On Serre's conjecture II for groups of type E_7

Philippe Gille

Institut Camille Jordan, Lyon

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G-torsors

Let k be a field.

- Let G be an affine algebraic group. A G-torsor is an non-empty affine (right) G-variety X such that the action map

$$X \times_k G \to X \times_k X, \quad (x, g) \mapsto (x, x.g)$$

  is an isomorphism.
- The variety G itself equipped with the right translation is a G-torsor ; it is called the trivial G-torsor.
- A G-torsor X is isomorphic to G if and only if $X(k) \neq \emptyset$.
- We denote by $H^1(k, G)$ the set of isomorphism classes of G–torsors. It is a pointed set.

G-torsors, II

- We consider an exact sequence of affine algebraic groups $1 \to G_1 \xrightarrow{i} G_2 \xrightarrow{f} G_3 \to 1$.
- Then we have an exact sequence of pointed sets

$$1 \to G_1(k) \to G_2(k) \to G_3(k) \xrightarrow{\varphi}$$

$$H^1(k, G_1) \to H^1(k, G_2) \to H^1(k, G_3).$$

- The map $\varphi : G_3(k) \to H^1(k, G_1)$ is called the characteristic map and is defined by $\varphi(g_3) = \left[f^{-1}(g_3)\right]$.
- A classical example is that of μ_n where we have the Kummer isomorphism $k^\times/(k^\times)^n \cong H^1(k, \mu_n)$.

Forms

- If G is the automorphism group of an affine k-variety Y (having eventually more structures), there is a way to construct G-torsors.
- Let Y' be a k–form of Y, that is a k-variety such that $Y \times \overline{k} \cong Y' \times \overline{k}$. Then the k-functor

$$R \mapsto \mathrm{Isom}_R(Y_R, Y'_R)$$

  is representable by a G-torsor denoted by $\mathrm{Isom}(Y, Y')$.
- In this case $H^1(k, G)$ classifies isomorphism classes of k-forms of Y.
- Artificial exemple : Hilbert 90 theorem, i.e. $H^1(k, \mathrm{GL}_n) = 1$.
- Let q be a regular quadratic form. Then $H^1(k, O(q))$ classifies isometry classes of regular quadratic forms of dimension $\dim(q)$.
- G is the split group of type G_2. Then $H^1(k, G)$ classifies octonions k–algebras and semisimple k-groups of type G_2.

Serre's conjecture II

- **Conjecture** (1962) : Assume that k is perfect of cohomological dimension ≤ 2. Let G be a semisimple simply connected k–group. Then $H^1(k, G) = 1$.
- The assumption on k means that the Galois cohomology groups $H^i(k, A)$ vanish for each finite Galois module A and each $i \geq 3$.
- The assumption on G means first that G is smooth connected with trivial solvable radical and secondly there are no non trivial separable isogeny of algebraic groups $G' \to G$.
- If k embeds in $\mathbb{C}$, it is equivalent to say that $G(\mathbb{C})$ is simply connected.
- Examples : SL_n, Sp_{2n}, Spin_{2n},...
- Examples of fields : $\mathbb{Q}_p$, $\mathbb{Q}(i)$, $\mathbb{F}_p(x)$, $\mathbb{C}((x, y))$, $\mathbb{C}(x, y)$.

Comments on Serre's conjecture II

- The conjecture has strong consequences on the classification on semisimple algebraic groups. More precisely, k-forms of G are classified by their quasi-split form and some Brauer invariants (the Tits class).
- One may dream about stronger conjectures. One is to say that G has enough cohomological invariants, that is one can detect the triviality of a given class of $H^1(k,G)$ by successive Galois cohomology classes.
- This would answer another question by Serre namely whether a G-torsor having a 0-cycle of degree one has a rational point.
- For classical groups, this is known (Bayer/Lenstra, Black) as well for quasi-split groups excepted E_8 (Chernousov, Garibaldi, G.).
- The case of other exceptional groups is widely open.

Approaches to the conjecture

- There are essentially two approaches. The first one is to fix the kind of fields and the other one the kind of groups.
- The first evidence has been the case of p-adic fields done by Kneser but also by Bruhat/Tits in an uniform way.
- For totally imaginary fields except E_8, this is due to Kneser-Harder in the late 60's. Chernousov solved the E_8 case in 1989.
- The case of global geometric fields is due to Harder in 1975.
- The case of $\mathbb{C}((x, y))$ is due to Colliot-Thélène/Ojanguren/Parimala in 2000.

