Isospectralité et cohomologie galoisienne

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The Laplace-Beltrami operator

Let (M, g) be a Riemannian manifold of dimension n, for example an open bounded subset Ω of $\mathbb{R}^n$.

- We consider the Laplace-Beltrami operator

$$\Delta_M : C^\infty(M, g) \to C^\infty(M, g).$$

- We have

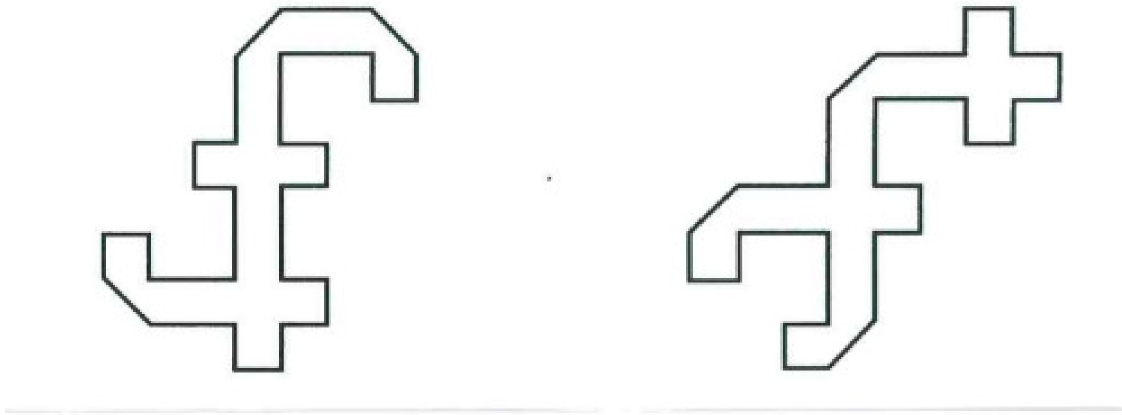
$$\Delta_M(f) = \frac{-1}{\sqrt{\det(g)}} \sum_{i,j} \frac{\partial}{\partial_{x_j}} \Big(g^{i,j} \sqrt{\det(g)} \frac{\partial}{\partial_{x_i}} f \Big) \quad .$$

- For $\Omega \subset \mathbb{R}^n$, this is the standard Laplacian

$$\Delta(f) = - \sum_{i=1,\dots,n} \frac{\partial^2}{\partial^2_{x_i}} f \quad .$$

- If M is compact, it is a self-adjoint operator with respect to the L^2 norm.
- The spectrum of (M, g) is the set of eigenvalues of Δ_M.

Can one hear the shape of a drum ?

- For the bounded domain $\Omega \subset \mathbb{R}^n$, the *Dirichlet spectrum* consists of the eigenvalues λ such that there exists a function f satisfying $\Delta(f) = \lambda f$ and $f_{|\partial\Omega} = 0$.
- The Dirichlet spectrum is what we hear.
- Two domains Ω_1, Ω_2 are said to be isospectral (or homophonic), if they have the same Dirichlet spectrum.
- The first example of non-isometric Dirichlet isospectral domains of $\mathbb{R}^2$ is due to Gordon-Webb-Wolpert in 1992.
- 

Isospectrality, the compact case

- The compact case was considered before (Weyl,...). Following Milnor, we start with the case of a torus $X = \mathbb{R}^n/L$ where L is a lattice.
- The eigenfunctions for X are the $\exp(2\pi i\phi(x))$ for ϕ running over the dual lattice $L^\vee = \mathrm{Hom}(L, \mathbb{Z})$.
- The respective eigenvalues are the $(2\pi)^2 \parallel \phi \parallel^2$.
- According to Witt, there exist two self-dual lattices $L_1, L_2 \subseteq \mathbb{R}^{16}$ which are distinct (no orthogonal transformation carries L_1 to L_2) and such that the lengths of L_1 and L_2 coincide.

The case of tori

- We define $X_1 = \mathbb{R}^{16}/L_1$ and $X_2 = \mathbb{R}^{16}/L_2$ where $L_1, L_2 \subseteq \mathbb{R}^{16}$ are the Witt self-dual lattices.

- By construction X_1 and X_2 have same spectrum and are not isometric.

- *Observation 1 :* Equivalently, the Riemannian manifolds X_1, X_2 have also the same length spectrum, which consists of the lengths of closed geodesics.

- *Observation* 2 *:* Since $L_1 \cap L_2$ is of finite index in L_1 (resp. L_2), it follows that X_1 and X_2 are *commensurable*.

