_{n,\mathrm{exc}=j}} 2^{\mathrm{cyc}(\sigma)-\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)} t^{\mathrm{fix}(\sigma)}; \tag{6b}$$

The (α, t)-binomial-Eulerian polynomials $\widetilde{A}_n(x, y, t \mid \alpha)$ by

$$\widetilde{A}_n(x, y, t \mid \alpha) = \sum_{\sigma \in \mathcal{M}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} t^{\mathrm{LRmaxda}(\sigma)+\mathrm{RLmaxdd}(\sigma)} \alpha^{\mathrm{LRmax}(\sigma)+\mathrm{RLmax}(\sigma)-2},$$

which is equal to $A_n(x, y, x, y, 0, 1, t \mid \alpha, \alpha)$ because a permutation $\sigma \in \mathfrak{S}_n$ is an element of $\mathcal{M}_n$ if and only if $\mathrm{LRmaxp}(\sigma) = 1$.

Theorem

For $n \geq 1$, we have

$$\widetilde{A}_n(x, y, t \mid \alpha) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \widetilde{\gamma}_{n,j}(\alpha, t)(xy)^j (x + y)^{n-2j},$$

where

$$\widetilde{\gamma}_{n,j}(\alpha, t) = \sum_{\sigma \in \mathcal{M}^{\mathrm{da}=0}_{n+1,\mathrm{asc}=j}} \alpha^{\mathrm{RLmax}(\sigma)-1} t^{\mathrm{RLmaxdd}(\sigma)} = \sum_{\sigma \in \mathfrak{S}^{\mathrm{cda}=0}_{n,\mathrm{exc}=j}} \alpha^{\mathrm{cyc}\,(\sigma)} t^{\mathrm{fix}(\sigma)}.$$

(α, t)-Stirling-Eulerian numbers

Define the cyclic (α, t)-*Stirling-Eulerian numbers* by

$$\left\langle {n \atop k} \right\rangle_{\alpha,t}^{\mathrm{cyc}} := \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=n-k}} \alpha^{\mathrm{cyc}(\sigma)} t^{\mathrm{fix}(\sigma)} \qquad (1 \le k \le n).$$

Partition the set of fixed points of each permutation in $\mathfrak{S}_n$ in two categories, say *blue ones* and *red ones*, we have

$$\begin{aligned} A_n^{\mathrm{cyc}}(x,y,t(x+y)\,|\,\alpha) &= \sum_{m=0}^{n} \binom{n}{m} (\alpha t x)^{n-m} A_m^{\mathrm{cyc}}(x,y,ty\,|\,\alpha) \\ &= \sum_{m=0}^{n} \binom{n}{m} (\alpha t x)^{n-m} \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle_{\alpha,t}^{\mathrm{cyc}} x^{m-k} y^k, \end{aligned}$$

where $\left\langle {0 \atop k} \right\rangle_{\alpha,t}^{\mathrm{cyc}} = \left\langle {k \atop 0} \right\rangle_{\alpha,t}^{\mathrm{cyc}} = \delta_{k,0}$.

Comparing the coefficients of $x^a y^b$ and $x^b y^a$ with $n = a + b$.

(α, t)-analogue of Chung-Graham-Knuth's formula

Hence, combining the symmetry of $A_n^{\mathrm{cyc}}(x, y, t(x+y)|\alpha)$ in x and y, we have

Theorem

For integers $a, b \geq 0$, we have

$$\sum_{k\geq 0}(\alpha t)^{a+b-k}\binom{a+b}{k}\left\langle {k \atop a} \right\rangle_{\alpha,t}^{\mathrm{cyc}} = \sum_{k\geq 0}(\alpha t)^{a+b-k}\binom{a+b}{k}\left\langle {k \atop b} \right\rangle_{\alpha,t}^{\mathrm{cyc}},$$

where $\left\langle {0 \atop k} \right\rangle_{\alpha,t}^{\mathrm{cyc}} = \left\langle {k \atop 0} \right\rangle_{\alpha,t}^{\mathrm{cyc}} = \delta_{k,0}$.

When $\alpha = t = 1$, it reduces to Chung-Graham-Knuth's symmetric Eulerian identity

$$\sum_{k\geq 0}\binom{a+b}{k}\left\langle {k \atop a-1} \right\rangle = \sum_{k\geq 0}\binom{a+b}{k}\left\langle {k \atop b-1} \right\rangle.$$

Generating functions of γ-coefficients

We can derive the exponential generating function for these two γ-coefficients, respectively.