Geometric fields, index and period

- The case of field of functions of surfaces excluded E_8 was obtained by Colliot-Thélène/G/Parimala in 2003.
- The general case (including E_8) was solved in 2013 by He/de Jong/Starr using deformation techniques of algebraic geometry. This is essentially uniform !
- One common thing between those fields is that they share the "index=period" property for algebras.
- It means that for any central division k-algebra D, its index is the same that its period, that is its exponent in the Brauer group $\mathrm{Br}(k)$.
- For number fields, it is a consequence of the Brauer/Hasse/Noether's theorem. For fields of functions of surfaces, it is a result by de Jong (2002).

There are other fields

- Let $n \geq 1$ be an integer. Merkurjev constructed in 1991 a field k of cohomological dimension 2 with quaternion algebras $Q_1, \dots, Q_n$ such that $A = Q_1 \otimes Q_2 \cdots \otimes Q_n$ is a division algebra.
- Then A is of period 2 and of index 2^n.
- This field is contructed by an infinite tower of fields.
- Excepted the E_8 case, conjecture II is proven for fields of cohomological dimension ≤ 2 satisfying the "period=index" property for period 2 and 3.
- Next we discuss mainly then exotic fields by the second viewpoint, that is by selecting a class of algebraic groups.

Separable cohomological dimension

- By using Kato's Galois cohomology groups, there is a way to extend the setting also in the positive characteristic. The relevant cohomological dimension is called the separable cohomological dimension.

- Serre refined his conjecture in 1994 as follows.

  Refined conjecture Let G be a semisimple simply connected k–group. Assume that k satisfies $\mathrm{scd}_l(k) \leq 2$ for each $l \in S(G)$. Then $H^1(k, G) = 1$.

- The finite set $S(G)$ is called the set of torsion primes of G.

Reduced norms

- Let A be a central simple algebra and consider its special linear group $\mathrm{SL}_1(A)$. Then $H^1(k, \mathrm{SL}_1(A)) = k^\times / \mathrm{Nrd}(A^\times)$.
- **Theorem** (Suslin, G in positive characteristic) Let p be a prime number. The following are equivalent :

  (i) $\mathrm{scd}_p(k) \leq 2$;

  (ii) for each finite separable field extension L/k and each central simple L–algebra B of p-primary index, we have $L^\times = \mathrm{Nrd}(B^\times)$.
- In other words, the surjectivity of reduced norms characterizes the property $\mathrm{scd}(k) \leq 2$.

Classical groups

- Bayer/Parimala proved Serre's conjecture II for classical groups (and G_2, F_4) in 1995 for perfect fields.
- It was extended by Berhuy/Frings/Tignol in free characteristic in 2008.
- The simplest example is that of $\mathrm{Spin}(q)$. The underlying classification fact is that quadratic forms are classified by their discriminant and their Hasse invariant for a field satisfying $\mathrm{scd}_2(k) \leq 2$.
- The most complicated case is that of outer groups of type A.

Norm groups

- Let X be a k-variety. Its (separable) norm group $N_X(k)$ is the subgroup of $k^\times$ generated by the $N_{L/k}(L^\times)$ for L running over the finite separable finite extensions of k satisfying $X(L) \neq \emptyset$.
- Example 1 : If X has a k-point or more generally a separable 0-cycle of degree one, then $N_X(k) = k^\times$. Note also the property $N_{L/k}\big(N_{X_L}(L)\big) \subseteq N_X(k)$ for each finite separable finite extension L/k.
- Example 2 : If $X = \mathrm{SB}(A)$, then $N_X(k) = \mathrm{Nrd}(A^\times)$.
- Example 3 : If $X = \{q = 0\}$ is a smooth projective quadric, then $N_X(k)$ is the image of the spinor norm $\Gamma(q)(k) \to k^\times$, that is the subgroup of $k^\times$ generated by even products $q(v_1)q(v_2)$ of non zero values of q.

Norm groups, II

The following statement generalizes Suslin's theorem.

- **Proposition** Let G be a k-group satisfying the hypothesis of Serre's conjecture II. Let X be the Borel variety of G. Then $N_X(k) = k^\times$.
- In particular for the quadratic form q (even, trivial discriminant), we have $N_{X_q}(k) = k^\times$, so that the map $k^\times \to H^1\big(k, \mathrm{Spin}(q)\big)$ is trivial.
- More generally, under the conditions of conjecture II, we are interested in the image of the map $f_* : k^\times/(k^\times)^n \cong H^1(k, \mu_n) \to H^1(k, G)$ for $f : \mu_n \to G$ a k–homomorphism.
- If $f(\mu_n)$ is central, then using the "norm principle" and the proposition, we have that f_* is the trivial map.

