Exercises, March 6, 2025

Let R be a base ring (commutative, unital).

1) Write the Hopf algebra structure for the multiplicative group $\mathbb{G}_{m,R} = \mathrm{Spec}(R[t,t^{-1}])$.

2) (i) Prove that there are no non-trivial homomorphisms from $\mathbb{G}_{m,R}$ to $\mathbb{G}_{a,R}$.
(ii) If R is reduced, prove that there are no non-trivial homomorphisms from $\mathbb{G}_{a,R}$ to $\mathbb{G}_{m,R}$.
(iii) If $\epsilon \in R$ is nonzero and $\epsilon^2 = 0$, use it to construct a non-trivial homomorphism from $\mathbb{G}_{a,R}$ to $\mathbb{G}_{m,R}$.

3) Let I be a set and consider the constant R-sheaf F_I defined by $F_I(S) = I$ for each R–ring S. Is the functor F_I representable by an affine R–scheme?

4) Assume that R is a discrete valuation ring (DVR for short) of fraction field K and of residue field k. Let $G = (\mathbb{Z}/2\mathbb{Z})_R$ be the constant R-group scheme associated to the abstract group $\mathbb{Z}/2\mathbb{Z}$. Consider the open subscheme

$$G' = \mathrm{Spec}(R) \sqcup \mathrm{Spec}(K)$$

of G and the closed subscheme

$$G'' = \mathrm{Spec}(R) \sqcup \mathrm{Spec}(k)$$

of G. Is G' (resp. G'') a R-subgroup scheme of G?

5) Let $\mathfrak{X}, \mathfrak{Y}$ be R–schemes and assume that $\mathfrak{Y}$ is separated. Using our version of Yoneda's lemma, shows that $\mathrm{Hom}_{\mathrm{Spec}(R)}(\mathfrak{X}, \mathfrak{Y}) \cong \mathrm{Hom}_R(h_{\mathfrak{X}}, h_{\mathfrak{Y}})$.