Theorem

Let $u = \sqrt{1-4x}$. We have

$$1 + \sum_{n\geq 1} \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j}(\alpha, t) x^j \frac{z^n}{n!} = \left(\frac{u\, e^{\frac{1}{2}(t-1)z}}{u \cosh(uz/2) - \sinh(uz/2)} \right)^{2\alpha},$$

$$1 + \sum_{n\geq 1} \sum_{j=0}^{\lfloor n/2 \rfloor} \widetilde{\gamma}_{n,j}(\alpha, t) x^j \frac{z^n}{n!} = \left(\frac{u\, e^{(t-\frac{1}{2})z}}{u \cosh(uz/2) - \sinh(uz/2)} \right)^{\alpha}.$$

Table

$n\backslash j$	0	1	2
0	1		
1	αt		
2	α^2t^2	2α	
3	α^3t^3	$2\alpha+6\alpha^2t$	
4	α^4t^4	$2\alpha+8\alpha^2t+12\alpha^3t^2$	$4\alpha+12\alpha^2$
5	α^5t^5	$2\alpha+10\alpha^2t+20\alpha^3t^2+20\alpha^4t^3$	$16\alpha+40\alpha^2+20\alpha^2t+60\alpha^3t$

Table: The first values of $\gamma_{n,j}(\alpha,t)$ for $0\le 2j<n\le 5$.

For $0\le j\le\lfloor n/2\rfloor$, let $d_{n,j}(\alpha,t)=\gamma_{n,j}(\alpha,t)/2^j$, then,

$$A_n(x,y,t\,|\,\alpha)=\sum_{j=0}^{\lfloor n/2\rfloor}2^jd_{n,j}(\alpha,t)(xy)^j(x+y)^{n-2j}.$$

Cycle André permutations

Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. Say that σ is an **André permutation of the first kind (resp. second kind)** if σ has no double descents, i.e., $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, and each x-factorisation $u\,\lambda(x)\,x\,\rho(x)\,v$ of σ has property

- $\lambda(x) = \emptyset$ if $\rho(x) = \emptyset$,
- $\max(\lambda(x)) < \max(\rho(x))$ (resp. $\min(\rho(x)) < \min(\lambda(x))$) if $\rho(x) \neq \emptyset$ and $\lambda(x) \neq \emptyset$,

where $\lambda(x)$ and $\rho(x)$ are the the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all greater than x. (Note different from the x-factorisation before.) Let $\mathcal{A}_n^1$ (resp. $\mathcal{A}_n^2$) be the set of André permutations of the first (resp. second) kind in $\mathfrak{S}_n$. It is known that the cardinality of $\mathcal{A}_n^1$ (resp. $\mathcal{A}_n^2$) is the *Euler number* E_n.

Let $A := \{a_1, \ldots, a_k\}$ be a set of k positive integers. Let $C = (a_1, \ldots, a_k)$ be a cycle (cyclic permutation) of A with $a_1 = \min\{a_1, \ldots, a_k\}$. Then, cycle C is called an **André cycle** if the word $a_2 \ldots a_k$ is an André permutation of the first kind. We say that a permutation is a **cycle André permutation** if it is a product of disjoint André cycles. Let $\mathcal{CA}_n$ be the set of cycle André permutations of $[n]$.

Combinatorial interpretation

Theorem

For $0 \le j \le \lfloor n/2 \rfloor$*, we have*

$$d_{n,j}(\alpha, t) = \sum_{\substack{\sigma \in \mathcal{CA}_n \\ \mathrm{drop}(\sigma)=j}} t^{\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)}, \tag{7}$$

$$d_{n,j}(\alpha, t) = \sum_{\substack{\sigma \in \mathcal{A}^{(i)}_{n+1} \\ \mathrm{des}(\sigma)=j}} t^{\mathrm{rlminda}(\sigma)} \alpha^{\mathrm{rmin}(\sigma)-1}, \quad (i = 1, 2). \tag{8}$$

We prove this theorem by computing the exponential generating functions of both sides, and derive the $i = 1$ case of (8) from (7) by a bijection from $\mathcal{CA}_n$ to $\mathcal{A}^1_{n+1}$, and the $i = 2$ case by constructing another bijection from $\mathcal{A}^1_{n+1}$ to $\mathcal{A}^2_{n+1}$ via André trees,