The non-central case

Again G satisfies the hypothesis of conjecture II. We are given $f : \mu_n \to G$ and denote by $H = Z_G(f)$ its centralizer.

- By Steinberg-Springer's connectedness theorem, H is connected so that H is reductive.
- The map $f_* : H^1(k, \mu_n) \to H^1(k, G)$ factorizes then by $H^1(k, H)$ and $f(\mu_n)$ is central in H. The vanishing of f_* boils then to the vanishing of $H^1(k, \mu_n) \to H^1(k, H)$.
- The problem is that H can be semisimple not simply connected and even not semisimple. In the split case, conjugacy classes of subgroups μ_n of G are classified by means of Kac coordinates.
- It provides then a precise combinatorial classification of the centralizers H occuring there. A consequence is that the quasi-split form H^{qs} of H contains a maximal quasi-trivial torus.
- This is enough to ensure the vanishing of $H^1(k, \mu_n) \to H^1(k, H)$. To summarize the map $f_* : H^1(k, \mu_n) \to H^1(k, G)$ is trivial.

Harder quadratic trick

Let L/k be a quadratic separable field extension and denote by $\Gamma = \langle \sigma \rangle = \mathrm{Gal}\,(L/k)$.

- Let G be split semisimple simply connected algebraic group of rank r and let B be a Borel subgroup of G.

- If $g \in G(L)$ is general enough, then ${}^g B_L \cap \sigma({}^g B_L)$ is a maximal k–torus T of G.

- T is isomorphic to a product of r norm one tori $R^1_{L/k}(\mathbb{G}_m)$, that is defined by the equation $N_{L/k}(y) = 1$ so that T contains μ_2^r and all maps $H^1(k, \mu_2)^r \to H^1(L/k, T) \to H^1(L/k, G)$ are onto.

- If $\mathrm{scd}_2(k) \leq 2$, it follows that $H^1(L/k, G) = 1$.

Sivatski theorem

Here k is a field of characteristic $\neq 2$ and satisfying $\mathrm{cd}_2(k) \leq 2$.

- From the vanishing of $H^1(k, \mathrm{Spin}(\mathbb{H}^n))$, we know that quadratic forms are classified by their discriminant and their Hasse-Witt invariant.

- Depending of your background, it can be easier to proceed in the way around and to classify quadratic forms first. The Merkurjev isomorphism $I^2(k)/I^3(k) \xrightarrow{\sim} {}_2\mathrm{Br}(k)$ provides an isomorphism $I^2(k) \xrightarrow{\sim} {}_2\mathrm{Br}(k)$ which maps the quadratic form n_Q to $[Q]$ for each quaternion algebra Q.

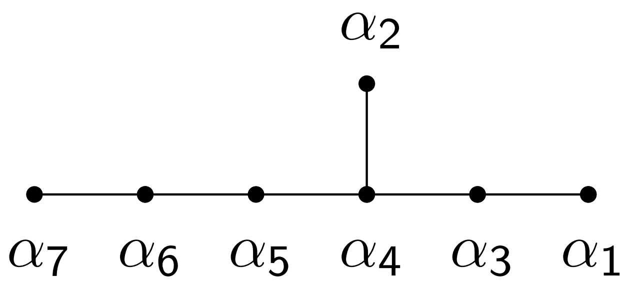
- This can be made more precise. For each central simple division algebra D of degree $2^n \geq 2$ and of period 2, there exists an unique (anisotropic) quadratic form $q_D \in I^2(k)$ of dimension $2n+2$. Furthermore D is a tensor product of n quaternion algebras.

Sivatski theorem II

- Another consequence is the following linkage property.
- Assume that q_D and $q_{D'}$ become isomorphic after a quadratic extension L/k, then we can decompose $D \cong D_0 \otimes_k Q$ and $D' = D_0 \otimes_k Q'$ where Q and Q' are quaternion algebras (D and D' are of degree ≥ 4).
- There are also classification results for odd dimensional quadratic forms and even dimensional quadratic forms with non trivial discriminant.