- We have a diagram

$$X_1 \xleftarrow{\;p_1\;} \mathbb{R}^{16}/(L_1 \cap L_2) \xrightarrow{\;p_2\;} X_2.$$

where p_1, p_2 are finite covers.

Hyperbolic compact surfaces

- The next example is that of $X_\Gamma = \Gamma \backslash \mathbb{H}$ where $\mathbb{H}$ stands for the Poincaré half-plane and $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a discrete torsion-free cocompact subgroup.

- Here also the knowledge of the spectrum is the same as the length spectrum (Huber, 1957).

- The length spectrum consists of the

$$d_\gamma = \mathrm{Inf}_{P \in \mathbb{H}} \big\{ d_{\mathbb{H}}(P, \gamma P) \big\}$$

where γ runs over $\Gamma \setminus \{0\}$.

Hyperbolic compact surfaces II

- Using quaternion algebras defined over some number field, Vigneras constructed in 1980 two isospectral non-isometric surfaces $X_1 = X_{\Gamma_1} = \Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$ and $X_2 = X_{\Gamma_2} = \Gamma_2 \backslash \mathrm{PSL}_2(\mathbb{R})$ where Γ_1, Γ_2 are subgroups of $\mathrm{PSL}_2(\mathbb{R})$ as above.

- Once again X_1 and X_2 are commensurable.

- More precisely, we take a totally real number field k, that is $k \otimes_k \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{[k:\mathbb{Q}]}$, a quaternion k–algebra Q satisfying $Q \otimes_k \mathbb{R} = M_2(\mathbb{R}) \times \mathbf{H}^{[k:\mathbb{Q}]-1}$.

- Now take a maximal order $\mathscr{Q}$ of Q. Then

$$\Gamma = \mathrm{SL}_1(\mathscr{Q})/\pm 1 = \Big\{ q \in \mathscr{Q} \mid q\overline{q} = 1 \Big\} / \pm 1$$

  is an arithmetic subgroup of $\mathrm{PSL}_2(\mathbb{R})$ of the desired shape.

Prasad-Rapinchuk's results

Their idea is to work up to commensurability.

- **Theorem 1.** We are given two compact locally symmetric spaces X_1, X_2 with nonpositive sectional curvature. If they have same Laplace-Beltrami spectrum with multiplicities, then they share the same length spectrum, that is $L(X_1) = L(X_2)$.
- **Theorem 2.** Let X_1, X_2 be arithmetically defined hyperbolic manifolds of dimension $d \not\equiv 1[4]$. If X_1 and X_2 are *length commensurable*, (i.e. $\mathbb{Q}\,.\,L(X_1) = \mathbb{Q}\,.\,L(X_2)$), then X_1 and X_2 are commensurable.
- The theorem is wrong in each dimension $d \equiv 1[4]$.
- We are considering manifolds of the shape $\Gamma \backslash \mathbb{H}^d$ where Γ is an arithmetic subgroup of $\mathrm{PO}(d, 1)$.
- There are a bunch of other results.

Algebraic interlude

We are interested in maximal tori of a given reductive group G defined over a field k, e.g. the special orthogonal group $\mathrm{SO}(q)$ of a quadratic form q.

- Maximal tori occur as centralizer of semisimple regular elements of $G(k)$. Geometrically (i.e. on the algebraic closure) there are isomorphic to $(\mathbb{G}_m)^r$ and are all $G(k_s)$–conjugated.
- The maximal tori of $\mathrm{GL}_2(\mathbb{R})$ are $(\mathbb{R}^\times)^2$ and $\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts on $\mathbb{C} \cong \mathbb{R}^2$.
- More generally, let A be an étale algebra of degree n, that is $A = k_1 \times \cdots \times k_r$ where the k_i's are finite separable extensions of k. Since A is a k–vector space, the (left) multiplication of $A^\times$ on A induces an embbedding $A^\times \subseteq \mathrm{GL}(A) \cong \mathrm{GL}_n(k)$.
- All maximal k–tori occur in this way and are denoted by $R_{A/k}(\mathbb{G}_m)$; this is the Weil restriction.
- We need to pay attention to the fact that the algebra structure of A/k is not encoded in the k–torus $R_{A/k}(\mathbb{G}_m)$.

Type of maximal tori

- Let T be a maximal k–torus of G. We can consider the set of root $\Phi(G_{k_s}, T_{k_s})$ with respect to the adjoint representation. It is a Galois subset of the character group $X^*(T_{k_s})$.

- We have then a finite set with an action of $\Gamma_k = \mathrm{Gal}\,(k_s/k)$, this is the **type** of $T \subset G$.

- It can be defined as a Galois cohomology class in $H^1(k, W)$ where W is the Weyl group of the Chevalley form of G.