Recall Carlitz-Scoville's (α, β)-Euerlian polynomials

Recall that

Theorem (Carlitz and Scoville)

$$\sum_{n\geq 0} A_n(x,y\,|\,\alpha,\beta)\frac{z^n}{n!} = (1+xF(x,y;z))^\alpha(1+yF(x,y;z))^\beta,$$

where $F(x,y;z)$ is given by

$$F(x,y;z) = \frac{e^{xz}-e^{yz}}{xe^{yz}-ye^{ez}}.$$

This is equivalent to the following rational generating function

$$\frac{A_n(x,y\,|\,\alpha,\beta)}{(1-x)^{n+\alpha+\beta}} = \sum_{j\geq 0} x^j y^{n-j}(j+\beta)^n\binom{\alpha+\beta+j-1}{j}, \tag{9}$$

where $A_n(x,y\,|\,\alpha,\beta) := \sum_{\sigma\in\mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)}y^{\mathrm{des}(\sigma)}\alpha^{\mathrm{LRmax}(\sigma)-1}\beta^{\mathrm{RLmax}(\sigma)-1}$.

Eulerian polynomials: from Euler's definition

Euler's definition $(D \cdot X)$

Let us consider the following two well-known linear operators on $\mathbb{C}[x]$:

X: multiplication by x;
D: the usual derivative.

$$[D, X] := DX - XD = 1.$$

Leonhard Euler (1707-1783)

$$\sum_{i \geq 0} x^i = \frac{1}{1-x}$$

$$\sum_{i \geq 0} (i+1) x^i = \frac{1}{(1-x)^2}$$

$$\sum_{i \geq 0} (i+1)^2 x^i = \frac{(1+x)}{(1-x)^3}$$

$$\sum_{i \geq 0} (i+1)^3 x^i = \frac{(1+4x+x^2)}{(1-x)^4}$$

Euler's

$$(D \cdot X)^n \left(\frac{1}{1-x}\right) = \sum_{i \geq 0} (i+1)^n x^i = \frac{A_n(x)}{(1-x)^{n+1}}.$$

q-derivative

For a fixed real number q with $0 < q < 1$ define the q-derivative operator δ_x by

$$\delta_x\, p(x) = \frac{p(qx) - p(x)}{(q-1)x},$$

where $p(x) \in \mathbf{K}[[x]]$. Thus $\delta_{x,q}\, x^k = [k]x^{k-1}$. Let

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}).$$

Recall the following q-binomial formulas

$$(x;q)_N = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix} (-1)^i x^i q^{i(i-1)/2},$$

$$\frac{1}{(x;q)_N} = \sum_{j\geq 0} x^j \begin{bmatrix} N+j-1 \\ j \end{bmatrix}.$$

Carlitz's q-Eulerian polynomials

Theorem (Carlitz 1975)

For $n \geq 0$, we have

$$\frac{\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}}{(x;q)_{n+1}} = \sum_{j \geq 0} x^j [j+1]^n = (\delta_x \cdot X)^n \left(\frac{1}{1-x} \right),$$

where $\mathrm{maj}(\sigma) := \sum_i \sigma_i > \sigma_{i+1}$.

How do we change the $\frac{1}{1-x}$ with another formal series? For instance, $\frac{1}{(x;q)_N}$? What happens?

A q-analogue of (α, β)-Eulerian polynomials

For positive integers α, β and $n \in \mathbb{N}$ we apply the operator $X^{1-\beta} \cdot \delta_x \cdot X^{\beta}$, we have

$$\sum_{j\geq 0} x^j \begin{bmatrix} j+\alpha+\beta-1 \\ j \end{bmatrix} = \frac{1}{(x;q)_{\alpha+\beta}};$$

$$\sum_{j\geq 0} x^j \begin{bmatrix} j+\alpha+\beta-1 \\ j \end{bmatrix} [j+\beta] = \frac{[\beta]+q^{\beta}[\alpha]x}{(x;q)_{\alpha+\beta+1}};$$

$$\sum_{j\geq 0} x^j \begin{bmatrix} j+\alpha+\beta-1 \\ j \end{bmatrix} [j+\beta]^2 = \frac{[\beta]^2+q^{\beta}([1+\alpha][\beta]+[\alpha][\beta+1])x+q^{2\beta+1}[\alpha]^2x^2}{(x;q)_{\alpha+\beta+2}}$$

Is there any combinatorial interpretation for these numerators?