Groups of type E_7

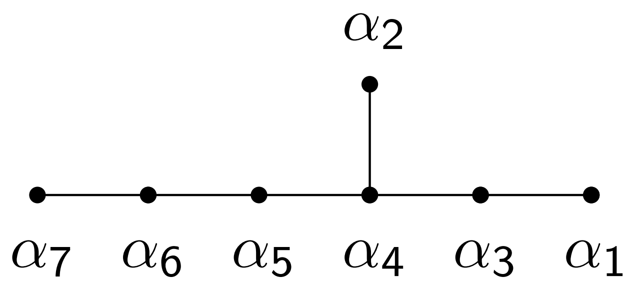
We deal with a semisimple simply connected algebraic group G of type E_7



- Let G_0 be the split form of G. It carries a linear representation $G_0 \hookrightarrow \mathrm{GL}_{56}$.
- This representation descends to a representation $G \hookrightarrow \mathrm{GL}_7(A)$ where A is central simple algebra of degree 8 and of period 2.
- A is called the Tits algebra of G.
- In general, we know that the algebra A is Brauer equivalent to a tensor product of 4 quaternion algebras. If $\mathrm{scd}_2(k) \leq 2$, Sivatski result states that A is a tensor product of 3 quaternion algebras.

The main result

Again G is a semisimple simply connected k-group of type



Theorem. We assume that $\mathrm{scd}_2(k) \leq 2$ and that $\mathrm{scd}_3(k) \leq 2$.

- (1) We have $H^1(k, G) = 1$ if A is of index ≤ 4.
- (2) If A is of index 1 (resp. 2, 4) then G is split (resp. has k–rang 4, 1) and admits a parabolic k-subgroup of type $\emptyset$ (resp. $\{2, 5, 7\}$, $\{2, 3, 4, 5, 6, 7\}$).
- (3) Assume that A is of index 8. Let $[z] \in H^1(k, G)$. Then $[z] = 1$ if and only if there exists a quaternion algebra Q and embeddings $\mathrm{SL}_1(Q) \to G$ and $\mathrm{SL}_1(Q) \to {}_zG$ which are geometrically "coroot embeddings".

A bunch of remarks

- Remarks : If A is of index 8, then G is anisotropic. Also there exists an embedding $\mathrm{SL}_1(Q) \to G$ but we do not know how to show that $\mathrm{SL}_1(Q)$ embeds in the twisted k–group ${}_zG$. We can attack this question by hermitian forms over biquaternion algebras.
- It follows that groups of type E_7 are classified by their Tits algebra in index $1, 2, 4$ for fields satisfying $\mathrm{scd}_2(k) \leq 2$ and $\mathrm{scd}_3(k) \leq 2$.
- We do not know whether all triquaternion algebras occur as Tits algebra of groups of some group of type E_7. A partial result is that biquaternion algebras occur (Quéguiner-Mathieu).
- What about exceptional groups of type E_6, E_8 ?
- There is no Sivatski's theorem for odd primes, this is a first difficulty in type E_6 for the relevant algebras of index 27 and period 3. Also groups of type E_6 with Tits algebra of index 9 are always anisotropic.

The "three quaternions" problem

- Interesting groups inside groups of type E_7 are those of type $D_6 \times A_1$ and D_6.
- **Question.** Let Q_1, Q_2, Q_3 be quaternion algebras. Does it exist an inner semisimple k–group H of type D_4 such that the Tits algebras of H are Brauer-equivalent to $Q_1 \otimes_k Q_2$, $Q_2 \otimes_k Q_3$, $Q_1 \otimes_k Q_3$?
- If the question admits a positive answer for k satisfying the assumptions of Serre's conjecture II for E_7, then one can show that Serre's conjecture II holds for E_7.

A statement for E_8

For simplicity, we assume that k is of characteristic zero.

- Bogolomov conjectured that $\mathrm{cd}(F^{ab}) = 1$ for any field F (possibly containing an algebraically closed field). It would imply that a F–group of type E_8 is split by an abelian extension of degree $2^a 3^b 5^c$. This is an open question which implies conjecture II for E_8.

- Under the hypothesis of conjecture II, we know that a k-torsor X under E_8 has a 0-cycle of degree one. So answering Serre's question on 0-cycles of degree one for torsors would conjecture II for E_8. We end with the following statement.

- **Theorem.** Assume that $\mathrm{cd}_2(k) \leq 2$, $\mathrm{cd}_3(k) \leq 2$ and $\mathrm{cd}_5(k) \leq 2$. Let $[z] \in H^1(k, E_8)$ (where E_8 stands for the Chevalley group of type E_8). Then $[z] = 1$ if and only if the group ${}_zE_8(k)$ has torsion.

Merci.