- For GL_n the type is nothing but the isomorphism class of the étale algebra we have seen for $W = S_n$.

- A natural problem is the following : let G, G' be reductive groups of the same shape (i.e. isomorphic over k_s) such that G and G' have same types of maximal tori. Are G and G' isomorphic ?

Relations with isopectrality

We assume here that k is a number field.

- Prasad and Rapinchuk showed that there are finitely many k–groups G' (of same geometrical shape as G) having same the same types of maximal tori.
- This finite set is denoted by $\mathrm{gen}(G)$ and is called the genus of G.
- If G is simple not of type A_n, D_{n+1} and E_6, then $\mathrm{gen}(G) = \{G\}$, that is, G is determined by its maximal tori.
- For the types A_n, D_{n+1} and E_6, $\mid \mathrm{gen}(G) \mid \geq 2$ in general.
- This is the way to construct noncommensurable length–commensurable arithmetically defined locally symmetric spaces of type A_n, D_{2n+1} and E_6.
- In all other cases (except possibly D_4), length-commensurable arithmetically defined locally symmetric spaces are commensurable.

Relations with isopectrality, II

We assume that k is a number field.

- Apart from the cases type A_n, D_{2n+1} and E_6 (and possibly D_4), length-commensurable arithmetically defined locally symmetric spaces are commensurable.

- This extends Theorem 2 on arithmetically defined hyperbolic d–manifolds for $d \not\equiv 1[4]$ since $\mathrm{PO}(d,1)$ is of type $B_{d/2}$ (resp. $D_{\frac{d+1}{2}}$) if d is even (resp. odd).

- For simplicity, assume that $k = \mathbb{Q}$. Given such a manifold $X = \Gamma \backslash G(\mathbb{R})/K$, the core of Prasad-Rapinchuk's paper is to relate the lengths of the elements of Γ with the maximal tori of the $\mathbb{Q}$–algebraic group G.

Remarkable groups

For the remaining part of the talk, we remain on the algebraic side.

- Let A be a central simple k–algebra of degree d and consider the linear algebraic k–group $\mathrm{GL}_1(A)$.
- Given an étale k–algebra K of degree d, the k–torus $R_{K/k}(\mathbb{G}_m)$ embeds in $\mathrm{GL}_1(A)$ with type $[K]$ if and only if $A \otimes_k K$ is isomorphic to $M_d(K)$.
- In this case, the *embedding problem* is then easy.
- Given another central simple algebra A' of degree d, it follows that if A and A' generate the same class in the Brauer group, then G and G' are toral equivalent.
- What about the converse ?
- It holds over a number field, but is false in general already in degree 2 (Garibaldi-Saltman, 2010).
- With N. Beli and T.-Y. Lee, we adressed similar questions for octonions algebras.

Case of octonions

- $G = \mathrm{Aut}(C)$ and $G' = \mathrm{Aut}(C')$ for C, C' octonion k–algebras. These are k–groups of type G_2 of rank 2, the root system has 12 elements and its Weyl group is $\mathbb{Z}/2\mathbb{Z} \times S_3$.

- If T is a maximal torus of G, its type is the data (k_2, k_3) where k_2 (resp. k_3) is a quadratic (resp. cubic) algebra.

- If k is a number field, C is isomorphic to C' iff G is toral-equivalent to G'.

- This is false over a general field.

- More precisely we extended the method of Garibaldi-Saltman involving 3-Pfister forms.

Case of octonions, II

- The embedding problem is to determine in terms of C whether a given couple (k_2, k_3) occurs in the list of types arising from G where k_i is an étale algebra of degree i).
- There is no nice criterion if k_3 is a cubic field. In any event the problem can be rephrased by saying that a certain algebraic affine k–variety X has a k-point.
- More precisely, that variety X may have quadratic and cubic points but no rational point. This is against *Springer's principle*.
- Geometrically speaking X is a G-homogeneous space whose geometric stabilizers are maximal tori. It is the first counterexample to the Springer principle of this shape after Florence (resp. Parimala) with finite geometric stabilizers (resp. parabolic stabilizers).

What is next ?

- Cases of other groups : results of Fiori-Scavia on groups of type F_4 (2019).
- Study of embedding problems in the arithmetic case : Bayer-Fluckiger, Lee and Parimala for classical groups. This leads to very subtle computations of Class Field Theory.
- Study of finiteness for the genus for algebraic groups over finitely generated fields over $\mathbb{Q}$: work of Chernousov, Rapinchuk and Rapinchuk on the $\mathrm{GL}_1(A)$-case and of spinor groups of quadratic forms.
- *Merci.*

References

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