Exercises, March 13, 2025

Let R be a base ring (commutative, unital).

1) (i) Show that the map $R[X]/(X^2-1) \to R \times R$ sending X to $(1,-1)$ defines a homomorphism $f : (\mathbb{Z}/2\mathbb{Z})_R \to \mu_{2,R}$ of group schemes.

(ii) If $2 \in R^\times$, show this is an isomorphism.

(iii) Show that $(\mathbb{Z}/2\mathbb{Z})(R)$ corresponds to idempotents in R. In these terms write out the map $(\mathbb{Z}/2\mathbb{Z})(R) \to \mu_2(R)$.

(iv) If $2 \notin R^\times$, show that f is not a monomorphism.

2) Assume that R is connected and consider a diagonalizable R–group scheme $\mathfrak{G} = \mathfrak{D}(A)$. Show that the R–scheme $\mathfrak{G}$ is finite over R if and only if A is finite (Hint: start with the case A finitely generated).

3) Let $\mathbb{C}$ be the field of complex numbers. Let $\alpha := 2025 \in \mathbb{C}$. Show that there is a unique $\mathbb{C}$-group scheme homomorphism $f : (\mathbb{Z})_{\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$ sending $1 \in (\mathbb{Z})_{\mathbb{C}}(\mathbb{C}) = \mathbb{Z}$ to $2025 \in \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) = \mathbb{C}^\times$. Show also that f is a monomorphism but not a closed embedding. (*Hint:* establish that $\mathrm{Hom}_{R-sch}(\mathbb{Z}_R, X) = \prod_{n\in\mathbb{Z}} X(R)$ for any R–scheme X.).