Bi-Stirling-Macmahon Eulerian polynomial

For $\sigma \in \mathfrak{S}_n$ define the (α, β)-*major index*

$$\widetilde{\mathrm{maj}}(\sigma) = (n - \mathrm{rlmin}(\sigma))(\beta - 1) + (\mathrm{des}(\sigma) - \mathrm{lrmin}(\sigma) + 1)(\alpha - 1) + \mathrm{maj}(\sigma).$$

Theorem

We have

$$\frac{\sum_{k=1}^{n+1} E_{n+1,k}(\alpha, \beta, q) x^{k-1}}{(x;q)_{n+\alpha+\beta}} = \sum_{j \geq 0} x^j \begin{bmatrix} j+\alpha+\beta-1 \\ j \end{bmatrix} [j+\beta]^n$$

and

$$E_{n,k}(\alpha, \beta, q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{des}(\sigma) = k-1}} [\alpha]^{\mathrm{lrmin}(\sigma)-1} [\beta]^{\mathrm{rlmin}(\sigma)-1} q^{\widetilde{\mathrm{maj}}(\sigma)}.$$

We can prove it by induction and Carlitz's insertion method!

Recurrence

Lemma

For $n \geq 1$, we have

$$E_{n+1}(x;q) = \big((1-x)[\beta] + [n+\alpha+\beta-1]x\big)E_n(x;q) + (1-x)xq^{\beta}\delta_x(E_n(x;q)) \quad (10)$$

Extracting the coefficient of x^{k-1} $(1 \leq k \leq n)$ in Eq. (10), we have

$$E_{n+1,k}(\alpha,\beta,q) = q^{\beta+k-2}[n+\alpha+1-k]E_{n,k-1}(\alpha,\beta,q) + [\beta+k-1]E_{n,k}(\alpha,\beta,q),$$

which can be proved by a similar argument for the recurrence of Carlitz's q-Eulerian polynomials.

Note that If $x=1$ Eq. (10) reduces to $E_{n+1}(1;q) = [n+\alpha+\beta-1]E_n(1;q)$, which yields

$$E_{n+1}(1;q) = \prod_{i=0}^{n-1}[\alpha+\beta+i].$$

Some relevant known results

- When $\alpha = \beta = 1$, it reduces to Carlitz's q-Eulerian polynomials;
- When $\alpha = 1$, Butlerstudied these polynomials $E_{n,k}(1, \beta, q)$ using the q-Rook theory with a cycle counting parameter, see [Butler, 2004]
- When $\alpha = q = 1$, it reduces to Savage-Viswanathan's $1/k$-Eulerian polynomials, see [Savage and Viswanathan, 2012];
- These polynomials appear also in Nadeau-Tewariand Gaudin as a sepecial case of the remixed Eulerian numbers, see [Nadeau and Tewari, 2023, Gaudin, 2024].

Some problems and further directions for q-analogue

- Is there a combinatorial proof for our q-identity? even $q = 1$, NO (Probability proof exists)
- Carlitz-Scoville also considered another model called (α, β)-sequence, which has same recurrence with $A(x, y \mid \alpha, \beta)$, in case $q = 1$, using this model and Ball in Box method (barred permutations) is easy to prove! For q-version?
- How can prove our theorem using the P-partitions? Or P-partition can be used to prove some Carlitz's rational function with a Stirling statistics?
- Varvak studied the connection between (q-) normal ordering problem and (q-) Rook number, so can we extend our results in the context of q-rook theory.

References

Butler, F. (2004).
Rook theory and cycle-counting permutation statistics.
Adv. in Appl. Math., 33(4):655–675.

Dong, Y., Lin, Z., and Pan, Q. (2024).
A q-analog of the stirling–eulerian polynomials.
The Ramanujan Journal, 65(3):1295–1311.

Gaudin, S. (2024).
Remixed eulerian numbers: beyond the connected case.
arXiv preprint arXiv:2411.03466.

Ji, K. Q. (2023).
The (α, β)-eulerian polynomials and descent-stirling statistics on permutations.
arXiv preprint arXiv:2310.01053.

Ji, K. Q. and Lin, Z. (2024).
The binomial-stirling–eulerian polynomials.
European Journal of Combinatorics, 120:103962.

Nadeau, P. and Tewari, V. (2023).

Merci!